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HEAT KERNEL COEFFICIENT E_4 FOR NONMINIMAL OPERATOR IN CURVED SPACE

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1 Introduction

In this paper we continue a computer study of the heat kernel expansion of elliptic differential operators on curved manifolds and in the presence of arbitrary gauge fields started in [1]. The coefficients in the asymptotic expansion of the diagonal heat kernel elements, called DeWitt-Seeley-Gilkey (DWSG) coefficients, are of fundamental importance in quantum field theory, quantum gravity, spectral geometry and topology of manifolds. Many quantities of interest (such as the effective action, Green function, anomalies in quantum field theory, the invariants of manifolds in spectral geometry) are expressed in terms of DWSG coefficients [2].

The heat kernel of a positive elliptic differential operator A of the order $2r$, acting on a bundle of k-tensors whose base is a compact closed n-dimensional manifold M, has a short-time expansion [3, 4, 5] in terms of geometrical invariants

$$\langle x|e^{-tA}|x\rangle \sim \sum_{m\geq 0} \mathrm{E_m}(\mathrm{x}|\mathrm{A})\,\mathrm{t}^{\frac{\mathrm{m-n}}{\mathrm{2r}}}, \qquad \mathrm{t} \to 0_+ \,. \tag{1}$$

Up to now the most complete results were obtained for the second-order operator $A = -\Box + X$ (X is a matrix in internal space), based on the DeWitt ansatz for heat kernel matrix elements. However, the DeWitt method does not apply to higher-order operators and *nonminimal* operators whose leading term is not a power of the Laplace operator. The simplest example of nonminimal operator is the Navier-Lamé operator of classical elasticity, $\mu\Delta\vec{V} + (\lambda+\mu)\nabla(\nabla\vec{V})$, involving a coupling among the components of a vector-valued function (the Lamé constants, λ and μ, characterize the material).

In recent years nonminimal operators of a similar sort have been encountered by physicists studying the quantization of gauge and gravitational fields in arbitrary gauges [2]. For example, the quantization of Yang-Mills field in an arbitrary covariant background gauge leads to the operator

$$A^{ab}_{\mu\nu} = -\delta_{\mu\nu}\Box^{ab} - (\frac{1}{\alpha} - 1)D^{ac}_{\mu}D^{cb}_{\nu} - 2f^{acb}G^{c}_{\mu\nu}, \tag{2}$$

where $\Box \equiv D_\mu D^\mu$ is the Laplace operator, D_μ is a covariant derivative containing the external field potential A_μ, $G_{\mu\nu}$ is a corresponding field strength and f^{abc} are the structure constants of a corresponding Lie algebra. For an analogous operator in quantum gravity see [6, 7]. The nontrivial question worth of study is the interplay between the dependence of a heat kernel on the gauge parameter α and various invariants appearing in its expansion. This in turn is important for calculating the gravitational conformal anomaly for gauge fields in a general covariant gauge and for investigating the possible gauge parameter dependence of the anomaly [8].

The main aim of this paper is to study the heat kernel expansion for a generic operator with the structure of (2),

$$-g^{\mu\nu}\Box + aD^{\mu}D^{\nu} + X^{\mu\nu}, \tag{3}$$

where $\Box$ is now a covariant generalization of the Laplace operator, D^{μ} is the covariant derivative involving affine and bundle connections, X is a tensor field and a is a scalar parameter (bundle indices are assumed implicitly). The expansion (1) is valid for nonminimal operators as well as minimal ones [3]. In view of inapplicability of the DeWitt method we follow the approach based on the technique of Widom's covariant pseudodifferential symbolic calculus developed in [9]. At present this is the most general method permitting one to handle operators of general type and of arbitrary order on curved manifolds.

There are several computer implementations of the algorithms for second-order operators based on general purpose computer algebra systems (for example, in [10] *Mathematica* and in [11] REDUCE and FORM were used). However, the calculation of the DWSG coefficients for nonminimal operators is much more complicated than for second-order minimal ones and is out of abilities of these programs for nontrivial orders of coefficients.

The algorithm developed in [9] was implemented as a C program [1]. Recently we rewrote this program. The new version allowed us to get for the first time the complete expression of the coefficient E_4 for the nonminimal operator (3) (earlier only partial results for E_4 were known [2, 12, 13].)

2 General Outline of Algorithm and Its Implementation

The heat operator $\exp(-tA)$ can be expressed in terms of the resolvent $(A-\lambda)^{-1}$:

$$e^{-tA} = \int_C \frac{id\lambda}{2\pi} e^{-t\lambda}(A-\lambda)^{-1}, \tag{4}$$

where the contour C goes counterclockwise around the spectrum of A.
For the matrix elements of the resolvent we take the following representation

$$G(x,x',\lambda) \equiv \langle x|\frac{1}{A-\lambda}|x'\rangle = \int \frac{d^n k}{(2\pi)^n\sqrt{g(x')}} e^{il(x,x',k)}\sigma(x,x',k;\lambda), \tag{5}$$

where σ is an amplitude, l is a phase function, $g(x) = \det g_{ab}(x)$, k is a wave vector.

The resolvent of operator A satisfies the equation $(A-\lambda)G = 1$ which leads to an equation for the amplitude:

$$(A(x, D_a + iD_a l) - \lambda)\sigma(x,x',k;\lambda) = I(x,x'), \tag{6}$$

where $I(x,x')$ is a transport function having both bundle and Lorentz indices. Expanding the amplitude σ in degrees of homogeneity of k : $\sigma = \sum_{m=1}^{\infty}\sigma_m(x,x',k;\lambda)$ we obtain the recursion equations for σ_m from Eq.(6). Solving these equations we can express DWSG coefficients by integrals of $[\sigma_m]$

$$E_m(x|A) = \int \frac{d^n k}{(2\pi)^n\sqrt{g}} \int_C \frac{id\lambda}{2\pi} e^{-\lambda}[\sigma_m](x,k,\lambda), \tag{7}$$

where $[\ldots]$ means transition to coincidence limit ($x = x'$). The integrals in Eq.(7) can be expressed in terms of gamma and Gauss hypergeometric functions for a wide class of operators A [1].

The covariant generalization of the properties of the flat space phase and transport functions is reduced to the following relations

$$[\{D_{a_1}\ldots D_{a_m}\}l] = 0, \quad m > 1; \qquad [\{D_{a_1}\ldots D_{a_m}\}I] = 0, \quad m \geq 1, \tag{8}$$

where $\{\ldots\}$ means symmetrizing in all indices. Eqs.(8) together with the "initial conditions" $[D_a l] = k_a$ and $[I] = E$ (E is the unit matrix) allow one to compute the coincidence limits for nonsymmetrized covariant derivatives $[D_{a_1}\ldots D_{a_m}l]$ and $[D_{a_1}\ldots D_{a_m}I]$. These all are universal polynomials in the torsion T^a_{bc}, curvature tensors R^a_{bcd}, W_{ab} and their covariant derivatives and are obtained directly from (8) by reducing all terms to a unified index ordering with the help of the Ricci identity. Substituting these coincidence limits into E_m we get the final result in terms of geometric quantities.

During computation various tensor simplifications must be done using Ricci, Bianchi and cyclic identities and other symmetry properties of tensors.

The above algorithm has been implemented in the C language. The C code of total length about 10000 lines contains about 200 functions for different manipulations with tensors and scalars. These functions are gathered into two programs DWSGCOEF and COLIM.

The COLIM program computes coincidence limits of the l and I functions and writes them to the disk. Once computed and stored the coincidence limits can be used in many calculations for different operators A.

The DWSGCOEF program computes E_m coefficients by the following steps:

1. *Reading input information (operator, order m, etc.)*
2. *Computing a set of asymptotic operators for constructing recursion equations*
3. *Computing σ_m with the help of the recursion equations*
4. *Taking the coincidence limit $[\sigma_m]$*
5. *Integrating $[\sigma_m]$ to obtain the coefficient E_m*
6. *Substituting tensor expressions for $[D_{a_1}\ldots D_{a_k}l]$ and $[D_{a_1}\ldots D_{a_k}I]$ into E_m*
7. *Reducing hypergeometric to elementary functions in the nonminimal case*
8. *Output E_m (and its trace in the nonminimal case)*

To cut down the intermediate swelling we use *term-by-term* strategy, i.e., Steps 4-6 are applied consecutively to single terms of σ_m generated at Step 3.

3 Computation of E_4

We computed the coefficient E_4 for operator (3) on a Pentium-75 PC. The full expression consists of 74 tensor terms with 56 different scalar coefficients and will be given elsewhere. Due to paper size limitations we can reproduce here only the trace of E_4. The expression for the trace still contains some novelties in comparison with the result of Branson et al. [13] who computed $\mathrm{tr}E_4$ without gauge field.

$$
\begin{aligned}
\mathrm{tr}E_4 = {} & (4\pi)^{-\frac{n}{2}}\{-C_1 D_i D^i X_j{}^j - C_2 D_i D_j X^{ij} - C_3 D_i D_j X^{ji} \\
& + \frac{C_4}{2}(X_i{}^i X_j{}^j + X_{ij}X^{ij}) + C_5 X_{ij}X^{ji} + C_6 X_{ij}W^{ij} - C_7 W_{ij}X^{ij} \\
& + C_8 W_{ij}W^{ij} + C_9 R_{ijkl}R^{ijkl} - C_{10}R_{ij}X^{ij} - C_{11}R_{ij}R^{ij} + C_{12}D_i D^i R \\
& + C_{13}R^2 - C_{14}RX_i{}^i\},
\end{aligned}
$$

where

$$
\begin{aligned}
C_1 = {} & \frac{1}{a^2(\frac{n}{2}-2)_4}\left\{(1-a)^{1-\frac{n}{2}}\left(1-a-\frac{na}{2}+\frac{na^2}{12}+\frac{n^2a^2}{16}+\frac{n^3a^2}{96}\right)\right. \\
& \left. -(1-a)^2+\frac{na^2}{3}+\frac{n^2a^2}{48}-\frac{5n^3a^2}{96}+\frac{n^4a^2}{96}\right\},
\end{aligned}
$$

$$
\begin{aligned}
C_2 = {} & \frac{1}{a^2(\frac{n}{2}-2)_4}\left\{(1-a)^{-\frac{n}{2}}\left(\frac{a(1-a)}{2}-2n+\frac{9na}{4}-\frac{13na^2}{24}+\frac{na^3}{24}+\frac{3n^2a}{8}\right.\right. \\
& \left. -\frac{3n^2a^2}{16}-\frac{n^3a^2}{48}-\frac{n^3a^3}{96}\right)-\frac{a(1-a)}{2}+2n-\frac{9na}{4}+\frac{7na^2}{24}+\frac{5n^2a}{8} \\
& \left. -\frac{7n^2a^2}{16}+\frac{n^3a^2}{12}\right\},
\end{aligned}
$$

$$
\begin{aligned}
C_3 = {} & \frac{1}{a^2(\frac{n}{2}-2)_4}\left\{(1-a)^{-\frac{n}{2}}\left(\frac{a(1-a)}{2}+n-\frac{3na}{4}-\frac{na^2}{24}+\frac{na^3}{24}-\frac{3n^2a}{8}\right.\right. \\
& \left. +\frac{3n^2a^2}{16}+\frac{n^3a^2}{24}-\frac{n^3a^3}{96}\right)-\frac{a(1-a)}{2}-n+\frac{3na}{4}-\frac{5na^2}{24}-\frac{n^2a}{8} \\
& \left. -\frac{n^2a^2}{16}+\frac{n^3a^2}{48}\right\},
\end{aligned}
$$

$$
C_4 = \frac{1}{a(\frac{n}{2}-1)_3}\left\{(1-a)^{-\frac{n}{2}}\left(-\frac{1}{2}+\frac{a}{4}+\frac{na}{8}\right)+\frac{1}{2}-\frac{a}{4}+\frac{na}{8}\right\},
$$

$$
\begin{aligned}
C_5 = {} & \frac{1}{a(\frac{n}{2}-1)_3}\left\{(1-a)^{-\frac{n}{2}}\left(\frac{1}{4}-\frac{3a}{8}+\frac{n}{4}-\frac{3na}{16}\right)\right. \\
& \left. -\frac{1}{4}+\frac{3a}{8}-\frac{n}{4}-\frac{3na}{16}-\frac{n^2a}{8}+\frac{n^3a}{16}\right\},
\end{aligned}
$$

$$
\begin{aligned}
C_6 = {} & \frac{1}{a^2(\frac{n}{2}-2)_5}\left\{(1-a)^{-1-\frac{n}{2}}\left(4a(1-a)^2+3n-\frac{9na}{4}-4na^2+\frac{31na^3}{8}\right.\right. \\
& -\frac{5na^4}{8}+\frac{3n^2}{4}-\frac{11n^2a}{4}+\frac{35n^2a^2}{16}+\frac{11n^2a^3}{96}-\frac{11n^2a^4}{32}-\frac{5n^3a}{16} \\
& \left. +\frac{11n^3a^2}{16}-\frac{23n^3a^3}{64}+\frac{n^3a^4}{64}+\frac{n^4a^2}{32}-\frac{5n^4a^3}{96}+\frac{n^4a^4}{64}\right)-4a(1-a) \\
& -3n-\frac{3na}{4}+\frac{5na^2}{4}-\frac{na^3}{8}-\frac{3n^2}{4}+\frac{n^2a}{2}-\frac{21n^2a^2}{16}+\frac{13n^2a^3}{96}-\frac{n^3a}{16} \\
& \left. +\frac{n^3a^2}{16}-\frac{3n^3a^3}{64}+\frac{n^4a^2}{32}+\frac{n^4a^3}{192}\right\},
\end{aligned}
$$

$$
\begin{aligned}
C_7 = {} & \frac{1}{a^2(\frac{n}{2}-2)_5}\left\{(1-a)^{-1-\frac{n}{2}}\left(-4a(1-a)^2+3n-\frac{25na}{4}+5na^2\right.\right. \\
& -\frac{17na^3}{8}+\frac{3na^4}{8}+\frac{3n^2}{4}-\frac{9n^2a}{4}+\frac{27n^2a^2}{16}-\frac{37n^2a^3}{96}+\frac{5n^2a^4}{32}-\frac{n^3a}{16} \\
& \left. +\frac{n^3a^2}{8}+\frac{n^3a^3}{64}-\frac{3n^3a^4}{64}+\frac{n^4a^3}{96}-\frac{n^4a^4}{64}\right)+4a(1-a)-3n+\frac{13na}{4} \\
& +\frac{na^2}{4}-\frac{na^3}{8}-\frac{3n^2}{4}+\frac{11n^2a^2}{16}+\frac{13n^2a^3}{96}-\frac{5n^3a}{16}+\frac{n^3a^2}{8}-\frac{3n^3a^3}{64} \\
& \left. -\frac{n^4a^2}{16}+\frac{n^4a^3}{192}\right\},
\end{aligned}
$$

$$
\begin{aligned}
C_8 = {} & \frac{1}{a(\frac{n}{2}-1)_2}\left\{(1-a)^{1-\frac{n}{2}}\left(-2+\frac{11na}{24}+\frac{na^2}{24}+\frac{n^2a}{48}-\frac{n^2a^2}{48}\right)\right. \\
& \left. +2-2a+\frac{13na}{24}-\frac{n^2a}{16}+\frac{n^3a}{48}\right\},
\end{aligned}
$$

$$
C_9 = \frac{(1-a)^{2-\frac{n}{2}}-16+n}{180};
$$

$$C_{10} = \frac{1}{a(\frac{n}{2}-1)_3}\left\{(1-a)^{-\frac{n}{2}}\left(-\frac{a}{2}+\frac{n}{2}-\frac{na}{3}+\frac{na^2}{12}-\frac{n^2a}{24}+\frac{n^2a^2}{24}\right)\right.$$
$$\left.+\frac{a}{2}-\frac{n}{2}+\frac{na}{3}-\frac{5n^2a}{24}\right\},$$

$$C_{11} = \frac{1}{a(\frac{n}{2}-1)_3}\left\{(1-a)^{-\frac{n}{2}}\left(\frac{a}{4}-\frac{n}{4}+\frac{37na}{180}-\frac{7na^2}{90}-\frac{na^3}{360}+\frac{n^2a}{24}\right.\right.$$
$$\left.-\frac{n^2a^2}{24}+\frac{n^3a}{1440}-\frac{n^3a^2}{720}+\frac{n^3a^3}{1440}\right)-\frac{a}{4}+\frac{n}{4}-\frac{37na}{180}+\frac{29n^2a}{360}$$
$$\left.-\frac{n^3a}{1440}+\frac{n^4a}{1440}\right\},$$

$$C_{12} = \frac{1}{a^2(\frac{n}{2}-2)_4}\left\{(1-a)^{1-\frac{n}{2}}\left(1-\frac{a}{2}-\frac{n}{2}-\frac{na}{4}+\frac{na^2}{30}+\frac{na^3}{120}\right.\right.$$
$$\left.+\frac{13n^2a^2}{240}+\frac{n^2a^3}{120}+\frac{11n^3a^2}{480}-\frac{n^3a^3}{480}+\frac{n^4a^2}{480}-\frac{n^4a^3}{480}\right)$$
$$-1+\frac{3a}{2}-\frac{a^2}{2}+\frac{n}{2}-\frac{3na}{4}+\frac{13na^2}{60}+\frac{n^2a}{4}-\frac{7n^2a^2}{48}+\frac{n^3a^2}{32}$$
$$\left.-\frac{n^4a^2}{96}+\frac{n^5a^2}{480}\right\},$$

$$C_{13} = \frac{1}{a(\frac{n}{2}-1)_3}\left\{(1-a)^{-\frac{n}{2}}\left(-\frac{1}{4}+\frac{a}{8}+\frac{7na}{72}-\frac{na^2}{36}-\frac{na^3}{144}+\frac{n^2a}{48}-\right.\right.$$
$$\left.\left.\frac{n^2a^2}{48}+\frac{n^3a}{576}-\frac{n^3a^2}{288}+\frac{n^3a^3}{576}\right)+\frac{1}{4}-\frac{a}{8}+\frac{na}{36}-\frac{n^2a}{36}-\frac{n^3a}{576}+\frac{n^4a}{576}\right\},$$

$$C_{14} = \frac{1}{a(\frac{n}{2}-1)_3}\left\{(1-a)^{-\frac{n}{2}}\left(-\frac{1}{2}+\frac{a}{4}+\frac{na}{6}-\frac{na^2}{24}+\frac{n^2a}{48}-\frac{n^2a^2}{48}\right)\right.$$
$$\left.+\frac{1}{2}-\frac{a}{4}-\frac{n^2a}{48}+\frac{n^3a}{48}\right\}.$$

4 Conclusion

The current version of the program computes E_4 for operator (3) on a Pentium-75 PC for 4 h 5 min, whereas computation of E_2 (and E_4 for minimal operator) takes trivial time (< 1 sec). The main reason for such a difference is tremendous swelling due to tensor submonomials of the form

$$(A_0^{-1})_{ai}(A_0^{-1})^i_j\cdots(A_0^{-1})^k_l(A_0^{-1})^l_b.$$

For operator (3) A_0^{-1} is a matrix $(A_0^{-1})_{ab} = \frac{1}{k^2-\lambda}\{g_{ab} + \frac{a\,k_a k_b}{(1-a)\,k^2-\lambda}\}$, whereas for a minimal operator $A_0^{-1} = \frac{1}{k^{2r}-\lambda}$, i.e., a single scalar. However we can cope in part with this swelling by writing $(A_0^{-1})_{ab}$ in terms of *projectors* [10] $P_{1ab} = g_{ab} - \frac{k_a k_b}{k^2}$ and $P_{2ab} = \frac{k_a k_b}{k^2}$. We hope to increase considerably the performance of the next version of the program using standard properties of projectors. Besides, some additional work on structuring large output expressions is needed to make them as short and readable as possible.

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