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THE DISCRETE SYMMETRY OF THE $N=2$ SUPERSYMMETRIC MODIFIED NLS HIERARCHY

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1 Introduction

Quite recently the minimal $N = 2$ supersymmetric extension of the one dimensional Toda mapping - f-Toda, has been proposed [1]. It acts as the symmetry transformation of the $N = 2$ supersymmetric Nonlinear Schrödinger (NLS) hierarchy, and almost all information concerning the hierarchy is encoded in the mapping.

The goal of the present Letter is to continue the list of the $N = 2$ superintegrable mappings in the $(1|2)$ superspace in order both to get new results for the corresponding integrable hierarchies of evolution equations, and to enlarge the reserve of the mappings for the analysis of their relations and origin. Besides the connection to integrable hierarchies, there is one more reason stimulating interest in such mappings - they are integrable themselves, i.e. every new mapping gives us a new example of a one-dimensional integrable system.

2 Note on the $N = 2$ supersymmetric mappings

Before introducing the new mappings, we would like to retrace the origin of the f-Toda mapping[1] [1]

$$\frac{1}{2}(\overleftarrow{f}\,\overleftarrow{\overline{f}} - f\overline{f}) = (\ln(\overline{D}\overleftarrow{f} \cdot D\overline{f}))' \tag{2.1}$$

for the pair of chiral and antichiral fermionic superfields $f(x,\theta,\overline{\theta})$ and $\overline{f}(x,\theta,\overline{\theta})$

$$Df = \overline{D}\,\overline{f} = 0, \tag{2.2}$$

respectively. The notation $\overleftarrow{f}$ ($\overrightarrow{f}$) means that the index of variable f is shifted by $+1$ (-1) (e.g., $\overleftarrow{f}_n = f_{n+1}$), and we use the standard representation for the $N = 2$ supersymmetric fermionic covariant derivatives

$$D = \frac{\partial}{\partial\theta} - \frac{1}{2}\overline{\theta}\frac{\partial}{\partial x}, \quad \overline{D} = \frac{\partial}{\partial\overline{\theta}} - \frac{1}{2}\theta\frac{\partial}{\partial x},$$
$$D^2 = (\overline{D})^2 = 0, \quad \{D,\overline{D}\} = -\frac{\partial}{\partial x} \equiv -\partial. \tag{2.3}$$

[1]The sign $'$ means the derivative with respect to x.

To do this, let us recall the existence of the mapping [2] which connects the $N=2$ super-NLS and the $a=4$ $N=2$ super-KdV hierarchies [3]. For example, their second flow equations

$$\frac{\partial}{\partial t_2}f = f'' + D(f\overline{f}\,\overline{D}f)', \quad \frac{\partial}{\partial t_2}\overline{f} = -\overline{f}'' + \overline{D}(f\overline{f}D\overline{f}), \tag{2.4}$$

$$\frac{\partial}{\partial t_2}J = [D,\overline{D}]J' + 4J'J, \tag{2.5}$$

respectively, are related by the mapping

$$J = -\frac{1}{2}(\frac{1}{2}f\overline{f} + (\ln D\overline{f})'). \tag{2.6}$$

Equations (2.4) and (2.5) admit the complex structures *

$$J^* = -J, \quad \theta^* = \overline{\theta}, \quad \overline{\theta}^{\,*} = \theta, \quad t_2^* = -t_2, \quad x^* = x, \tag{2.7}$$

and

$$f^* = \overline{f}, \quad \overline{f}^{\,*} = f, \quad \theta^* = \overline{\theta}, \quad \overline{\theta}^{\,*} = \theta, \quad t_2^* = -t_2, \quad x^* = x, \tag{2.8}$$

respectively, which being applied to the mapping (2.6) produce another admitted mapping

$$J = -\frac{1}{2}(\frac{1}{2}f\overline{f} - (\ln \overline{D}f)'). \tag{2.9}$$

Denoting the superfields f and $\overline{f}$ in (2.9) by the new symbols $\overleftarrow{f}$ and $\overleftarrow{\overline{f}}$, respectively, and equating the superfields J belonging to the mappings (2.6) and (2.9), we reproduce the f-Toda chain equations (2.1).

The described scheme of getting integrable mappings is quite general. To illustrate it, let us consider a few new examples.

In [4] the mapping

$$J = \frac{1}{2}(\frac{1}{2}b\overline{b} + (\ln \overline{b})'). \tag{2.10}$$

connecting the $N=2$ supersymmetric modified NLS (mNLS) and the $a=4$ $N=2$ super-KdV hierarchies has been constructed. Here, $b(x,\theta,\overline{\theta})$ and $\overline{b}(x,\theta,\overline{\theta})$ are chiral and antichiral bosonic superfields

$$Db = \overline{D}\,\overline{b} = 0, \tag{2.11}$$

respectively. The second flow equations of $N=2$ super-mNLS hierarhy

$$\frac{\partial}{\partial t_2}b = b'' + D(b\overline{b}\,\overline{D}b)', \quad \frac{\partial}{\partial t_2}\overline{b} = -\overline{b}'' + \overline{D}(b\overline{b}D\overline{b}) \tag{2.12}$$

admit the complex structure

$$b^* = i\overline{b}, \quad \overline{b}^{\,*} = ib, \quad \theta^* = \overline{\theta}, \quad \overline{\theta}^{\,*} = \theta, \quad t_2^* = -t_2, \quad x^* = x, \tag{2.13}$$

where i is the imaginary unity. Applying (2.13) and (2.7) to (2.10), we observe that besides the mapping (2.10) there exists one more

$$J = -\frac{1}{2}(\frac{1}{2}b\overline{b} - (\ln b)'), \tag{2.14}$$

and, therefore, the mapping

$$\frac{1}{2}(\overleftarrow{b}\,\overleftarrow{\overline{b}} - b\overline{b}) = (\ln(\overleftarrow{b}\,\overline{b}))' \tag{2.15}$$

acts as the symmetry transformation of the $N=2$ super-mNLS hierarchy.

Quite recently in [5] the relationship between "quasi" $N=4$ SKdV and the $\alpha=-2$, $N=2$ super Boussinesq [6] hierarchies has been established (for detail, see [5]). Using the relations constructed there and following the above-discussed line, one can produce the mapping

$$\begin{aligned}
&[D,\overline{D}\,]\overleftarrow{V} + \overleftarrow{V}^{\,2} + \overleftarrow{\Phi}_+\overleftarrow{\overline{\Phi}}_- - [D,\overline{D}\,]V - V^2 - \Phi_+\Phi_- \\
&= -(2\overline{D}\,\overleftarrow{V}\cdot D + 2DV\cdot\overline{D})\ln(\overleftarrow{\Phi}_-\Phi_+), \\
&\overleftarrow{V} - V = -(\ln(\overleftarrow{\Phi}_-\Phi_+))',
\end{aligned} \tag{2.16}$$

which acts as the symmetry transformation of the "quasi" $N=4$ SKdV hierarchy, where V is an unconstrained bosonic superfield, and Φ_+ and Φ_- are bosonic chiral and antichiral superfields

$$D\Phi_+ = \overline{D}\,\Phi_- = 0, \tag{2.17}$$

respectively.

In what follows we restrict ourselves to a concrete example of the mapping (2.15).

The mapping (2.15) possesses the inner automorphism σ with the properties

$$\sigma b\sigma^{-1} = -i\overleftarrow{\overline{b}}, \quad \sigma\overline{b}\sigma^{-1} = -i\overleftarrow{b}, \quad \sigma\overleftarrow{\overline{b}}\sigma^{-1} = ib, \quad \sigma\overleftarrow{b}\sigma^{-1} = i\overline{b}$$
$$\sigma x\sigma^{-1} = x, \quad \sigma\theta\sigma^{-1} = \overline{\theta}, \quad \sigma\overline{\theta}\sigma^{-1} = \theta. \tag{2.18}$$

One can rewrite (2.15) in a form more similar to (2.1)

$$\frac{1}{2}(D\overleftarrow{\overline{\xi}} \cdot \overline{D}\overleftarrow{\xi} - D\xi \cdot \overline{D}\xi) = (\ln(D\overleftarrow{\overline{\xi}} \cdot \overline{D}\xi))', \tag{2.19}$$

if one introduces a new pair of chiral and antichiral fermionic superfields $\xi(x,\theta,\overline{\theta})$ and $\overline{\xi}(x,\theta,\overline{\theta})$

$$D\xi = \overline{D}\,\overline{\xi} = 0, \tag{2.20}$$

respectively, by the following invertible relations

$$\xi = -\partial^{-1}D\overline{b}, \quad \overline{\xi} = -\partial^{-1}\overline{D}\xi;$$
$$b = D\overline{\xi}, \quad \overline{b} = \overline{D}\xi. \tag{2.21}$$

Relations (2.15), (2.19) and (2.21) fix only the scaling dimensions of the products $[b\overline{b}] = cm^{-1}$ and $[\xi\overline{\xi}] = cm^{0}$. It is interesting to note that despite the nonlocal character of the mutual relation between the superfields $b, \overline{b}$ and $\xi, \overline{\xi}$, it will be demonstrated that the integrable equations of the $N = 2$ super-mNLS hierarchy are local both in terms of the $b, \overline{b}$ and $\xi, \overline{\xi}$ superfields.

In spite of seeming similarity of the mappings (2.15) and (2.19) with the f-Toda mapping (2.1), there is an essential difference between them: neither the mappings (2.15) and (2.19) nor their bosonic limits are algebraically solvable with respect to the involved superfields b, $\overline{b}$ and $\xi, \overline{\xi}$, and their bosonic components[2].

Invariant with respect to the mapping (2.15), the hierarchy of the evolution equations

$$\frac{\partial}{\partial t}\begin{pmatrix} b \\ \overline{b} \end{pmatrix} = \begin{pmatrix} B \\ \overline{B} \end{pmatrix}, \tag{2.22}$$

[2]Let us remember that in the bosonic limit the f-Toda mapping is algebraically solvable.

with B and $\overline{B}$ being functionals whose independent arguments are $b, \overline{b}, b\,', \overline{b}\,', \overline{D}b, D\overline{b}, ...$, can be obtained by solving the symmetry equation, corresponding to the mapping (2.15),

$$\frac{1}{2}(\overleftarrow{B}_p\overleftarrow{\overline{b}} + \overleftarrow{b}\overleftarrow{\overline{B}}_p - B_p\overline{b} - b\overline{B}_p) = (\frac{\overleftarrow{B}_p}{\overleftarrow{b}} + \frac{\overline{B}_p}{\overline{b}})' \tag{2.23}$$

providing this invariance. In accordance with [7, 8], we got (2.23) taking the derivative of the mapping (2.15) with respect to the time t and substituting eqs.(2.22) into the result.

3 The list of the results for the mapping (2.15)

With short comments, we present the list of the results for the mapping (2.15) in the framework of the general scheme given in [1].

3.1. Solutions of the symmetry equation. The obvious nontrivial solution of the symmetry equation (2.23) is

$$\begin{pmatrix} B_0 \\ \overline{B}_0 \end{pmatrix} = \begin{pmatrix} b \\ -\overline{b} \end{pmatrix}. \tag{3.1}$$

The infinite tower of partial solutions B_p and $\overline{B}_p$ $(p = 0, 1, 2, ...)$ is generated by the recursion operator R

$$\begin{pmatrix} B_p \\ \overline{B}_p \end{pmatrix} = R^p \begin{pmatrix} B_0 \\ \overline{B}_0 \end{pmatrix}, \tag{3.2}$$

$$R = \Pi\partial \begin{pmatrix} 1 - \frac{1}{2}b\partial^{-1}\overline{b}, & -\frac{1}{2}b\partial^{-1}b \\ -\frac{1}{2}\overline{b}\partial^{-1}\overline{b}, & -1 - \frac{1}{2}\overline{b}\partial^{-1}b \end{pmatrix}\Pi, \tag{3.3}$$

which is defined modulo an arbitrary operator $C\overline{\Pi}$ annihilating the column in the r.h.s. of relation (3.2), where C is an arbitrary matrix-valued pseudo-differential operator. Here, Π $(\overline{\Pi})$

$$\Pi \equiv -\begin{pmatrix} D\overline{D}\partial^{-1}, & 0 \\ 0, & \overline{D}D\partial^{-1} \end{pmatrix}, \quad \overline{\Pi} \equiv -\begin{pmatrix} \overline{D}D\partial^{-1}, & 0 \\ 0, & D\overline{D}\partial^{-1} \end{pmatrix},$$
$$\Pi\Pi = \Pi, \quad \overline{\Pi}\,\overline{\Pi} = \overline{\Pi}, \quad \Pi\overline{\Pi} = \overline{\Pi}\Pi = 0, \quad \Pi + \overline{\Pi} = 1 \tag{3.4}$$

is the matrix that projects up and down elements of a column on the chiral (antichiral) and antichiral (chiral) subspaces, respectively. To prove (3.2), (3.3) by induction, it is necessary to use relations (2.15) and (2.23) as well as their direct consequences: the two identities which can be obtained from (2.15) by the action of derivatives D and $\overline{D}$, respectively; the identity which can be produced from (2.23) by the action of $[D, \overline{D}]$; and the following identity

$$
\begin{aligned}
&(\frac{1}{2}\overleftarrow{b}\,\overleftarrow{\bar b} - (\ln \overleftarrow{\bar b})')(\frac{1}{2}\partial^{-1}(\overleftarrow{B}\,\overleftarrow{\bar b} + \overleftarrow{b}\,\overleftarrow{\overline{B}}) - \frac{\overleftarrow{B}}{\overleftarrow{b}}) \\
&= (\frac{1}{2}b\bar b + (\ln \bar b)')(\frac{1}{2}\partial^{-1}(B\bar b + b\overline{B}) + \frac{\overline{B}}{\bar b}),
\end{aligned}
\tag{3.5}
$$

which one can derive by rewriting relations (2.15) and (2.23) in the following equivalent form:

$$
\begin{aligned}
&\frac{1}{2}\overleftarrow{b}\,\overleftarrow{\bar b} - (\ln \overleftarrow{\bar b})' = \frac{1}{2}b\bar b + (\ln \bar b)', \\
&\frac{1}{2}\partial^{-1}(\overleftarrow{B}\,\overleftarrow{\bar b} + \overleftarrow{b}\,\overleftarrow{\overline{B}}) - \frac{\overleftarrow{B}}{\overleftarrow{b}} = \frac{1}{2}\partial^{-1}(B\bar b + b\overline{B}) + \frac{\overline{B}}{\bar b},
\end{aligned}
\tag{3.6}
$$

respectively, and equating the product of their left-hand sides to the product of their right-hand sides.

For example, we present here the first four solutions

$$
\begin{aligned}
&B_0 = b\,, \quad \overline{B}_0 = -\bar b; \quad B_1 = b'\,, \quad \overline{B}_1 = \bar b'; \\
&B_2 = b'' + \frac{1}{2}D\overline{D}(b\bar b b), \quad \overline{B}_2 = -\bar b'' + \frac{1}{2}\overline{D}D(b\bar b\,\bar b); \\
&B_3 = b''' + D\overline{D}(\frac{3}{2}b'b\bar b - \frac{1}{4}(b\bar b)^2 b), \\
&\overline{B}_3 = \bar b''' - \overline{D}D(\frac{3}{2}\bar b' b\bar b + \frac{1}{4}(b\bar b)^2\bar b).
\end{aligned}
\tag{3.7}
$$

These expressions coincide with the corresponding ones for the $N = 2$ super-mNLS hierarchy [4].

It is instructive to consider the first nontrivial equations (2.12) belonging to $N = 2$ super-mNLS hierarchy in terms of superfield components defined as

$$
r = b|, \quad \bar r = \bar b|, \quad \psi = \overline{D}b|, \quad \overline{\psi} = D\bar b|,
\tag{3.8}
$$

where $|$ means the $(\theta, \bar\theta) \to 0$ limit. In terms of such components the second flow equations become

$$
\begin{aligned}
&\frac{\partial}{\partial t_2}r = r'' - r\bar r r' - r\psi\overline{\psi}, \quad \frac{\partial}{\partial t_2}\psi = (\psi' - r\bar r\psi)', \\
&\frac{\partial}{\partial t_2}\bar r = -\bar r'' - r\bar r\,\bar r' + \bar r\psi\overline{\psi}, \quad \frac{\partial}{\partial t_2}\overline{\psi} = -(\overline{\psi}' + r\bar r\overline{\psi})',
\end{aligned}
\tag{3.9}
$$

and in the bosonic limit, i.e. when the fermionic fields ψ and $\overline{\psi}$ are equal to zero, they coincide with mNLS equations of Ref. [9]. After passing to the new fields g and $\bar g$ defined by the following invertible nonlocal transformations:

$$
\begin{aligned}
&g = r\,\exp(\frac{1}{2}\partial^{-1}(r\bar r)), \quad \bar g = \bar r\,\exp(-\frac{1}{2}\partial^{-1}(r\bar r)); \\
&r = g\,\exp(-\frac{1}{2}\partial^{-1}(g\bar g)), \quad \bar r = \bar g\,\exp(\frac{1}{2}\partial^{-1}(g\bar g)),
\end{aligned}
\tag{3.10}
$$

eqs.(3.9) become

$$
\begin{aligned}
&\frac{\partial}{\partial t_2}g = g'' - (g\bar g g)' - g\psi\overline{\psi}, \quad \frac{\partial}{\partial t_2}\psi = (\psi' - g\bar g\psi)', \\
&\frac{\partial}{\partial t_2}\bar g = -\bar r'' - (g\bar g\,\bar g)' + \bar g\psi\overline{\psi}, \quad \frac{\partial}{\partial t_2}\overline{\psi} = -(\overline{\psi}' + g\bar g\overline{\psi})'.
\end{aligned}
\tag{3.11}
$$

Simple inspection of eqs.(3.11) shows that they are also local, and their bosonic limit coincides with the derivative NLS (dNLS) equations of Ref. [10]. Of course, eqs.(3.11) also possess the $N = 2$ supersymmetry, however, due to the nonlocal character of the transformations (3.10), it is realized nonlocally. Thus, in terms of the fields $g, \bar g, \psi$ and $\overline{\psi}$, the hierarchy (3.2), (3.3) can be called the $N = 2$ supersymmetric dNLS hierarchy reflecting the name of its first nontrivial bosonic representative.

3.2. Hamiltonians.

The infinite set of hamiltonians H_p with the scale dimension p and invariant with respect to the mapping (2.15) can be derived from the formula

$$
H_p = \int dx d\theta d\bar\theta \mathcal{H}_p \equiv \int dx d\theta d\bar\theta \partial^{-1}(B_p\bar b + b\overline{B}_p),
\tag{3.12}
$$

which is a direct consequence of the symmetry equation (2.23) like its counterpart in the case of the f-Toda mapping (2.1) (for detail, see [1]).

The equation of motion for the first hamiltonian density $\mathcal{H}_1$ is also the same

$$\frac{\partial}{\partial t_p}\mathcal{H}_1 = \mathcal{H}'_p \,, \tag{3.13}$$

and produces the additional superfield integral of motion

$$\widetilde{H}_1 = \int dx \mathcal{H}_1. \tag{3.14}$$

Using (3.12) and (3.7) we get, for example, the following expressions for the first four hamiltonian densities

$$\begin{aligned} &\mathcal{H}_0 = 0, \quad \mathcal{H}_1 = b\overline{b}, \quad \mathcal{H}_2 = 2(b'\overline{b} - \frac{1}{2}(b\overline{b})' - \frac{1}{2}(b\overline{b})^2), \\ &\mathcal{H}_3 = b''\overline{b} + b\overline{b}\,'' - b'\overline{b}\,' - \frac{3}{2}b\overline{b}\,\overline{D}b \cdot D\overline{b} \\ &\qquad + \frac{3}{2}b\overline{b}(b\overline{b}\,' - b'\overline{b}) + \frac{3}{4}(b\overline{b})^3, \end{aligned} \tag{3.15}$$

which coincide with the corresponding hamiltonian densities for the $N = 2$ super-mNLS hierarchy [4], which again confirms the above-mentioned relation between the mapping (2.15) and $N = 2$ super-mNLS hierarchy.

3.3. The mapping (2.19).

Let us very briefly discuss the solutions of the symmetry equation corresponding to the mapping (2.19).

Representing the solutions (3.2), (3.3) of the symmetry equation (2.23) in the form of recurrent relations

$$\begin{aligned} &B_{p+1} = D\overline{D}(-B_p + b\mathcal{H}_p), \qquad \overline{B}_{p+1} = \overline{D}D(\overline{B}_p + \overline{b}\mathcal{H}_p), \\ &\begin{pmatrix} B_0 \\ \overline{B}_0 \end{pmatrix} = \begin{pmatrix} b \\ -\overline{b} \end{pmatrix} \end{aligned} \tag{3.16}$$

using (2.22) and substituting into (3.16) the transformation (2.21) to the fermionic superfields $\xi, \overline{\xi}$ as well as introducing the definition of their time derivatives

$$\frac{\partial}{\partial t_p}\begin{pmatrix} \xi \\ \overline{\xi} \end{pmatrix} = \begin{pmatrix} \Xi_p \\ \overline{\Xi}_p \end{pmatrix}, \tag{3.17}$$

one can easily find the following recurrent relation:

$$\begin{aligned} &\Xi_{p+1} = D(\overline{D}\Xi_p + \overline{D}\xi \cdot \mathcal{H}_p), \qquad \overline{\Xi}_{p+1} = \overline{D}(-D\overline{\Xi}_p + D\overline{\xi} \cdot \mathcal{H}_p), \\ &\begin{pmatrix} \Xi_0 \\ \overline{\Xi}_0 \end{pmatrix} = \begin{pmatrix} -\xi \\ \overline{\xi} \end{pmatrix} \end{aligned} \tag{3.18}$$

for the solutions of the symmetry equation corresponding to the mapping (2.19). From (3.18) it is evident that in terms of the superfields $\xi, \overline{\xi}$ the evolution equations of the $N = 2$ super-mNLS hierarchy are also local.

3.4. Hamiltonian structures.

For the chiral-antichiral bosonic superfields b and $\overline{b}$, a hamiltonian structure J should be a skew symmetric[3] $J^T = -J$ pseudo-differential 2x2 matrix operator, which besides the Jacobi identity and the chiral consistency conditions

$$J\Pi = \overline{\Pi}J = 0, \quad J\overline{\Pi} = \Pi J = J, \tag{3.19}$$

should also satisfy the following additional constraint [11]:

$$J(b, \overline{b}) = \Phi J(\overleftarrow{b}, \overleftarrow{\overline{b}})\Phi^T, \tag{3.20}$$

which provides the invariance of the hamiltonian equations

$$\frac{\partial}{\partial t}\begin{pmatrix} b \\ \overline{b} \end{pmatrix} = J\begin{pmatrix} \delta/\delta b \\ \delta/\delta\overline{b} \end{pmatrix} H \tag{3.21}$$

with respect to the mapping (2.15). Here, (Φ) $\hat{\Phi}$ is the (invert) matrix of Fréchet derivatives, corresponding to the mapping (2.15),[4]

$$\begin{aligned} &\Phi = \phi_1 \otimes \phi_2 \equiv \Pi \begin{pmatrix} 1 + 1/2bL^{-1}\overline{b}\partial^{-1}\overline{D}D \\ -1/2\overline{b}L^{-1}\overline{b}\partial^{-1}\overline{D}D \end{pmatrix} \overline{b}^{\,-1} \otimes \left(\overleftarrow{\overline{b}} - 2\partial \overleftarrow{b}^{\,-1}, \ \overleftarrow{b} \right) \Pi, \\ &\hat{\Phi} = A\sigma\Phi\sigma^{-1}A \equiv \hat{\phi}_2 \otimes \hat{\phi}_1 \equiv (-A\sigma\phi_1\sigma^{-1}) \otimes (-\sigma\phi_2\sigma^{-1}A), \end{aligned} \tag{3.22}$$

where the sign '⊗' stands for the tensor product and the operator L

$$L \equiv \partial + \frac{1}{2}b\overline{b} + \frac{1}{2}\overline{b}\partial^{-1}\, D[\overline{D}, b]\,, \quad [\overline{D}, L] = 0 \tag{3.23}$$

[3] Let us remember the rules of the adjoint conjugation operation 'T': $D^T = -D$, $\overline{D}^T = -\overline{D}$, $(MN)^T = (-1)^{d_M d_N} N^T M^T$, where d_M (d_N) is the Grassman parity of the operator M (N), equal to 0 (1) for bosonic (fermionic) operators. In addition, for matrices it is necessary to take the operation of the matrix transposition. All other rules can be derived by using these ones.

[4] Here, the derivatives ∂, $\overline{D}$ and D act like operators, i.e. must be commuted with b and $\overline{b}$.

coincides with the Lax operator of $N = 2$ super-mNLS hierarchy [4], σ is the automorphism (2.18), and the matrix A

$$A \equiv \begin{pmatrix} 0, & i \\ i, & 0 \end{pmatrix} \tag{3.24}$$

is its matrix of Fréchet derivatives. The matrices $\hat{\phi}_{1,2}$, $\phi_{1,2}$, Φ and $\hat{\Phi}$ possess the following properties:

$$\begin{aligned} &\hat{\phi}_1\phi_1 = \phi_2\hat{\phi}_2 = 1, \qquad \phi_1 \otimes \hat{\phi}_1 = \hat{\phi}_2 \otimes \phi_2 = \Phi\hat{\Phi} = \hat{\Phi}\Phi = \Pi, \\ &\Phi\overleftarrow{R} = R\Phi, \quad \hat{\Phi}R = \overleftarrow{R}\hat{\Phi}, \\ &\Phi \begin{pmatrix} \overleftarrow{B} \\ \overleftarrow{\overline{B}} \end{pmatrix} = \begin{pmatrix} B \\ \overline{B} \end{pmatrix}, \quad \hat{\Phi} \begin{pmatrix} B \\ \overline{B} \end{pmatrix} = \begin{pmatrix} \overleftarrow{B} \\ \overleftarrow{\overline{B}} \end{pmatrix}, \end{aligned} \tag{3.25}$$

where B and $\overline{B}$ are arbitrary solutions of the symmetry equation (2.23)[5].

We have a solution of eq.(3.20) similar to [1] for the first hamiltonian structure

$$J_1 = \phi_1 \otimes \partial\phi_1^T, \tag{3.26}$$

as well as for the k-th hamiltonian structure

$$J_k = R^k J_1. \tag{3.27}$$

At $k = 2$ eq.(3.27) reproduces the second hamiltonian structure J_2 [4]

$$J_2 = \frac{i}{2}\Pi A \partial \tag{3.28}$$

of $N = 2$ super-mNLS hierarchy.

One can check also that expression (3.3) for the recursion operator can be represented in the following form:

$$R = J_2 J_1^*, \tag{3.29}$$

where the matrix J_1^* is defined as

$$J_1^*(b, \overline{b}) = \hat{\phi}_1^T \otimes \partial^{-1}\hat{\phi}_1 \equiv \overline{\Pi} \begin{pmatrix} \overline{b}\partial^{-1}\overline{b}, & \overline{b}\partial^{-1}b + 2 \\ b\partial^{-1}\overline{b} - 2, & b\partial^{-1}b \end{pmatrix} \Pi \tag{3.30}$$

[5]To check relation (3.25) for the first nontrivial solution (3.1), it is necessary to remove ambiguity in the operator $\partial^{-1}\partial 1$, to appear in calculations, by setting $\partial^{-1}\partial 1 = (\partial^{-1}\partial)1 \equiv 1$.

and possesses the following properties:

$$\{J_1^*, J_1\} = 1, \quad J_1^*(\overleftarrow{b}, \overleftarrow{\overline{b}}) = A\sigma J_1^*(b, \overline{b})\sigma^{-1}A. \tag{3.31}$$

Let us note that to get the hamiltonian structures, which are invariant with respect to the mapping (2.19), one can apply the transformation (2.21) to the hamiltonian structures (3.26), (3.27). Thanks to a very simple structure of its matrix of Fréchet derivatives

$$\begin{pmatrix} 0, & D \\ \overline{D}, & 0 \end{pmatrix} \tag{3.32}$$

one can easily find the corresponding formulas, which are not presented here.

4 Conclusion

In this Letter, we proposed a few supersymmetric mappings in the (1|2) superspace, which act as the symmetry transformation of the integrable hierarchies corresponding to them, and analyzed their common origin. Using one of them as an example, we constructed the recursion operator and hamiltonian structures of the $N = 2$ supersymmetric modified NLS hierarchy. We believe that the same approach can be realized for other superintegrable mappings and hope to return to it in future.

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Дискретная симметрия $N=2$ суперсимметричной модифицированной иерархии НУШ

Предложено несколько новых $N=2$ суперинтегрируемых подстановок в $(1|2)$ суперпространстве. С использованием одной из них, действующей как дискретная симметрия $N=2$ суперсимметричной модифицированной иерархии НУШ, построены рекурсионный оператор и гамильтоновы структуры иерархии.

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The Discrete Symmetry of the $N=2$ Supersymmetric Modified NLS Hierarchy

A few new $N=2$ superintegrable mappings in the $(1|2)$ superspace are proposed and their origin in analyzed. Using one of them, acting like the discrete symmetry transformation of the $N=2$ supersymmetric modified NLS hierarchy, the recursion operator and hamiltonian structures of the hierarchy are constructed.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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