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ON ONE PROPERTY OF THE KÄHLER FERMIONS

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In studying spaces whose geometrical structure is defined by the integrable connection

$$\Gamma^i_{jk} = h^i_a \partial_j h^a_k \tag{1}$$

Einstein has discovered a remarkable identity analogous to the Bianchi one. In formula (1) h^i_a are linear independent vector fields. Numbering indices are taken from the beginning of the latin alphabet and run over four labels $a = 0, 1, 2, 3$. The reciprocal system of covector fields h^a_i is defined by the equations $h^i_a\, h^a_j = \delta^i_j$. As it is shown in [1], from quantities h^i_a and their partial derivatives one can construct a tensor E^{ij}, analogous to the Einstein tensor $G^{ij} = R^{ij} - \frac{1}{2} G^{ij} R$, and satisfying the equations

$$(\nabla_i - K_i) E^{ij} = 0 \tag{2}$$

identically, where ∇_i is a covariant derivative with respect to the connection (1), and the covector K_i is equal to the contraction of the tensor $K^i_{jk} = h^i_a(\partial_j h^a_k - \partial_k h^a_j)$, $K_i = K^j_{ij}$. Identity (2) is analogous to the identity $G^{ij}{}_{;i} = 0$ for the Einstein tensor. This is an explicit expression for the tensor E_{ij}

$$E^{ij} = (\nabla_i - K_i) A^{ilj} + K^i_{lm} A^{lmj},$$

where

$$A^{ijl} = K^{ijl} + \mu(g^{il} K^j - g^{jl} K^i) + \nu(K^{ilj} - K^{jli}),$$

μ, ν are constants. In view of such a deep similarity one can suggest the equations

$$E^{ij} = l\, S^{ij}, \tag{3}$$

where l is a constant, which are analogous to the Einstein equations

$$G_{ij} = kT_{ij}.$$

From the Einstein identity (2) it follows that the tensor S^{ij} analogous to the tensor of energy-momentum T_{ij} should satisfy the equations

$$(\nabla_i - K_i)S^{ij} = 0. \tag{4}$$

So, the problem is to find a field such that can be characterized as follows. From the components of this field one can construct a tensor S^{ij} which satisfies equations (4) on the solutions of the equations for this field . Below we shall show that such unusual field really exists.

A transcription of the Dirac equation as a set of equations for antisymmetric tensor fields was introduced by mathematician E. Kähler [2](see also [3]).This Kähler-Dirac equation has been studied in connection with the lattice fermions [4],[5] and other remarkable properties [6]-[11]. As Graf has suggested [6], the Kähler field might be more fundamental than the Dirac spinor. This is an appealing idea because it conforms to the Einstein methodology of associating all physical fields with geometrical objects. Here we shall establish

how to construct the tensor S^{ij} from the components of the Kähler field. The covariant antisymmetric tensor field $U_{i_1 \cdots i_p}$ $(p = 0, 1, 2, 3, 4)$ is called the p-form. If

$$U = (U, U_i, U_{ij}, U_{ijk}, U_{ijkl})$$

is a form, the generalized curl operator D_e is defined by

$$D_e U = (0, D_i U, 2D_{[i}U_{j]}, 3D_{[i}U_{jk]}, 4D_{[i}U_{jkl]}),$$

where $D_i = \nabla_i - K_i$, square brackets denote alternation. For the operator D_i of generalized divergence we have the following definition

$$D_i U = (-D^m U_m, -D^m U_{mi}, -D^m U_{mij}, -D^m U_{mijk}, 0).$$

The Kähler-Dirac equation in the spaces with connection (1) has the form

$$DU = mU, \tag{5}$$

where $D = D_i + D_e$. Similarly to the operators D_i and D_e, one can introduce the operators Q_i and Q_e, defined by the vector field as follows

$$Q_e U = (0, V_i U, 2V_{[i}U_{j]}, 3V_{[i}U_{jk]}, 4V_{[i}U_{jkl]}),$$

$$Q_i U = (-V^m U_m, -V^m U_{mi}, -V^m U_{mij}, -V^m U_{mijk}, 0).$$

If we introduce, in addition to these operators, a numerical operator Λ such that

$$\Lambda U = (U, -U_i, U_{ij}, -U_{ijk}, U_{ijkl}),$$

then it can be shown that the operator $Q = (Q_i - Q_e)\Lambda$ commutes with the operator D under the condition $\nabla_i V^j = 0$. Since the curvature tensor of the connection (1) is equal to zero identically, then the equations $\nabla_i V^j = 0$ are integrable. Thus, operator Q acts in the space of the solutions of the Kähler-Dirac equation (5). This symmetry of equation (5) immediately gives the tensor

$$S^{ij} = \sum_{p=0}^{4} \frac{(-1)^p}{p!} (\frac{1}{2} g^{ij} U_{i_1 \cdots i_p} \bar{U}^{i_1 \cdots i_p} + U^{ii_1 \cdots i_p} \bar{U}^{j}_{.i_1 \cdots i_p} + \\ + U^{iji_1 \cdots i_p} \bar{U}_{i_1 \cdots i_p}) + c.c., \qquad (6)$$

which on the solutions of wave equation (5) satisfies equations (4). Hence it follows that system of equations (3), (5) and (6) is consistent. Since tensors S^{ij} and E^{ij} have dimensions cm^{-3} and cm^{-2}, respectively, then the constant l in equation (3) has dimension of length. It is natural to suppose that l is equal to the Planck length. It should be noted also that the 00-component of tensor (6) is positive definite.

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