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# $N=2$ SUPERINTEGRABLE f-TODA MAPPING AND SUPER-NLS HIERARCHY IN $(1|2)$ SUPERSPACE

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# 1 Introduction

Recently, a manifestly $N = 2$ supersymmetric formulation of the Nonlinear Schrödinger (NLS) hierarchy (see, e.g., [1] and references therein) has been constructed [2, 3]. The method is based on the super-Hamiltonian formalism, sufficiently cumbersome for some concrete calculations. Quite recently, the advantage of an alternative approach to the theory of supersymmetric integrable hierarchies has been demonstrated in [4], where the formalism of integrable mappings [5, 6] was applied to the problem of constructing hierarchies of $(1+2)$-dimensional integrable systems in $(2|2)$ superspace. In [4], a few two-dimensional superintegrable mappings were proposed. It would be interesting to find some new examples manifesting the benefit of using supersymmetric mappings.

The goal of the present Letter is to present the results of Refs. [1, 2, 3] (including the derivation of some new results) as a direct corollary to the existence of a new integrable $N = 2$ supersymmetric mapping (we also call it a substitution) acting in $(1|2)$ superspace. This mapping relates two pairs of chiral-antichiral fermionic superfields and, in the bosonic limit, it is equivalent to the one-dimensional Toda mapping. Taking into account that the Toda-mapping is responsible for the existence and properties of the bosonic NLS hierarchy, we call our supersymmetric mapping Fermi-Toda (f-Toda), reflecting the existence of the fermionic fields in its background. However, this name may be considered to have a deeper foundation if one remembers that Fermi was one of the authors of [7], where equations of a nonlinear chain were applied for the first time to the solution of the physical problem of establishing the heat equilibrium in a short-range interacting dynamic system.

# 2 $N = 2$ supersymmetric f-Toda mapping

In this section, we introduce the f-Toda mapping and show its integrability.

We work in $(1|2)$ superspace with one bosonic $x$ and two fermionic $\theta, \overline{\theta}$ coordinates and use standard representation for the $N = 2$ supersymmetric fermionic covariant derivatives

$$D = \frac{\partial}{\partial\theta} - \frac{1}{2}\overline{\theta}\frac{\partial}{\partial x}, \quad \overline{D} = \frac{\partial}{\partial\overline{\theta}} - \frac{1}{2}\theta\frac{\partial}{\partial x},$$

$$D^2 = (\overline{D})^2 = 0, \quad \{D, \overline{D}\} = -\frac{\partial}{\partial x} \equiv -\partial. \tag{2.1}$$

Let us introduce a pair of chiral and antichiral fermionic superfields $f(x,\theta,\overline{\theta})$ and $\overline{f}(x,\theta,\overline{\theta})$, respectively,

$$Df = \overline{D}\,\overline{f} = 0, \tag{2.2}$$

and the following relation[1]:

$$\frac{1}{2}(\overleftarrow{f}\,\overleftarrow{\overline{f}} - f\overline{f}) = (\ln(\overline{D}\overleftarrow{f} \cdot D\overline{f}))', \tag{2.3}$$

which is the definition of the mapping or the rule determining the correspondence between two initial functions, $f$ and $\overline{f}$, and two final ones, $\overleftarrow{f}$ and $\overleftarrow{\overline{f}}$. The action of the inverse transformation is denoted by $\overrightarrow{f}$ and $\overrightarrow{\overline{f}}$, and the corresponding mapping takes the form

$$\frac{1}{2}(f\overline{f} - \overrightarrow{f}\,\overrightarrow{\overline{f}}) = (\ln(\overline{D}f \cdot D\overrightarrow{\overline{f}}))'. \tag{2.4}$$

The notation $\overleftarrow{f}$ ($\overrightarrow{f}$) means that the index of variable $f$ is shifted by $+1$ ($-1$) (e.g., $\overleftarrow{f}_n = f_{n+1}$). Relations (2.3) and (2.4) fix only the scaling dimension of the product $[f\overline{f}] = cm^{-1}$.

We would like to remark that the mapping (2.3) is not algebraically solvable with respect to the superfields $f$ and $\overline{f}$, or $\overleftarrow{f}$ and $\overleftarrow{\overline{f}}$, in contrast to its bosonic limit[2]—the Toda chain—where the bosonic components of the superfields $f$ and $\overline{f}$ can be expressed pure algebraically in terms of their counterparts belonging to the superfields $\overleftarrow{f}$ and $\overleftarrow{\overline{f}}$, and vice versa.

Substitution (2.3) possesses the inner automorphism $\sigma$ with the properties

$$\sigma f \sigma^{-1} = \overleftarrow{\overline{f}}, \quad \sigma \overline{f} \sigma^{-1} = \overleftarrow{f}, \quad \sigma \overleftarrow{f} \sigma^{-1} = f, \quad \sigma \overleftarrow{\overline{f}} \sigma^{-1} = \overline{f},$$
$$\sigma D \sigma^{-1} = \overline{D}, \quad \sigma \overline{D} \sigma^{-1} = D, \tag{2.5}$$

[1]The sign $'$ means the derivative with respect to $x$.

[2]Let us remember that in the bosonic limit, all fermionic components must be set equal to zero.



which will be useful in what follows. The action of $\sigma$ on the covariant derivatives $D$ and $\overline{D}$ can be induced by the following transformation of the $(1|2)$ superspace coordinates

$$\sigma x \sigma^{-1} = x, \quad \sigma\theta\sigma^{-1} = \overline{\theta}, \quad \sigma\overline{\theta}\sigma^{-1} = \theta. \tag{2.6}$$

To establish the connection of the mapping (2.3) with the theory of integrable systems, one can consider the general representation for an evolution-type system

$$\frac{\partial f}{\partial t} = F(f,\overline{f},f',\overline{f}',\overline{D}f,D\overline{f}...), \quad \frac{\partial \overline{f}}{\partial t} = \overline{F}(f,\overline{f},f',\overline{f}',\overline{D}f,D\overline{f}...), \tag{2.7}$$

with the additional requirement of its invariance with respect to the mapping. The system (2.7) is invariant with respect to the transformation (2.3) if the functions $F$ and $\overline{F}$ are subjected to a set of constraints called the symmetry equations of the mapping [5, 6]. In other words, the symmetry equation may be treated as the condition ensuring the invariance of the evolution-type system (2.7) with respect to the mapping (2.3). It appears that it strictly determines some class of partial solutions corresponding to some hierarchy of integrable systems. In what follows, we show that the symmetry equation of the mapping (2.3) does extract the $N = 2$ super-NLS hierarchy of integrable equations [2, 3].

The symmetry equation for a given mapping can be obtained by taking its derivative with respect to an arbitrary parameter and denoting the derivatives of the independent functions involved in a substitution by correspondingly new symbols [5, 6]. In the case under consideration, these symbols are $F = \dot{f}$ and $\overline{F} = \dot{\overline{f}}$, which are the chiral and antichiral superfields, respectively,

$$DF = \overline{D}\,\overline{F} = 0, \tag{2.8}$$

as $f$ and $\overline{f}$ (2.2). Using this method, one can obtain a symmetry equation corresponding to the substitution (2.3) in the following form:

$$\frac{1}{2}(\overleftarrow{F}\,\overleftarrow{\overline{f}} + \overleftarrow{f}\,\overleftarrow{\overline{F}} - F\overline{f} - f\overline{F}) = (\frac{\overline{D}\overleftarrow{F}}{\overline{D}\overleftarrow{f}} + \frac{D\overline{F}}{D\overline{f}})'. \tag{2.9}$$

It is obvious that (2.9) possesses the trivial partial solution $F = f', \overline{F} = \overline{f}'$. To understand this, it is sufficient to choose the bosonic coordinate $x$ as the parameter of differentiation. The next obvious but nontrivial solution is

$$F = f, \quad \overline{F} = -\overline{f}. \tag{2.10}$$

In Refs. [5, 6], a mapping was called integrable if its symmetry equation possessed at least one nontrivial solution. Thus, in this sense, the f-Toda substitution (2.3) is integrable.

To conclude this section, we state the following proposition.

*The f-Toda mapping (2.3) is integrable and each solution of its symmetry equation (2.9) is connected with an evolution-type system (2.7) invariant with respect to its transformation.*

# 3 The symmetry of the symmetry equation

In this section, we construct a recurrent procedure for finding an infinite set of partial solutions to the symmetry equation (2.9) and establish their connection to the $N = 2$ super-NLS hierarchy.

Let us present the explicit form of the transformation that generates a new solution to the symmetry equation from an arbitrarily given one. We make the following assertion.

*If the pair $F$ and $\overline{F}$ is a solution of the symmetry equation (2.9), the pair $\widetilde{F}$ and $\widetilde{\overline{F}}$, defined as*[3]

$$\begin{aligned} \widetilde{F} &= F' + D(f\overline{D} + \frac{1}{2}\overline{D}f)\partial^{-1}(f\overline{F} + F\overline{f}), \\ \widetilde{\overline{F}} &= -\overline{F}' + \overline{D}(\overline{f}D + \frac{1}{2}D\overline{f})\partial^{-1}(f\overline{F} + F\overline{f}), \end{aligned} \tag{3.1}$$

*is also a solution.*

One can prove (3.1) by straightforward but rather tedious calculations. The main steps to prove this statement are given in the appendix.

Representing (3.1) in the form

$$\begin{pmatrix} \widetilde{F} \\ \widetilde{\overline{F}} \end{pmatrix} = R \begin{pmatrix} F \\ \overline{F} \end{pmatrix}, \tag{3.2}$$

[3]Here, the derivatives $\partial$, $\overline{D}$ and $D$ act like operators, i.e., they must be commuted with $f$ and $\overline{f}$.



one can obtain the following expression for the recursion operator $R$ of the integrable hierarchy corresponding to the substitution (2.3)[3]:

$$R = \Pi \begin{pmatrix} \partial + fD\overline{D}\partial^{-1}\overline{f} + \frac{1}{2}\partial f\partial^{-1}\overline{f}, & -fD\overline{D}\partial^{-1}f - \frac{1}{2}\partial f\partial^{-1}f \\ \overline{f}\,\overline{D}D\partial^{-1}\overline{f} + \frac{1}{2}\partial\overline{f}\partial^{-1}\overline{f}, & -\partial - \overline{f}\,\overline{D}D\partial^{-1}f - \frac{1}{2}\partial\overline{f}\partial^{-1}f \end{pmatrix} \Pi, \tag{3.3}$$

where $\Pi$ ($\overline{\Pi}$)

$$\Pi \equiv -\begin{pmatrix} D\overline{D}\partial^{-1}, & 0 \\ 0, & \overline{D}D\partial^{-1} \end{pmatrix}, \quad \overline{\Pi} \equiv -\begin{pmatrix} \overline{D}D\partial^{-1}, & 0 \\ 0, & D\overline{D}\partial^{-1} \end{pmatrix},$$
$$\Pi\Pi = \Pi, \quad \overline{\Pi}\,\overline{\Pi} = \overline{\Pi}, \quad \Pi\overline{\Pi} = \overline{\Pi}\Pi = 0, \quad \Pi + \overline{\Pi} = 1 \tag{3.4}$$

is the matrix that projects the up and down elements of a column on the chiral (antichiral) and antichiral (chiral) subspaces, respectively. Let us stress that the expression for $R$ is defined up to an arbitrary additive operator which annihilates the column on the r.h.s. of relation (3.2). It is clear that such an operator can be represented in the following general form: $C\overline{\Pi}$, where $C$ is an arbitrary matrix-valued pseudo-differential operator.

Simple inspection of $R$ (3.3) shows that it possesses the following properties:

$$\Pi R = R\Pi = R, \quad \overline{\Pi}R = R\overline{\Pi} = 0, \tag{3.5}$$

and, therefore, its action preserves the chiral structure (2.8) of the evolution equations (2.7). Because of this chiral structure, all expressions for the recursion operator, which differ by the above-mentioned operator $C\overline{\Pi}$, are equivalent.

Acting $p$-times ($p = 0, 1, 2, \ldots$) by the recursion operator on the first nontrivial solution (2.10) of the symmetry equation, we can generate the new solutions $F_p$ and $\overline{F}_p$,

$$\begin{pmatrix} F_p \\ \overline{F}_p \end{pmatrix} = R^{\,p} \begin{pmatrix} f \\ -\overline{f} \end{pmatrix}, \tag{3.6}$$

and the corresponding evolution equations

$$\frac{\partial}{\partial t_p} \begin{pmatrix} f \\ \overline{f} \end{pmatrix} = \begin{pmatrix} F_p \\ \overline{F}_p \end{pmatrix} \tag{3.7}$$

belonging to the integrable hierarchy. Taking into account the scaling dimension of the recursion operator $[R] = cm^{-1}$, it is easy to observe that for the $p$-th solution, the maximal order of the bosonic derivative linearly appearing on the right-hand side of eqs. (3.7) is equal to $p$. Using the terminology of inverse scattering theory, one can say that the $p$-th solution corresponds to the $p$-th flow.

The first five solutions to the symmetry equation have the following form:

$$\begin{aligned}
&F_0 = f\,, \quad \overline{F}_0 = -\overline{f}; \quad F_1 = f'\,, \quad \overline{F}_1 = \overline{f}';\\
&F_2 = f'' + D(f\overline{f}\,\overline{D}f), \quad \overline{F}_2 = -\overline{f}'' + \overline{D}(f\overline{f}D\overline{f});\\
&F_3 = f''' + \frac{3}{2}D((f\overline{D}f)'\overline{f}), \quad \overline{F}_3 = \overline{f}''' + \frac{3}{2}\overline{D}((\overline{f}D\overline{f})'f);\\
&F_4 = f'''' + \frac{1}{2}D[3(f\overline{D}f)''\overline{f} - 3f\overline{f}D\overline{f}\cdot(\overline{D}f)^2 + f(\overline{f}\,\overline{D}f)'' + (f\overline{f})''\overline{D}f],\\
&\overline{F}_4 = -\overline{f}'''' - \frac{1}{2}\overline{D}[3(\overline{f}D\overline{f})''f + 3f\overline{f}\,\overline{D}f\cdot(D\overline{f})^2 + \overline{f}(fD\overline{f})'' - (f\overline{f})''D\overline{f}].
\end{aligned} \tag{3.8}$$

These expressions coincide with the corresponding ones for the $N = 2$ supersymmetric NLS hierarchy [2, 3], therefore, we can recognize that the f-Toda mapping (2.3) is related to the $N = 2$ supersymmetric NLS hierarchy, which justifies its name.

Thus, the following proposition summarizes this section.

*The f-Toda mapping (2.3) acts like the symmetry transformation of the $N = 2$ super-NLS hierarchy.*

# 4 The f-Toda invariant Hamiltonian structures

In this section, we construct the Hamiltonian structures which are invariant with respect to the f-Toda mapping (2.3).

By definition, for the chiral-antichiral fermionic superfields $f$ and $\overline{f}$, the f-Toda invariant Hamiltonian structure $J$ is a symmetric[4] $J^T = J$ pseudo-differential 2x2 matrix operator which, in addition to the Jacobi identity

[4]Let us recall the rules of the adjoint conjugation operation "$T$": $D^T = -D$, $\overline{D}^T = -\overline{D}$, $(MN)^T = (-1)^{d_M d_N} N^T M^T$, where $d_M$ ($d_N$) is the Grassman parity of the operator $M$ ($N$), equal to 0 (1) for bosonic (fermionic) operators. In addition, for matrices, it is necessary to take the operation of the matrix transposition. All other rules can be derived using these.



and the chiral consistency conditions

$$J\Pi = \overline{\Pi}J = 0, \quad J\overline{\Pi} = \Pi J = J, \tag{4.1}$$

should also satisfy the following additional constraint [8, 9]:

$$J(f,\overline{f}) = \Phi J(\overleftarrow{f},\overleftarrow{\overline{f}})\Phi^T, \tag{4.2}$$

which provides its invariance with respect to the f-Toda mapping. Here[3],

$$\begin{aligned}
\Phi &= \phi_1 \otimes \phi_2\\
&\equiv \Pi \begin{pmatrix} D(1 - \frac{1}{2}fL^{-1}\overline{f}) \\ -\frac{1}{2}\{D,\overline{f}\}L^{-1}\overline{f} \end{pmatrix} \{D,\overline{f}\}^{-1} \otimes \left( \overleftarrow{\overline{f}} + 2\overline{D}\partial\{\overline{D},\overleftarrow{f}\}^{-1}, \ -\overleftarrow{f} \right) \Pi
\end{aligned} \tag{4.3}$$

is the inverse matrix of Fréchet derivatives corresponding to the mapping (2.3), where the notation '$\otimes$' stands for the tensor product and the operator $L$,

$$L \equiv \partial - \frac{1}{2}f\overline{f} - \frac{1}{2}f\partial^{-1}\overline{D}\{D,\overline{f}\}, \quad [D, L] = 0 \tag{4.4}$$

coincides with the Lax operator of the $N = 2$ supersymmetric NLS hierarchy [3]. One can easily invert relation (4.2) by applying the automorphism $\sigma$ (2.5). As a result, we obtain

$$J(\overleftarrow{f},\overleftarrow{\overline{f}}) = \hat{\Phi}J(f,\overline{f})\hat{\Phi}^T, \tag{4.5}$$

where

$$\hat{\Phi} = A\sigma\Phi\sigma^{-1}A \equiv \hat{\phi}_2 \otimes \hat{\phi}_1 \equiv (-A\sigma\phi_1\sigma^{-1}) \otimes (-\sigma\phi_2\sigma^{-1}A), \tag{4.6}$$

and

$$A \equiv \begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix} \tag{4.7}$$

is the matrix of Fréchet derivatives for the automorphism $\sigma$ (2.5). The matrices $\hat{\phi}_{1,2}$, $\phi_{1,2}$, $\Phi$, and $\hat{\Phi}$ possess the following useful properties:

$$\begin{aligned}
&\hat{\phi}_1\phi_1 = \phi_2\hat{\phi}_2 = 1, \quad \phi_1 \otimes \hat{\phi}_1 = \hat{\phi}_2 \otimes \phi_2 = \Phi\hat{\Phi} = \hat{\Phi}\Phi = \Pi,\\
&\Phi\overleftarrow{R} = R\Phi, \quad \hat{\Phi}R = \overleftarrow{R}\hat{\Phi}, \quad \Phi\begin{pmatrix} \overleftarrow{F} \\ \overleftarrow{\overline{F}} \end{pmatrix} = \begin{pmatrix} F \\ \overline{F} \end{pmatrix}, \quad \hat{\Phi}\begin{pmatrix} F \\ \overline{F} \end{pmatrix} = \begin{pmatrix} \overleftarrow{F} \\ \overleftarrow{\overline{F}} \end{pmatrix},
\end{aligned} \tag{4.8}$$

where $F$ and $\overline{F}$ are arbitrary solutions to the symmetry equation (2.9)[5]. Acting by the projectors $\Pi$ and $\overline{\Pi}$ (3.4) on the condition (4.2), one can immediately check that the constraint (4.2) is consistent with the constraint (4.1).

Let us present an infinite set of partial solutions of the condition (4.2).

Without going into detail, we state that the solution $J_1$ of the condition (4.2), with the scaling dimension $[cm^0]$ corresponding to the first Hamiltonian structure, has the following form:

$$J_1(f,\overline{f}) = \phi_1 \otimes \partial\phi_1^T, \tag{4.9}$$

where $\phi_1$ is defined by (4.3). It is obvious that $J_1$ is the symmetric operator satisfying the Jacobi identity. One can easily show that $J_1$ also satisfies the chiral constraints (4.1). As for the condition (4.2), it can be checked by direct though laborious calculations. However, there is an easier way to prove (4.9). Substituting $J_1$ (4.9) into (4.2) and making some obvious algebraic transformations, the condition (4.2) can be rewritten in the following equivalent form:

$$J_1(\overleftarrow{f},\overleftarrow{\overline{f}}) = \Lambda\sigma J_1(f,\overline{f})\sigma^{-1}A. \tag{4.10}$$

Let us introduce the matrix $J_1^*$ defined by the following equation:

$$\{J_1, J_1^*\} = 1, \tag{4.11}$$

which admits the unique solution. Using relations (4.8) and their adjoint, one can easily construct this solution[3],

$$J_1^*(f,\overline{f}) = \hat{\phi}_1^T \otimes \partial^{-1}\hat{\phi}_1 \equiv \overline{\Pi}\begin{pmatrix} \overline{f}\partial^{-1}\overline{f}, & -\overline{f}\partial^{-1}f - 2 \\ -f\partial^{-1}\overline{f} - 2, & f\partial^{-1}f \end{pmatrix}\Pi, \tag{4.12}$$

where $\hat{\phi}_1$ is determined by eq. (4.6). Applying the automorphism $\sigma$ (2.5) to eqs. (4.11) and having in mind the one-to-one correspondence between $J_1$ and $J_1^*$, one can conclude that the condition (4.10) for $J_1$ is satisfied if and only if, the similar condition for $J_1^*$,

$$J_1^*(\overleftarrow{f},\overleftarrow{\overline{f}}) = \Lambda\sigma J_1^*(f,\overline{f})\sigma^{-1}A, \tag{4.13}$$

[5] To check relation (4.8) for the first nontrivial solution (2.10), it is necessary to remove the ambiguity in the operator $\partial^{-1}\partial 1$ that appears in the calculations by setting $\partial^{-1}\partial 1 = (\partial^{-1}\partial)1 \equiv 1$.



is implemented. Because of the rather simple structure of $J_1^*$ (4.12), it is a simple exercise to check the correctness of eq. (4.13) and, therefore, eq. (4.10) for $J_1$ is also correct. This completes the proof of eq. (4.9).

Acting $k$-times by the recursion operator (3.3) on the first Hamiltonian structure (4.9) and taking into account that the scaling dimension $[R] = cm^{-1}$, it is easy to understand that

$$J_k = R^k J_1 \tag{4.14}$$

gives us the $k$-th Hamiltonian structure.

Using the general rule (4.14), we derive, for example, the following representation for the second Hamiltonian structure[3] $J_2$:

$$J_2(f,\overline{f}) = \frac{1}{2}\begin{pmatrix} fD\overline{D}\partial^{-1}f, & D\overline{D} - fD\overline{D}\partial^{-1}\overline{f} \\ -\overline{D}D + \overline{f}\,\overline{D}D\partial^{-1}f, & -\overline{f}\,\overline{D}D\partial^{-1}\overline{f} \end{pmatrix}, \tag{4.15}$$

which resembles the form of its bosonic counterpart and obviously satisfies the chiral consistency conditions (4.1).

In terms of $N = 1$ superfields, the explicit expressions for the first and second Hamiltonian structures of the super-NLS hierarchy were constructed in [10]; however, to our knowledge, the recursion operator (3.3) is presented here for the first time.

To conclude this section, we would like to stress that the consistency conditions (4.1) is satisfied for all the Hamiltonian structures (4.14). This is evident from the explicit form of the recursion operator $R$ (3.3) and the properties of the projectors (3.4). Thus, all of the hamiltonian structures are degenerate matrices. This is the peculiarity of a manifest $N = 2$ superinvariant description of the super-NLS hierarchy in terms of $N = 2$ superfields, which has no analogue in the description in terms of $N = 1$ superfields or components. This means that the standard representation of $R$ in terms of the first and second Hamiltonian structures $R = J_2 J_1^{-1}$ fails, though the relation $RJ_1 = J_2$ is correct. It is instructive to find its correct generalization in the case under consideration. Without going into detail, let us present the answer,

$$R = J_2 J_1^*, \tag{4.16}$$

where the matrix $J_1^*$ is defined by eq. (4.12). Using (adjoint) eqs. (4.8), it is easy to verify the mutual relations (4.14) and (4.16) between $R$, $J_1$, $J_1^*$, and $J_2$.

# 5 The f-Toda invariant Hamiltonians

In this section, we construct the Hamiltonians which are invariant with respect to the f-Toda mapping (2.3) and demonstrate that the evolution equations (3.7) can be represented in Hamiltonian form.

Let us recall that in the $N=2$ supersymmetric case, the Hamiltonian $H(f,\overline{f})$ can be expressed in terms of the Hamiltonian density $\mathcal{H}(f,\overline{f})$ as

$$H(f,\overline{f}) = \int dZ\,\mathcal{H}(f,\overline{f}), \tag{5.1}$$

where $Z=(x,\theta,\overline{\theta})$ is the coordinate of $N=2$ superspace and $dZ = dx d\theta d\overline{\theta}$ is an invariant $N=2$ supersymmetric measure.

The Hamiltonian $H(f,\overline{f})$ is invariant with respect to the f-Toda mapping (2.3)

$$H(\overleftarrow{f},\overleftarrow{\overline{f}}) = H(f,\overline{f}), \tag{5.2}$$

if the Hamiltonian density $\mathcal{H}(f,\overline{f})$ satisfies the following condition:

$$\mathcal{H}(\overleftarrow{f},\overleftarrow{\overline{f}}) - \mathcal{H}(f,\overline{f}) = \Psi + \overline{\Psi}, \tag{5.3}$$

where $\Psi$ ($\overline{\Psi}$) is an arbitrary local chiral (antichiral) function of $f$ and $\overline{f}$,

$$D\Psi = \overline{D}\,\overline{\Psi} = 0. \tag{5.4}$$

Let us note that the r.h.s. of the condition (5.3) admits a more general structure in comparison with its bosonic counterpart [8, 9], where only the derivative $\partial$ $(= -\{D,\overline{D}\})$ of an arbitrary local function is admitted. It is evident that through integration over the invariant supersymmetric measure $dZ$, due to (5.4), it becomes equal to zero, providing the invariance condition (5.2) for the Hamiltonian.

Now, we will construct an infinite set of partial solutions of the condition (5.3).

Acting by the operator $2\partial^{-1}$ on the symmetry equation (2.9),

$$\partial^{-1}(\overleftarrow{F}\overleftarrow{\overline{f}} + \overleftarrow{f}\overleftarrow{\overline{F}}) - \partial^{-1}(F\overline{f} + f\overline{F}) = 2\frac{\overline{D}\overleftarrow{F}}{\overline{D}\overleftarrow{f}} + 2\frac{D\overline{F}}{D\overline{f}} \tag{5.5}$$



and comparing the result (5.5) with (5.3), one can immediately find the solution to the condition (5.3) for the Hamiltonian density $\mathcal{H}$,

$$\mathcal{H} = \partial^{-1}(F\overline{f} + f\overline{F}). \tag{5.6}$$

Substituting the infinite set of the solutions (3.6) of the symmetry equation (2.9) into (5.6), one can generate the infinite set of Hamiltonian densities $\mathcal{H}_p$ with scale dimension $p$ (see the paragraph below formula (3.7)):

$$\mathcal{H}_p = \partial^{-1}(F_p\overline{f} + f\overline{F}_p). \tag{5.7}$$

Using the explicit expressions (3.8), we obtain, for example, the following five first Hamiltonian densities:

$$\begin{aligned}
&\mathcal{H}_0 = 0, \quad \mathcal{H}_1 = f\overline{f}, \quad \mathcal{H}_2 = 2(f'\overline{f} - \frac{1}{2}(f\overline{f})'),\\
&\mathcal{H}_3 = 3(f''\overline{f} - \frac{1}{2}f\overline{f}\,\overline{D}f\cdot D\overline{f} - (f'f)' + \frac{1}{3}(f\overline{f})''),\\
&\mathcal{H}_4 = f'''\overline{f} - f''\overline{f}' + f'\overline{f}'' - f\overline{f}''' + 2f\overline{f}f'\overline{f}'\\
&\qquad + f\overline{f}(\overline{D}f\cdot D\overline{f}' - \overline{D}f'\cdot D\overline{f}) + 2(f\overline{f}' - f'\overline{f})\overline{D}f\cdot D\overline{f}.
\end{aligned} \tag{5.8}$$

Up to unessential total derivatives and overall multipliers, these Hamiltonian densities coincide with the corresponding quantities of the $N=2$ super-NLS hierarchy [3], which confirms the above-mentioned interrelation of the f-Toda mapping (2.3) and the $N=2$ super-NLS hierarchy.

Using the explicit expressions for the f-Toda-invariant first and second Hamiltonian structures, (4.9), (4.12), and (4.15), as well as for the invariant Hamiltonians (5.8), one can construct the Hamiltonian system of evolution equations

$$\frac{\partial}{\partial t_p}\begin{pmatrix} f \\ \overline{f} \end{pmatrix} = J_1\begin{pmatrix} \delta/\delta f \\ \delta/\delta\overline{f} \end{pmatrix} H_{p+1} = J_2\begin{pmatrix} \delta/\delta f \\ \delta/\delta\overline{f} \end{pmatrix} H_p, \tag{5.9}$$

$$J_1^{*}\frac{\partial}{\partial t_p}\begin{pmatrix} f \\ \overline{f} \end{pmatrix} = \begin{pmatrix} \delta/\delta f \\ \delta/\delta\overline{f} \end{pmatrix} H_{p+1}, \tag{5.10}$$

which, by construction, are invariant with respect to the f-Toda mapping (2.3). Direct calculations show that they are equivalent to the evolution equations (3.7), (3.8), i.e. the following relation

$$\begin{pmatrix} F_p \\ \overline{F}_p \end{pmatrix} = J_1\begin{pmatrix} \delta/\delta f \\ \delta/\delta\overline{f} \end{pmatrix} H_{p+1} \tag{5.11}$$

is satisfied.

We would like to close this section with the remark that the first Hamiltonian density $\mathcal{H}_1$ satisfies the following equation of motion:

$$\frac{\partial}{\partial t_p}\mathcal{H}_1 = \mathcal{H}_p'. \tag{5.12}$$

Hence, there exists the additional integral of motion,

$$\widetilde{H}_1 = \int dx \mathcal{H}_1, \tag{5.13}$$

where we have only space integration, which means that $\widetilde{H}_1$ is the unconstrained superfield and, therefore, it contains four independent components, the Hamiltonians. To obtain relation (5.12), one can substitute expressions (3.7) for $F_p$ and $\overline{F}_p$, as well as for $\mathcal{H}_1$ (5.8) into (5.7). At $p = 2$, this property was observed in [11] for the wide class of $N = 2$ supersymmetric generalized NLS hierarchies constructed there. Taking this fact into account, as well as that $N = 2$ super-NLS is a particular representative of the class of $N = 2$ super-GNLS hierarchies, it seems plausible to assume that relation (5.12) is also satisfied for the entire class of $N = 2$ super-GNLS hierarchies at arbitrary values of the parameter $p = 1, 2, 3, \ldots$, as in the $N = 2$ super-NLS case.

# 6 Conclusion

In this paper, we have proposed the $N = 2$ supersymmetric f-Toda mapping (2.3) in $(1|2)$ superspace that can be considered as the minimal $N = 2$ superextension of the one-dimensional Toda chain. We demonstrated that the $N = 2$ super-NLS hierarchy is invariant with respect to the f-Toda mapping, and produced its manifestly $N = 2$ supersymmetric recursion operator and Hamiltonian structures, using only their symmetry properties. New general representations (5.7) and (5.12) for its Hamiltonians were observed.

We would like to note that the f-Toda substitution is not exotic. There is a wide class of such kinds of substitutions for which the approach developed in the present letter may be literally applied. We hope to present them together with the corresponding integrable hierarchies in future publications. It would also be interesting to construct a super-Hamiltonian



structure of the f-Toda chain and its explicit solutions. It seems to be very important to find its two-dimensional integrable counterparts that could admit the superconformal structure and give new example of consistent two-dimensional supersymmetric field theories. This work is under progress at the present time.

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# Appendix

Here, we shortly describe the main steps of the proof of the assertion (3.1) of section 3.

First, it is necessary to substitute $\widetilde{F}$ and $\widetilde{\overline{F}}$ (3.1) into the symmetry equation (2.9). Direct calculations give the following form for the different terms of the symmetry equation:

$$\begin{aligned}
\overline{D}\overleftarrow{\widetilde{F}} =& \overline{D}\overleftarrow{F}' + \frac{1}{2}\overline{D}(\overleftarrow{f}\overleftarrow{\overline{f}}\overleftarrow{F}) - [\overleftarrow{f}\overline{D} + ((\overline{D}\overleftarrow{f})D\overline{D} + \frac{1}{2}(\overline{D}\overleftarrow{f}') \\
&- \frac{1}{2}\overleftarrow{f}'\overline{D})\partial^{-1}](\overleftarrow{f}\overleftarrow{\overline{F}} + \overleftarrow{F}\overleftarrow{\overline{f}}), \\
D\widetilde{\overline{F}} =& -D\overline{F}' + \frac{1}{2}D(f\overline{f}\,\overline{F}) - [\overline{f}D + ((D\overline{f})\,\overline{D}D + \frac{1}{2}(D\overline{f}') \\
&- \frac{1}{2}\overline{f}'D)\partial^{-1}](f\overline{F} + F\overline{f}), \\
\overleftarrow{\widetilde{F}}&\overleftarrow{\overline{f}} + \overleftarrow{f}\overleftarrow{\widetilde{\overline{F}}} - \widetilde{F}\overline{f} - f\widetilde{\overline{F}} \\
&= \overleftarrow{F}'\overleftarrow{\overline{f}} - \overleftarrow{f}\overleftarrow{\overline{F}}' - \frac{1}{2}[(\overleftarrow{f}\overleftarrow{\overline{f}})' + (\overline{D}\overleftarrow{f})\overleftarrow{\overline{f}}D - \overleftarrow{f}(D\overleftarrow{\overline{f}})\overline{D}]\partial^{-1}(\overleftarrow{f}\overleftarrow{\overline{F}} + \overleftarrow{F}\overleftarrow{\overline{f}}) \\
&\quad - F'\overline{f} + f\overline{F}' + \frac{1}{2}[(f\overline{f})' + (\overline{D}f)\overline{f}D - f(D\overline{f})\overline{D}]\partial^{-1}(f\overline{F} + F\overline{f}).
\end{aligned} \tag{A.1}$$

Second, it is necessary to use relations (2.3) and (2.9) and their direct consequences: the two identities which can be obtained from (2.3) by the action of derivatives $D$ and $\overline{D}$, respectively; the identity which can be produced from (2.9) by the action of the operator $[D,\overline{D}]$; and the following identity:

$$(\frac{1}{2}\overleftarrow{f}\overleftarrow{\overline{f}} - (\ln\overline{D}\overleftarrow{f})')(\frac{1}{2}\partial^{-1}(\overleftarrow{F}\overleftarrow{\overline{f}} + \overleftarrow{f}\overleftarrow{\overline{F}}) - \frac{\overline{D}\overleftarrow{F}}{\overline{D}\overleftarrow{f}})$$
$$= (\frac{1}{2}f\overline{f} + (\ln D\overline{f})')(\frac{1}{2}\partial^{-1}(F\overline{f} + f\overline{F}) + \frac{D\overline{F}}{D\overline{f}}), \qquad \text{(A.2)}$$

which one can derive by rewriting relations (2.3) and (5.5) in the following equivalent form:

$$\frac{1}{2}\overleftarrow{f}\overleftarrow{\overline{f}} - (\ln\overline{D}\overleftarrow{f})' = \frac{1}{2}f\overline{f} + (\ln D\overline{f})',$$
$$\frac{1}{2}\partial^{-1}(\overleftarrow{F}\overleftarrow{\overline{f}} + \overleftarrow{f}\overleftarrow{\overline{F}}) - \frac{\overline{D}\overleftarrow{F}}{\overline{D}\overleftarrow{f}} = \frac{1}{2}\partial^{-1}(F\overline{f} + f\overline{F}) + \frac{D\overline{F}}{D\overline{f}}, \qquad \text{(A.3)}$$

respectively, and equating the product on their left-hand sides to the product on their right-hand sides.

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