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DYNAMIC RETARDATION CORRECTIONS TO THE MASS SPECTRUM OF HEAVY QUARKONIA

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Эффект динамического запаздывания
в спектре масс тяжелых кваркониев

В квазипотенциальном подходе Логунова—Тавхелидзе вычислены обусловленные динамическим запаздыванием поправки первого порядка к спектру масс тяжелых кварконйев. В ковариантном ядре уравнения Бете-Солпитера были использованы граничные условия стоячей волны. Как и ожидалось, эти поправки оказываются малыми для всех низколежащих мезонных уровней и стремятся к нулю в пределе бесконечно тяжелых кварков ($m_Q \to \infty$). Проведено сравнение предложенного подхода для вычисления поправок, обусловленных запаздыванием, с другими подходами, известными в литературе.

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Dynamic Retardation Corrections to the Mass Spectrum
of Heavy Quarkonia

In the framework of the Logunov—Tavkhelidze quasipotential approach the first-order retardation corrections to the heavy quarkonia mass spectrum are calculated using the stationary wave boundary condition in the covariant kernel of the Bethe-Salpeter equation. As has been expected, these corrections turn out to be small for all low-lying heavy meson states and vanish in the heavy quark limit ($m_Q \to \infty$). The comparison of the suggested approach to the calculation of retardation corrections with others, known in literature, is carried out.

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1 Introduction

Within the framework of the constituent quark model it is quite natural to study the properties of the bound $q\bar{q}$ systems on the basis of the Bethe–Salpeter (BS) equation. Despite the remarkable success in the quantitative description of bound–state masses and formfactors, achieved with the use of instantaneous (static) interaction kernel in this equation [1]-[4], a completely relativistic approach to the problem is still lacking. Namely, from the physical point of view one can expect that the dynamic retardation corrections (i.e. the corrections coming from the explicit dependence of the quark–quark interaction kernel on the relative energy variables in a 4-dimensional (4D) BS treatment) to the bound–state characteristics must be small for heavy quarkonia and may become significant in the light–quark sector. Moreover, one expects a smooth static limit when masses of constituent quarks tend to infinity. In practice, however, the situation is more complicated owing to the infrared–singular behaviour of the "confining" kernels which are present in the BS equation for the quark–antiquark wave function. Since at the present stage the exact derivation of such relativistic kernels directly from QCD is unknown, different prescriptions are assumed for the *ad hoc* relativistic generalization [5]-[11] of the phenomenological static potentials. As to the one–gluon exchange part of the interquark potential, which has proven to be significant in the quantitative description of meson data, it can be uniquely generalized to 4 dimensions with the use of the field–theoretical arguments. In contrast with the one–gluon exchange potential, the "relativization" of the confining potentials, in general, introduces a new mass parameter in the theory [7, 9, 10, 11], which may be fixed, using additional constraints either on the relativistic counterpart of this potential [7, 9, 10, 11] or the bound–state equation [8]. Neither of these constraints can be preferred from the physical point of view, rendering ambiguous the identification of the dynamic retardation effect in the observable characteristics.

The existence of an additional free parameter in the theory stems from the necessity of the infrared regularization of the "confining" kernel in 4 dimensions. As a result, the smooth static limit, in general, is lost [9, 11] and the dynamic retardation corrections to the bound-state characteristics turn out to be large in the heavy-quark sector [9, 11], rendering doubtful even the concept of the confining interaction in 4 dimensions [9].

It should be pointed out that a naive relativistic generalization of a given static confining potential does not, in general, lead to the BS kernel which is also "confining" in the sense that the resulting BS equation possesses only a discrete spectrum. This has been demonstrated e.g., in ref. [12] for the case of harmonic oscillator potential. On the other hand, in ref. [11] it was demonstrated that using the Logunov–Tavkhelidze quasipotential approach in the above-mentioned case, it is possible to "revive" the discrete energy levels below the two-particle threshold.

In the present paper, we study the quark-antiquark bound-state BS equation with the kernel, explicitly dependent on relative energy variables, in the framework of the first-order quasipotential approach. We use the stationary wave boundary conditions in the kernel, corresponding to the "confining" part of the $q\bar{q}$ interaction. It is demonstrated that the first-order quasipotential equation with a kernel like that possesses a discrete spectrum and has a smooth static limit when the constituent quark mass tends to infinity unlike the case when the conventional prescriptions are used for the regularization of this kernel [6]-[9]. The physical motivation for such an "unusual" boundary condition is discussed.

2 First – Order Quasipotential Equation for the $q\bar{q}$ Systems

Below we use the basic relations of the Logunov–Tavkhelidze quasipotential approach [13]. The first-order quasipotential in this approach is defined by the equation

$$\underline{\tilde{V}}^{(1)}(M_B;\mathbf{p},\mathbf{q}) =< \mathbf{p}|\underline{\tilde{G}_0}^{-1}\widetilde{G_0 K G_0}\underline{\tilde{G}_0}^{-1}|\mathbf{q} > \tag{1}$$

Here P_μ stands for the total four-momentum of the $q\bar{q}$ system ($P_\mu = (M_B, \mathbf{0})$ in the c.m.f., M_B being the bound-state mass); G_0 is the free two-fermion Green function, K is the Bethe–Salpeter equation kernel and the procedure $\tilde{A}$ for any operator A is defined as follows:

$$\tilde{A}(P;\mathbf{p},\mathbf{q}) = \int \frac{dp_0}{2\pi} A(P;p,q)\frac{dq_0}{2\pi} \tag{2}$$

and

$$\tilde{G}_0 = \underline{\tilde{G}}_0\gamma_1^0\gamma_2^0\Pi;\quad \Pi = (\Lambda_1^{(+)}\Lambda_2^{(+)} - \Lambda_1^{(-)}\Lambda_2^{(-)})\gamma_1^0\gamma_2^0;$$

$$\underline{\tilde{G}}_0 = [M_B - h_1(\mathbf{p}) - h_2(-\mathbf{p})]^{-1}$$

$$\Lambda_i^{(\pm)} = \frac{w_i \pm h_i}{2w_i};\quad h_i = \alpha_i\mathbf{p}_i + m\gamma_i^0;\quad w_i = \sqrt{m^2 + \mathbf{p}_i^2} \tag{3}$$

where m_i denotes the mass of the i-th constituent quark. Below for the simplicity, we restrict ourselves to the equal-mass case $m_1 = m_2 \equiv m$.

The equal-time wave function $\tilde{\varphi}(\mathbf{p})$ obeys the following equation:

$$[M_B - h_1(\mathbf{p}) - h_2(-\mathbf{p})]\tilde{\varphi}(\mathbf{p}) = -i\gamma_1^0\gamma_2^0\,\frac{4}{3}\int\frac{d^3\mathbf{q}}{(2\pi)^3}\underline{\tilde{V}}^{(1)}(M_B;\mathbf{p},\mathbf{q})\tilde{\varphi}(\mathbf{q}) \tag{4}$$

At the next step we project (4) onto the positive-energy states $\tilde{\varphi}(\mathbf{p}) \to \Lambda_1^{(+)}\Lambda_2^{(+)}\tilde{\varphi}(\mathbf{p})$, $\underline{\tilde{V}}^{(1)} \to \Lambda_1^{(+)}\Lambda_2^{(+)}\underline{\tilde{V}}^{(1)}\Lambda_1^{(+)}\Lambda_2^{(+)}$. As it has been demonstrated in ref. [2], the contribution from the negative-energy component of the wave function to eq. (4) can be neglected, provided the solutions of this equation exist. Further, we assume that the spin structure of the quasipotential is the equal-weight mixture of the scalar and the fourth component of the vector: $\hat{O}_c = \frac{1}{2}(I_1 \otimes I_2 + \gamma_1^0 \otimes \gamma_2^0)$ which is perhaps the simplest choice from the more general ones [1]-[4] and provides the existence of the stable discrete energy levels. (Note that the one-gluon exchange part of the potential in eq. (4) is completely neglected since we are interested in the retardation corrections coming from the confining part of the potential).

The double-positive component $\tilde{\varphi}^{(++)}(\mathbf{p}) = \Lambda_1^{(+)}\Lambda_2^{(+)}\tilde{\varphi}(\mathbf{p})$ of the wave function is expressed in terms of the Pauli spinor $\tilde{\chi}^{(+)}(\mathbf{p})$ [2]

$$\tilde{\varphi}^{(++)}(\mathbf{p}) = \left(\frac{w+m}{2w}\right)^{\frac{1}{2}} \begin{pmatrix} 1 \\ \frac{\sigma_1 \mathbf{p}}{w+m} \end{pmatrix} \otimes \left(\frac{w+m}{2w}\right)^{\frac{1}{2}} \begin{pmatrix} 1 \\ \frac{-\sigma_2 \mathbf{p}}{w+m} \end{pmatrix} \tilde{\chi}^{(+)}(\mathbf{p}) \tag{5}$$

Using the partial-wave expansion

$$\tilde{\chi}^{(+)}(\mathbf{p}) = \sum_{LSJM_J} <\hat{\mathbf{p}}|LSJM_J> \tilde{R}^{(+)}_{LSJ}(p);\;\; \hat{\mathbf{p}} = \frac{\mathbf{p}}{p};\;\; S = 0,1 \tag{6}$$

and neglecting, as in refs. [10, 11], the mixing between the $L = J \pm 1$ and $S = 0,1$ states, from (4) we obtain

$$[M_B - 2w(p)]\tilde{R}^{(+)}_{LSJ}(p) = \frac{4}{3}\int_0^\infty q^2 dq \left(\tilde{V}^{(+)}_L(p,q)\frac{1}{2}\left(1 + \frac{m^2}{w(p)w(q)}\right) + \right.$$

$$\left. +(L-J)\frac{4J(J+1)}{(2J+1)^2}(\tilde{V}^{(+)}_{J-1}(p,q) - \tilde{V}^{(+)}_{J+1}(p,q))\frac{(w(p)-m)(w(q)-m)}{4w(p)w(q)}\right) \tilde{R}^{(+)}_{LSJ}(q) \tag{7}$$

where the partial-wave expansion of the positive-energy projection of the quasipotential reads as

$$\Lambda^{(+)}_1\Lambda^{(+)}_2\underline{\tilde{V}}^{(1)}\Lambda^{(+)}_1\Lambda^{(+)}_2 = 4\pi^2 \sum_{LSJM_J} <\hat{\mathbf{p}}|LSJM_J> V^{(+)}_L(p,q) < LSJM_J|\hat{\mathbf{q}}> \tag{8}$$

and, for the case of local quasipotential

$$\tilde{V}^{(+)}_L(p,q) = \int r^2 dr j_L(pr)\tilde{V}_c(r;M_B)j_L(qr) \tag{9}$$

j_L being the spherical Bessel function.

3 Dynamic Input and Construction of the First-Order Quasipotential

We parametrize the confining part of the static $q\bar{q}$ potential in the following simple way:

$$V_{st}(r) = kr + c \tag{10}$$

In order to construct the quasipotential, one should find the 4D counterpart of the static interquark potential given by (10). The conventional prescription for the pure linear potential consists in the substitution of $V_{st}(\mathbf{q}) = |\mathbf{q}|^{-4}$ by $V(q) = (-q^2 - i0)^{-2} \equiv (-q_0^2 + \mathbf{q}^2 - i0)^{-2}$. However, it is well known that the kernel $(-q^2 - i0)^{-2}$ in 4 dimensions needs the infrared regularization. A commonly used regularization for any power-law potential (see, e.g. [6, 9]), is achieved by writing

$$\int d^3\mathbf{r} e^{-i\mathbf{q}\mathbf{r}}|\mathbf{r}|^n \longrightarrow (-)^n \lim_{\mu\to 0}\frac{\partial^{n+1}}{\partial\mu^{n+1}}\left(\frac{4\pi}{q_0^2 - \mathbf{q}^2 - \mu^2 + i0}\right) \tag{11}$$

For, e.g., $n = 1$ (linear potential case) the r.h.s. of eq. (11) contains a divergent piece proportional to $\ln\mu$ [9]. This divergent term can be removed at the expense of the explicit dependence of the renormalized kernel on the subtraction point [9]. Thus, an additional mass scale parameter necessarily appears in the covariant kernel owing to the infrared–singular behaviour of the "confining" interaction. It should be pointed out that the latter is a rather general property of "confining" kernels and holds, e.g. within the dimensional regularization scheme [7]. In its turn, the presence of such an additional scale in the BS kernel makes the transition to the static limit $m \to \infty$ less transparent and may lead to large retardation corrections even in this limit [9, 10, 11], which is completely unacceptable from the physical point of view.

To overcome the above mentioned difficulty, we note that it is shared by all the regularizations, known to us, which leave the *analytic structure* of the kernel untouched, i.e. in other words, the boundary condition in the kernel is given by the conventional causal prescription $q^2 \to q^2 + i0$ for all internal lines. On the other hand, in the kernel, which is assumed to confine the particles, one could *a priori* expect a different choice of the boundary condition other than to the conventional one corresponding to freely moving particles in the remote past and future. Bearing this in mind, in the present paper we have investigated the possibility of an "unconventional" choice of the boundary condition in the covariant BS kernel. Namely, instead of (11) we use the following prescription for the relativistic generalization of power–law potentials in the position space:

$$r^\alpha \longrightarrow K_\alpha(x) = \frac{\Gamma(1+\frac{\alpha}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}+\frac{\alpha}{2})}\theta(-x^2)(-x^2)^{(\alpha-1)/2} \tag{12}$$

where α is not, in general, an integer number and $\alpha \neq -1, -2, \cdots$ (note that the latter condition excludes from the consideration, e.g., the Coulombic kernel). The normalization in (12) is chosen so that in the static limit $\int_{-\infty}^{\infty} dx^0 K_\alpha(x^0, \mathbf{x}) = |\mathbf{x}|^\alpha$.

The prescription (12) can be extended even to a wider class of static potentials. To demonstrate this, let us first consider the case of the exponential potential. Expanding this potential in powers of r and using (12) in every order, it is easy to verify that

$$e^{-\mu r} \to \theta(-x^2)\left(-\frac{\mu}{2}J_0(\mu\sqrt{-x^2}) + \frac{1}{\pi}(-x^2)^{-1/2}\,{}_1F_2(1;\frac{1}{2},\frac{1}{2};-\frac{\mu^2x^2}{4})\right) \tag{13}$$

where J_ν and ${}_pF_q$ denote, respectively, the Bessel and hypergeometric functions. Eq. (13) enables one to apply the procedure of the relativistic generalization to a wide class of potentials which can be written in the following form:

$$V(r) = \int d\mu C(\mu) e^{-\mu r} \tag{14}$$

where $C(\mu)$ must obey certain conditions in order to provide the convergence of the integral over $d\mu$ in the relativistic case.

For some widely used confining potentials (constant kernel, linear and oscillator potentials, $\alpha = 0, 1, 2$, respectively) the Fourier transform of Eq. (12) reads as

$$K_\alpha(q) = \begin{pmatrix} 2 \\ -4 \\ 12 \end{pmatrix} \pi \left(\frac{1}{(q_0^2 - (|\vec{q}| + i0)^2)^{\frac{3+\alpha}{2}}} - (-)^\alpha \frac{1}{(q_0^2 - (|\vec{q}| - i0)^2)^{\frac{3+\alpha}{2}}}\right) \tag{15}$$

$$K_\alpha(q)|_{q_0=0} = \begin{cases} (2\pi)^3\delta^{(3)}(\vec{q}) \\ -4\pi\left((|\vec{q}| + i0)^{-4} + (|\vec{q}| - i0)^{-4}\right) \\ -(2\pi)^3\nabla_q^2\delta^{(3)}(\vec{q}) \end{cases} \tag{16}$$

Consequently, the stationary–wave boundary condition appears in the Fourier–transformed kernels instead of the conventional causal prescription. In fact, for the case of the pure linear case ($\alpha = 1$) this prescription has been known for a long time and has already been used for the relativistic generalization of the static linear potential (see, e.g. [14, 15]). Thus, eq. (12) can be understood merely as an extension of this prescription to any power–law potentials. It should be pointed out that, unlike the case with causal prescription, the "principal value" kernels introduced in the present paper are infrared finite and, hence, do not depend on an additional scale parameter. The limiting procedure $q_0 \to 0$ in (16) is unambiguous and leads to the well–defined distributions.

Next, we pass to the calculation of the first–order quasipotential (1), corresponding to the interaction kernel (12). The projection of this quasipotential onto the positive–energy states can be written in the following form:

$$\underline{\tilde{V}}^{(1),(++++)}(M_B;\mathbf{p},\mathbf{q}) = \int d^3\mathbf{x} e^{-i(\mathbf{p}-\mathbf{q})\mathbf{x}} \frac{\Gamma\left(1+\frac{\alpha}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{1}{2}+\frac{\alpha}{2}\right)} r^\alpha \times$$

$$\times \int_{-1}^{1} d\tau (1-\tau^2)^{(\alpha-1)/2}\left(\theta(\tau)e^{i(M_B - w(p) - w(q))r\tau} + \theta(-\tau)e^{-i(M_B - w(p) - w(q))r\tau}\right) \tag{17}$$

Neglecting relativistic corrections in the exponentials $M_B - w(p) - w(q) = M_B - 2m + O(\frac{1}{m}) = -\epsilon_B$, as well as the imaginary part of this expression in analogy with refs. [10, 11, 16], we arrive at the local first–order quasipotential

$$\underline{\tilde{V}}^{(1),(++++)}(r;\epsilon_B) = \frac{\Gamma\left(1+\frac{\alpha}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{1}{2}+\frac{\alpha}{2}\right)} r^\alpha \int_{-1}^{1} d\tau (1-\tau^2)^{(\alpha-1)/2} \cos\epsilon_B r\tau =$$

$$= \left|\frac{2r}{\epsilon_B}\right|^{\alpha/2} \Gamma\left(1+\frac{\alpha}{2}\right) J_{\alpha/2}(|\epsilon_B r|) \tag{18}$$

Hence, the relativistic generalization of the static potential $V_{st}(r) = kr + c$ gives the first-order local quasipotential

$$\tilde{V}_c(r, \epsilon_B) = k\frac{\sin \epsilon_B r}{\epsilon_B} + cJ_0(\epsilon_B r) \tag{19}$$

which accounts for the retardation effect and reduces to $V_{st}(r)$ in the limit $\epsilon_B \to 0$.

As it can be seen from eq. (19), the account for the dynamic retardation effect in the case of pure linear potential effectively leads to the colour screening at intermediate distances. It should be pointed out that such a behaviour qualitatively agrees with the results of calculations for the unquenched lattice fermions in QCD [17]. At larger distances, the deviation of the retarded potential from the static one becomes significant, and one can no further rely on the first-order calculations. Note, however, that for the case of heavy quarkonia the latter difficulty causes no trouble since the wave function of the $q\bar{q}$ bound system in this case rapidly vanishes with the increase of r and, therefore, does not "feel" the oscillating "tail" of the potential at large distances. To be more precise, let us recall, that the mean radius of the bound system in the nonrelativistic limit for the power-law potentials $V(r) \sim r^\alpha$ scales as $m^{-1/(\alpha+2)}$, where m is the mass of the constituent. Consequently, since the mass of the bound state in the heavy quark limit scales as $M_B = 2m + \mathrm{const} + o(1)$, the expression $\epsilon_B < r >$ vanishes in this limit and $\tilde{V}_c(r, \epsilon_B)$ (19) reduces to $V_{st}(r)$ (10).

4 Results

Next we turn to the numerical solution of the obtained equation with the retardation effect taken into account explicitly through the energy-dependence of the first-order quasipotential. It is not obvious from the beginning whether the potential (19) leads to the discrete energy levels due to its oscillating behaviour as $r \to \infty$. Let us, therefore, consider the equation (7) with the potential (19) in detail. Passing to the nonrelativistic limit and neglecting for a moment the

"constant" term in (19), proportional to c, in the configuration space, we obtain the following differential equation:

$$f''(z) + (a\cos 2z + b)f(z) = 0 \tag{20}$$

where $f(r) = rR(r)$, $R(r) \equiv R_0(r)$ are the radial wave function of the bound state in the configuration space (for simplicity we assume the angular momentum, $L = 0$), $z = \frac{1}{2}((M_B - 2m)r - \pi/2)$, $a = \frac{k(M_B-2m)}{3m^3}$, $b = \frac{(M_B-2m)}{4m^3}$, and the boundary conditions imposed on $f(z)$ are $f\left(-\frac{\pi}{4}\right) = 0$ and $f(+\infty) = 0$.

Equation (20) has been extensively studied in the mathematical physics (see, e.g. [18]). We shall remind some results of investigation. Namely, if $f_1(z)$ is a particular solution of eq. (20) with the following initial conditions:

$$f_1(0) = 1, \quad f_1'(0) = 0 \tag{21}$$

and

$$\cosh 2\pi\mu = f_1(\pi),$$

the general solution of eq. (20) has the form:

$$f(z) = \begin{cases} C_1 e^{2\mu z}\varphi_1(z) + C_2 e^{-2\mu z}\varphi_2(z); & \cosh 2\pi\mu > 1 \\ (C_1\cos 2\nu z + C_2 \sin 2\nu z)\varphi_1(z) + (C_2 \cos 2\nu z - C_1 \sin 2\nu z)\varphi_2(z); & \\ & |\cosh 2\pi\mu| < 1; \quad \mu = i\nu \\ C_1 e^{2\rho z}\varphi_1(z) + C_2 e^{-2\rho z}\varphi_2(z); & \cosh 2\pi\mu < -1; \quad \mu = \rho + \frac{i}{2}, \end{cases} \tag{22}$$

$\varphi_1(z)$ and $\varphi_2(z)$ being the periodic functions in z with the period π.

Due to the fact that we consider eq. (20) in the semi-infinite interval $-\frac{\pi}{4} \leq z < +\infty$, it is possible to find normalizable solutions decreasing exponentially as $z \to +\infty$ ($C_1 = 0$ and $|\cosh 2\pi\mu| > 1$, eq. (22)). The eigenvalue condition then reads as

$$\varphi_2\left(-\frac{\pi}{4}\right) = 0, \quad |\cosh 2\pi\mu| > 1 \tag{23}$$

Thus, equation (20), despite the oscillating behaviour of the potential at the spatial infinity, allows for the discrete spectrum provided $|\cosh 2\pi\mu| > 1$, corresponding to the condition $M_B - 2m < 0$ in the limit $|M_B - 2m| << 2m$. Adding the constant term, proportional to c, it is natural to suppose that, for a small $|M_B - 2m|$ the discrete energy levels exist for $M_B - 2m - \frac{4}{3}c < 0$. Thus, the potential (19) in the nonrelativistic limit acts like the potential well. Note that a similar potential (the rising potential screened at large distances, $r > 1$ Fm was successfully used for the description of the meson spectrum in the framework of the coupled Dyson–Schwinger and Bethe–Salpeter equations, e.g., in [19]. Therefore, we expect that equation (7) with the potential (19) gives reasonable description of the low–lying meson states.

At the next step we have attempted to solve eq. (7) numerically, expanding the unknown radial wave function $\tilde{R}^{(+)}_{LSJ}(p)$ in the complete orthonormalized basis of the nonrelativistic oscillator wave functions [1, 2, 10, 11]

$$\tilde{R}^{(+)}_{LSJ}(p) = p_0^{-3/2} \sum_{n=0}^{\infty} c^{(+)}_{nLSJ} R_{nL}(p/p_0) \tag{24}$$

where

$$R_{nL}(z) = \left(\frac{2\Gamma\left(n+L+\frac{3}{2}\right)}{\Gamma(n+1)} \right)^{\frac{1}{2}} \frac{1}{\Gamma\left(L+\frac{3}{2}\right)} z^L \exp\left(-\frac{1}{2}z^2\right)\ {}_1F_1\left(-n, L+\frac{3}{2}, z^2\right) \tag{25}$$

and p_0 is an arbitrary scale parameter. Substituting (25) in equation (7) and truncating the sum at some fixed value N_{max}, we arrive at a system of linear algebraic equations for the coefficients $c^{(+)}_{nLSJ}$. If the procedure converges with increasing N_{max}, the eigenvalues M_B are determined from this system of equations. The calculations show that the final results do not depend on the scale parameter p_0, but the appropriate choice of this parameter leads to faster convergence of the series (25). It should be stressed that if the solution of equation (7) does not exist (e.g., due to the behaviour of the potential at the spatial infinity), this reveals in the divergence of the procedure with increasing N_{max} despite the fact that the potential matrix elements are calculated in the exponentially damping wave function basis.

Since the potential (19) depends on the unknown binding energy, $\epsilon_B = 2m - M_B$, of the $q\bar{q}$ system, equation (7) is solved with the use of the iteration method. Namely, we solve the equation with the static potential $V_{st}(r) = kr + c$ and determine the eigenvalues $M_B^{(st)}$. At the next step, these static values are substituted into the potential (19) in order to determine the corrected spectrum which, in its turn, is used as an input in the next iteration. We have checked that typically after 10–15 steps the iteration procedure converges for most low-lying heavy quarkonia energy levels.

In table 1 the results of calculations of the dynamic retardation corrections to the heavy quarkonia mass spectrum are presented. In these calculations, the parameters k and c were taken to be $k = 0.21\ GeV^2$, $c = -1.0\ GeV$. The constituent quark masses were chosen to be $m_c = 1.78\ GeV$ and $m_b = 5.10\ GeV$ in order to fit the J/ψ and Υ masses. As we see from table 1, this set of parameters gives reasonable description of the heavy meson mass spectrum in the static approximation. As has been expected, the dynamic retardation corrections turn out to be small (typically a few per cent) for all low-lying quarkonia states given in this table.

To check the consistency of the numerical methods applied to solve the problem under study, we have repeated our calculations for the "truncated" quasipotential

$$\tilde{V}'_c(r, \epsilon_B) = \begin{cases} k/\epsilon_B \sin \epsilon_B r + cJ_0(\epsilon_B r), & r < r_0 \\ k'r + c', & r > r_0 \end{cases} \tag{26}$$

with $\epsilon_B r_0 < \pi/2$ and the choice for k' and c' guarantees that $\tilde{V}'_c(r, \epsilon_B)$ along with its first derivative is continuous at $r = r_0$. Equation (7) with the quasipotential (26) obviously has the solutions since it grows at the spatial infinity instead of the oscillating behaviour revealed by the quasipotential $\tilde{V}_c(r, \epsilon_B)$ (19). However, we have numerically checked that for the low–lying states the mass spectrum obtained with the use of the full quasipotential (19) almost coincides with that obtained from the truncated one (26), provided $\epsilon_B r_0$ is sufficiently close to $\pi/2$. Consequently, the existence of stable solutions of eq. (7) with the full quasipotential (19) has been verified independently.

Next, we have checked the consistency of the iterative method used to handle the nonlinear dependence of the obtained quasipotential on the binding

Mesons	J^{PC}	$N^{2S+1}L_J$	1	2	3	4
$c\bar{c}$						
$\eta_c(2.980)$	0^{-+}	1^1S_0	3.095	3.204	0.109	$3.5\cdot10^{-2}$
$\eta_c'(3.590)$	0^{-+}	2^1S_0	3.690	3.739	0.049	$1.3\cdot10^{-2}$
$J/\psi(3.097)$	1^{--}	1^3S_1	3.096	3.204	0.108	$3.5\cdot10^{-2}$
$\psi'(3.686)$	1^{--}	2^3S_1	3.691	3.742	0.051	$1.3\cdot10^{-2}$
$h_{c1}(3.526)$	1^{+-}	1^1P_1	3.445	3.460	0.014	$4.2\cdot10^{-3}$
$\chi_{c0}(3.414)$	0^{++}	1^3P_0	3.445	3.460	0.014	$4.2\cdot10^{-3}$
$\chi_{c1}(3.511)$	1^{++}	1^3P_1	3.445	3.460	0.014	$4.2\cdot10^{-3}$
$\chi_{c2}(3.556)$	2^{++}	1^3P_2	3.445	3.461	0.014	$4.2\cdot10^{-3}$
$b\bar{b}$						
η_b	0^{-+}	1^1S_0	9.463	9.619	0.156	$1.7\cdot10^{-2}$
η_b'	0^{-+}	2^1S_0	9.899	9.966	0.067	$6.8\cdot10^{-3}$
$\Upsilon(9.460)$	1^{--}	1^3S_1	9.463	9.619	0.156	$1.7\cdot10^{-2}$
$\Upsilon'(10.023)$	1^{--}	2^3S_1	9.899	9.966	0.067	$6.8\cdot10^{-3}$
Υ''	1^{--}	1^3D_1	9.938	9.993	0.055	$5.5\cdot10^{-3}$
$\Upsilon'''(10.355)$	1^{--}	3^3S_1	10.250	10.255	0.005	$5.1\cdot10^{-4}$
Υ^{IV}	1^{--}	2^3D_1	10.277	10.291	0.014	$1.4\cdot10^{-3}$
h_{b1}	1^{+-}	1^1P_1	9.720	9.835	0.115	$1.2\cdot10^{-2}$
$\chi_{b0}(9.860)$	0^{++}	1^3P_0	9.720	9.835	0.115	$1.2\cdot10^{-2}$
$\chi_{b1}(9.892)$	1^{++}	1^3P_1	9.720	9.835	0.115	$1.2\cdot10^{-2}$
$\chi_{b2}(9.913)$	2^{++}	1^3P_2	9.720	9.835	0.115	$1.2\cdot10^{-2}$
h_{b1}'	1^{+-}	2^1P_1	10.097	10.109	0.013	$1.3\cdot10^{-3}$
$\chi_{b0}'(10.232)$	0^{++}	2^3P_0	10.097	10.109	0.013	$1.3\cdot10^{-3}$
$\chi_{b1}'(10.255)$	1^{++}	2^3P_1	10.097	10.109	0.013	$1.3\cdot10^{-3}$
$\chi_{b2}'(10.268)$	2^{++}	2^3P_2	10.097	10.109	0.013	$1.3\cdot10^{-3}$

Table 1. The dynamic retardation corrections to the heavy quarkonia mass spectrum

1) The meson mass in the static approximation, $M_B^{(st)}$ (GeV)

2) The meson mass with an inclusion of the retardation effect, $M_B^{(ret)}$ (Gev)

3) The size of the retardation correction, $M_B^{(ret)}-M_B^{(st)}$ (GeV)

4) $|M_B^{(ret)}-M_B^{(st)}|/M_B^{(st)}$

energy of the meson. To this end, we have substituted the trial value for ϵ_B in the quasipotential and determined the output value of the same quantity from the equation. Varying the input energy by a small step, and solving the equation at every energy, one can determine a fixed point, where the input energy coincides with the output one. We have verified that the iterative solutions for the low-lying states, listed in Table 1, are obtained in the fixed point method as well.

Finally, we have checked the sensitivity of the approach to the choice of a concrete form of the confining potential. To this end, we have repeated all the calculations using the oscillator kernel

$$V_{st}^o(r)=\frac{\omega^2}{2}r^2+c \tag{27}$$

with $\omega^2=0.05\ GeV^2$, $c=-0.64\ GeV$, $m_c=1.75\ GeV$, $m_b=5.03\ GeV$. The results for the relative size of the retardation corrections (not listed in Table 1) are almost identical to those for the case of linear confinement. Consequently, within the prescription chosen for the relativistic generalization of the static confining kernels (12) the reasonable result for the magnitude of the retardation effect is obtained irrespective of the choice of a concrete confining kernel.

5 Discussion

In the present paper, we have investigated a possible way of relativistic generalization of the static $q\bar{q}$ interaction. As a result, a covariant Bethe-Salpeter kernel, depending on all the components of the quark relative 4-momenta, is obtained. It is shown that the choice of the stationary–wave boundary conditions in the "relativized" kernel instead of the conventional causal one, enables one to overcome some difficulties which are inherent in the approaches used previously for the treatment of this problem. Namely,

– The proposed prescription enables us to obtain the infrared–finite kernel which does not contain the dependence on the infrared subtraction scale. As a result, the static limit in the BS equation is unambiguous as well as the identification of the dynamic retardation corrections.

– The first-order quasipotential, obtained with the use of the Logunov-Tavkhelidze method, contains the dependence on the binding energy. This is a remnant of the dynamic retardation effect in the first order. We have demonstrated that the discrete solutions of the first order quasipotential equation exist at least for the lowest energy levels. Moreover, in the static limit, when $m \to \infty$, the energy dependence in the quasipotential is effectively eliminated resulting, as required, in the initial static potential.

– With the use of the numerical methods we have demonstrated that, unlike the results obtained in refs. [9, 10, 11], the retardation corrections to the mass spectrum of heavy bound $q\bar{q}$ systems are small and do not depend on the details of the $q\bar{q}$ potential.

However, the most important question which arises here consists in the interpretation of the obtained results. We would like to stress that all these results were obtained at the expense of the choice of an unconventional boundary condition in the "confining" kernel of the BS equation. It seems to us that neither of these results can be obtained in the Euclidean formulation of the BS approach to the $q\bar{q}$ bound-state problem. Further, in the ladder approximation the problem reduces to the choice of a boundary condition in the effective "dressed" gluon propagator, which is believed to confine quarks. In literature, we can find several examples when the confined particles are quantized with the use of the principal-value prescription [14, 15, 21, 22]. (For the discussion of the analytical properties of the Feynman amplitudes in the 2D QCD, see ref. [22], where it is demonstrated that, although the confinement restricts the analyticity domains of various Green functions corresponding to the coloured particles, the analytic properties of the colourless current correlators remain untouched). At the present stage, we can not claim that the presence of the long-range (confining) force in the system leads to, or requires the modification of the boundary condition in the propagator as compared to the usual case of short-range forces and with the particles, moving freely outside the interaction region. However, the results of the present investigation indicate that the conventional 3D picture of rising potentials etc. is directly obtained from the 4D BS approach in the static limit only provided the stationary wave boundary condition is used in the kernel which describes the covariant confining interaction in 4 dimensions.

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