ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E2-96-372

A.N.Sissakian, V.M.Ter-Antonyan*

QUANTIZATION ON THE CONE AND CYON-OSCILLATOR DUALITY

Submitted to «Physics Letters A»

*E-mail: terant@thsun1.jinr.dubna.su

1996

Сисакян А.Н., Тер-Антонян В.М. E2-96-372

Квантование на конусе и кайон-осцилляторная дуальность

Показано, что трехмерный изотропный осциллятор, координаты которого принадлежат верхней половине конуса, дуален кайону, т.е. связанной системе частица-вихрь, наделенной дробной статистикой.

Работа выполнена в Лаборатории теоретической физики им.Н.Н.Боголюбова ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 1996

Sissakian A.N., Ter-Antonyan V.M. E2-96-372

Quantization on the Cone and Cyon—Oscillator Duality

It is shown that the three-dimensional isotropic oscillator with coordinates belonging to the two-dimensional half-up cone is dual to the cyon, i.e. the planar particle-vortex bound system provided by fractional statistics.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 1996

Our interest in the paper will be to consider the three– dimensional isotropic oscillator with coordinates belonging to a two– dimensional cone without the tip. We show that this system is dual to the cyon [1], i.e. a planar particle–vortex bound system with the Coulomb attractive interaction. This cyon is a simple prototype for the objects with fractional statistics [2], known also as anyons [3]. Anyons play an important role in the field theory [4], in the fractional quantum Hall effect [5], and in high–T_c superconductivity [6].

Let u_μ and (u, θ, φ) be Cartesian and spherical coordinates in $\mathbb{R}^3$. Define points u_μ as belonging to the half-up cone C_2^+, if $\theta = \pi/6$ and $u \neq 0$. Consider the Schrödinger equation

$$\frac{\partial^2\Psi}{\partial u_\mu^2} + \frac{2M}{\hbar^2}\left(E - \frac{M\omega^2 u^2}{2}\right)\Psi = 0, \qquad u_\mu \in C_2^+ \tag{1}$$

It is easy to verify that

$$\left(\frac{\partial^2}{\partial u_\mu^2}\right)_{u_\mu \in C_2^+} = \frac{4}{r}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right) \tag{2}$$

where $r = u^2$.

Combining (1) and (2), we obtain the Schrödinger equation in $\mathbb{R}^2$

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\varphi} + \frac{2M}{\hbar^2}\left(\frac{M\omega^2}{8} + \frac{E}{4r}\right)\Psi = 0 \tag{3}$$

The tip of the cone has been removed $(u > 0)$ and hence it is not possible to deform one into the other two loops with different winding numbers. As a consequence, Ψ satisfies the twisted boundary condition [7]

$$\Psi(r, \varphi + 2\pi) = e^{i\pi\nu}\Psi(r, \varphi) \tag{4}$$

As a statistical parameter ν can be arbitrary $(\nu \in [0, 1])$, our system is an anyon.In particular, if $\nu = 0$ or $\nu = 1$, the wavefunction picks up a plus or a minus sign, and the system acquires bosonic or fermionic statistics respectively.

Let us introduce the new wavefunction ψ instead of the previous one

$$\Psi(r, \varphi) = \psi(r, \varphi)e^{i\nu\varphi/2} \tag{5}$$

Equation (3) transforms into

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\left(\frac{\partial}{\partial\varphi} + i\frac{\nu}{2}\right)^2\psi + \frac{2M}{\hbar^2}\left(\frac{M\omega^2}{8} + \frac{E}{4r}\right)\psi = 0 \tag{6}$$

with the periodic boundary condition

$$\psi(r, \varphi + 2\pi) = \psi(r, \varphi) \tag{7}$$

We can see from (3) and (6) that the canonical angular momentum operator $\hat{J}_c = -i\partial/\partial\varphi$ is replaced by the kinetic angular momentum operator $\hat{J} = \hat{J}_c + s$. Here $s = \nu/2$ has the meaning of the spin of our system. Thus, the spin s and the statistics ν appear to be related in the conventional way ($\nu = 2s$).

To clarify the physics associated with equation (6), let us introduce the Cartesian coordinates

$$x_1 = r\cos\varphi \qquad x_2 = r\sin\varphi \tag{8}$$

It is easy to verify by direct computation that (6) may be rewritten as a Pauli equation

$$\frac{1}{2M}\left(\hat{p} + \frac{e}{c}\vec{A}\right)^2 \psi - \frac{\alpha}{r}\psi = \epsilon\psi \tag{9}$$

with the vector potential

$$\vec{A}(\vec{x}) = \frac{\Phi}{2\pi}\left(-\frac{x_2}{x_1^2 + x_2^2}\hat{x}_1 + \frac{x_1}{x_1^2 + x_2^2}\hat{x}_2\right) \tag{10}$$

Here $\hat{x}_1$ and $\hat{x}_2$ are unit vectors along the x_1- and the x_2-axes, $\Phi = \hbar c\nu/2|e|$ and

$$\epsilon = \frac{M\omega^2}{8}, \qquad \alpha = -eq = \frac{E}{4} \tag{11}$$

Thus, our initial system is equivalent to the cyon,i.e. the system consisting of a point charge particle moving on the plane in the external static electric and magnetic fields of a vortex localized at the origin: $B = \Phi\delta^{(2)}(\vec{x})$. This system continuously interpolates the bosonic ($s = 0$) and the fermionic ($s = 1/2$) systems which have been obtained from the circular oscillator by the quantum Bohlin transformation [8].

Let us introduce the separation ansatz

$$\psi(r,\varphi) = R(r)e^{im\varphi}/\sqrt{2\pi} \tag{12}$$

Here m is an integer because of the boundary condition (6). This is true despite of the fact that the algebra of the two-dimensional rotations is abelian and in principle an arbitrary constant could be added to the angular eigenvalues [9] By substituting the wavefunction (12) into (6) we are led to the radial equation

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{(m+s)^2}{r^2}R + \frac{2\mu}{\hbar^2}\left(\epsilon + \frac{\alpha}{r}\right)R = 0 \tag{13}$$

and conclude that the radial function R and the energy spectrum ϵ of the bound states of the cyon may be estimated from eigenfunctions and eigenvalues of the two- dimensional Coulomb problem [10] by the substitution $m \to m + s$.

Thus, we obtain

$$R(r) = Ce^{-z/2}z^J F(-\sqrt{\alpha/(-2\epsilon r_0)} + |m+s| + 1/2, 2J+1, z) \tag{14}$$

where $z = 2r/(-2\epsilon)^{1/2}r_0$, $r_0 = \hbar^2/M\alpha$, $J = m + s$, $s = \nu/2$.

The corresponding energy is

$$\epsilon^s_{n_\rho m} = -\frac{M\alpha^2}{2\hbar^2(n_r + |m+s| + 1/2)^2} \tag{15}$$

The value of the normalization constant

$$C = \frac{2}{(-2\epsilon)^{1/2}}\frac{1}{\Gamma(2J+1)}\sqrt{\frac{\Gamma(2j + n_r + 1)}{(2n_r + 2J + 1)(n_r)!}} \tag{16}$$

follows from the condition

$$\int_0^\infty |R(r)|^2 r dr = 1 \tag{17}$$

Equations (15) and (11) may be combined to yield

$$E = \hbar\omega(2n_r + 2|m+s| + 1) \tag{18}$$

We conclude that the energy levels for the isotropic oscillator on the cone are identical with energy levels of the circular oscillator possessing, apart from the usual angular momentum m, also the intrinsic (topological) quantum number ν which gives rise to the spin $s = \nu/2$ of the cyon. The spin s is localized near cyon and the kinetic angular momentum $J = m + s$ is located at the spatial infinity,as for the cyon without attractive Coulomb interaction [11].

We are now in a position to give explanation for the meaning of the term "duality" which we have used in the title of our paper. Equations (1) and (9) are connected with each other by the ansatz $(E,\omega) \to (\epsilon,\alpha)$ and by the transformation exchanging the coordinates (u_1,u_2) and (x_1,x_2) in the following way

$$x_1 = 2uu_1, \qquad x_2 = 2uu_2$$

Now,in equation (1) the coupling constant ω of the oscillator is fixed and the energy E is quantized. The situation with equation (9) is inverse: here the cyon's coupling constant α (or E) is fixed and the energy ϵ (or ω) of the cyon is quantized. According to (11),these conditions are inconsistent among themselves and, therefore, the isotropic oscillator on the cone and the cyon are rather dual than identical to each other [12].

We would like to thank Levon Mardoyan and George Pogosyan for helpful discussion.

References

[1] A.S. Goldhaber, Phys. Rev. Lett., **49**, 905 (1982)

[2] J.M. Leinaas and J. Myrheim, Nuovo Cimento, **237B**, 1 (1977).

[3] F. Wilczek, Phys. Rev. Lett., **49**, 957 (1982).

[4] de Sousa Gerbert, Nucl. Phys., **B346**, 440 (1990)

[5] M. Stone (editor), Quantum Hall Effect, World Scientific, Singapore (1992).

[6] R.B. Laughlin, Phys. Rev. Lett., **60**, 2677 (1988)

[7] Stefano Forte, Rev. Mod. Phys., **64**, 194 (1992)

[8] A. Nersessian, V.M. Ter–Antonyan, M. Tsulaja, Mod. Phys. Lett., **A11**, 1605 (1996)

[9] E. Merzbacher, Am. J. Phys., **30**, 237 (1962)

[10] M. J. Englefield, Group theory and the Coulomb problem, Willey-Intersciense, New-York-Toronto (1972).

[11] A.S. Goldhaber and R. Mackenzie, Phys. Lett., **214B**, 471 (1988).

[12] Jeffrey A. Harvey, Magnetic Monopoles, Duality and Supersymmetry, EFI-96-06, hep-th/9603086..

Received by Publishing Department
on October 11, 1996.