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EXACT SOLUTIONS TO THE NONLINEAR SPINOR FIELD EQUATIONS IN THE GÖDEL UNIVERSE

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Точные решения нелинейных уравнений спинорного поля во Вселенной Гёделя

Получены точные решения нелинейных уравнений спинорного поля во внешнем гравитационном поле Вселенной Гёделя. Рассмотрены два типа нелинейных лагранжианов: $\mathcal{L}_{\mathrm{N}} = F(I_S)$ и $\mathcal{L}'_{\mathrm{N}} = G(I_P)$, где $F(I_S)$ и $G(I_P)$ являются произвольными функциями от спинорных инвариантов $I_S = S = \overline{\Psi}\Psi$ и $I_P = P^2 = (i\overline{\Psi}\gamma^5\Psi)^2$. Установлены условия, при которых имеются одномерные солитоноподобные решения, а также роль гравитационного поля в формировании этих конфигураций.

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Exact Solutions to the Nonlinear Spinor Field Equations in the Gödel Universe

The nonlinear spinor field in the external gravitational field of the Gödel universe is considered and exact static solutions to the field equations corresponding to the Lagrangians with the nonlinear terms $\mathcal{L}_{\mathrm{N}} = F(I_S)$ and $\mathcal{L}_{\mathrm{N}} = G(I_P)$ are obtained. Here $F(I_S)$ and $G(I_P)$ are arbitrary functions of the spinor invariants $I_S = S = \overline{\Psi}\Psi$ and $I_P = P^2 = (i\overline{\Psi}\gamma^5\Psi)^2$. The conditions under which one-dimensional soliton-like solutions exist are established and the role of gravity in the formation of these objects is determined.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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1. INTRODUCTION

Recently an increasing interest has been expressed to the search of soliton-like solutions because of the necessity to describe the elementary particles like extended objects. In this work, the nonlinear spinor field is considered in the external gravitational field of the Gödel universe. The Gödel universe exhibits a number of unusual properties associated with rotation [1]. It is homogeneous in space and time and is filled with a perfect fluid. The main role of rotation in this universe consists in the avoidance of the cosmological singularity in the early universe, when the centrifugate forces of rotation dominate over the gravitation and the collapse does not occur [2].

The paper is organized as follows: in Sec. 2 the nonlinear spinor field with $\mathcal{L}_{\rm N} = F(I_S)$ in the external gravitational field Gödel universe is considered and exact solutions to the corresponding field equations are obtained. In Sec. 3 the properties of the energy density are investigated. In Sec. 4 this field is considered in the flat space–time in order to determine the role of gravitation over the nonlinear spinor fields. In Sec. 5 we consider the nonlinear term $\mathcal{L}_{\rm N} = G(I_P)$ in the Lagrangian of the spinor field in the Gödel universe and we obtain exact solutions to the corresponding field equations. In Sec. 6 the analysis of the energy density distribution is made for three different functions $G(I_p)$. In Sec. 7 we study this field in the Minkowski space–time. A discussion and interpretation of our results is presented in Sec. 8, together with a summary of our research.

2. NONLINEAR SPINOR FIELD WITH $\mathcal{L}_{\rm N} = F(I_S)$

The Lagrangian of the nonlinear spinor field reads

$$\mathcal{L} = \frac{i}{2}(\overline{\Psi}\gamma^{\mu}\nabla_{\mu}\Psi - \nabla_{\mu}\overline{\Psi}\gamma^{\mu}\Psi) - m\overline{\Psi}\Psi + \mathcal{L}_{\rm N}, \qquad (1)$$

where the nonlinear term $\mathcal{L}_{\rm N} = F(I_S)$ represents an arbitrary function of the spinorial invariant $I_S = S = \overline{\Psi}\Psi$; γ^{μ} are the spinorial matrices in curve space–time defined by the tetradic vectors and the metric tensor components in the following form [3]

$$\gamma^{\mu} = g^{\mu\nu}e^{a}_{\nu}(x)\eta_{ab}\tilde{\gamma}^{b}, \qquad (2)$$

$$\eta_{ab} = \mathrm{diag}\,(+1, -1, -1, -1); \qquad (3)$$

where e^{a}_{ν} form a set of tetradic 4–vectors which is given by the expression [4]

$$g_{\mu\nu} = \eta_{ab}e^{a}_{\mu}e^{b}_{\nu} \qquad (4)$$

and $\tilde{\gamma}^a$ are the flat space–time Dirac matrices; we have chosen them according to Ref. [5]. ∇_μ are the covariant spinor derivatives; they are defined as follows [6]:

$$\nabla_\mu = \partial_\mu - \Gamma_\mu(x), \tag{5}$$

where $\Gamma_\mu(x)$ are the spinorial affine connection matrices of curve space–time, determined by the relation

$$\Gamma_\mu(x) = \frac{1}{4} g_{\rho\delta}(\partial_\mu e^b_\sigma \cdot e^\rho_b - \Gamma^\rho_{\mu\sigma})\gamma^\delta\gamma^\sigma; \tag{6}$$

$\Gamma^\rho_{\mu\sigma}$ being the Christoffel symbols.

The metric of Gödel universe is represented in the following form [7]

$$ds^2 = dt^2 - dx^2 + \frac{1}{2} e^{2\sqrt{2}\Omega x} dy^2 + 2e^{\sqrt{2}\Omega x} dy\, dt - dz^2; \tag{7}$$

here Ω is the rotating angular velocity of the universe.

For the γ^μ and γ_μ we have

$$\gamma_0 = \tilde{\gamma}_0; \quad \gamma_1 = \tilde{\gamma}_1; \quad \gamma_2 = \frac{1}{\sqrt{2}} e^{\sqrt{2}\Omega x}(\sqrt{2}\tilde{\gamma}_0 + \tilde{\gamma}_2); \quad \gamma_3 = \tilde{\gamma}_3; \quad \gamma_5 = \tilde{\gamma}_5;$$

$$\gamma^0 = \tilde{\gamma}^0 - \sqrt{2}\tilde{\gamma}^2; \quad \gamma^1 = \tilde{\gamma}^1; \quad \gamma^2 = \sqrt{2}\, e^{-\sqrt{2}\Omega x}\tilde{\gamma}^2; \quad \gamma^3 = \tilde{\gamma}^3;$$

$$\gamma^5 = -\frac{i}{4} E_{\mu\nu\lambda\tau}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\tau = \tilde{\gamma}^5; \tag{8}$$

where $E_{\mu\nu\lambda\tau} = \sqrt{-g}\varepsilon_{\mu\nu\lambda\tau}$ and $\varepsilon_{0123} = 1$.

Consequently, the spinorial affine connection matrices become

$$\Gamma_0 = \frac{\Omega}{2}\tilde{\gamma}^1\tilde{\gamma}^2; \quad \Gamma_1 = \frac{\Omega}{2}\tilde{\gamma}^0\tilde{\gamma}^2; \quad \Gamma_2 = \frac{\Omega}{2\sqrt{2}} e^{\sqrt{2}\Omega x}\tilde{\gamma}^1\tilde{\gamma}^0; \quad \Gamma_3 = 0. \tag{9}$$

The set of field equations corresponding to (1) is

$$i\gamma^\mu\nabla_\mu\Psi - m\Psi + \frac{\partial F}{\partial\overline{\Psi}} = 0,$$

$$i\nabla_\mu\overline{\Psi}\gamma^\mu + m\overline{\Psi} - \frac{\partial F}{\partial\Psi} = 0. \tag{10}$$

We choose $\Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\end{pmatrix}$ and $\Psi_\alpha(x, y, z, t) = v_\alpha(x)$, where $\alpha = 1, 2, 3, 4$.

For the first equation of (10) we obtain the resultant expression

$$\tilde{\gamma}^1\partial_x v + \frac{\Omega}{2}(\tilde{\gamma}^0\tilde{\gamma}^1\tilde{\gamma}^2 + \sqrt{2}\tilde{\gamma}^1)v + i(m - F')v = 0, \tag{11}$$

where $F' = \frac{dF}{dS}$. Consequently, for the $v_\alpha(x)$ one gets the next set of equations

$$\begin{cases} v_4' + \frac{\Omega}{\sqrt{2}} v_4 - i\left(\frac{\Omega}{2} - m + F'\right) v_1 = 0,\\ v_3' + \frac{\Omega}{\sqrt{2}} v_3 + i\left(\frac{\Omega}{2} + m - F'\right) v_2 = 0,\\ v_2' + \frac{\Omega}{\sqrt{2}} v_2 - i\left(\frac{\Omega}{2} + m - F'\right) v_3 = 0,\\ v_1' + \frac{\Omega}{\sqrt{2}} v_1 + i\left(\frac{\Omega}{2} - m + F'\right) v_4 = 0. \end{cases} \tag{12}$$

From (12) we obtain the following relation for the invariant $S = \overline{\Psi}\Psi = \overset{*}{v}_1 v_1 + \overset{*}{v}_2 v_2 - \overset{*}{v}_3 v_3 - \overset{*}{v}_4 v_4$:

$$\frac{dS}{dx} + \sqrt{2}\Omega S = 0, \tag{13}$$

which has the solution

$$S = C_1 e^{-\sqrt{2}\Omega x}, \quad C_1 = \text{const}. \tag{14}$$

From (14) it follows that $F[S(x)]$ is a function of the x variable.

With the transformation $u_\alpha(x) = v_\alpha(x)e^{-\Omega x/\sqrt{2}}$, the set of equations (12) becomes

$$\begin{cases} u_4' + i(m - \frac{\Omega}{2} - \Phi)u_1 = 0,\\ u_3' + i(m + \frac{\Omega}{2} - \Phi)u_2 = 0,\\ u_2' - i(m + \frac{\Omega}{2} - \Phi)u_3 = 0,\\ u_1' - i(m - \frac{\Omega}{2} - \Phi)u_4 = 0, \end{cases} \tag{15}$$

where $\Phi(x) = F'(S)$. From (15) we obtain

$$\begin{cases} u_4' \pm i(m - \frac{\Omega}{2} - \Phi)\sqrt{a_1^2 - u_4^2} = 0,\\ u_3' \pm i(m + \frac{\Omega}{2} - \Phi)\sqrt{a_2^2 - u_3^2} = 0,\\ u_2' \mp i(m + \frac{\Omega}{2} - \Phi)\sqrt{a_2^2 - u_2^2} = 0,\\ u_1' \mp i(m - \frac{\Omega}{2} - \Phi)\sqrt{a_1^2 - u_1^2} = 0, \end{cases}$$

where a_1 and a_2 are constants. The solutions of this system are

$$\begin{cases} u_1 = \pm a_1 \cosh[\theta_1(x)],\\ u_2 = \pm a_2 \cosh[\theta_2(x)],\\ u_3 = \pm i a_2 \sinh[\theta_2(x)],\\ u_4 = \pm i a_1 \sinh[\theta_1(x)]; \end{cases} \tag{16}$$

where $\theta_1 = \left(\frac{\Omega}{2} - m\right)x + \int\Phi\, dx + b_1$ and $\theta_2 = -\left(\frac{\Omega}{2} + m\right)x + \int\Phi\, dx + b_2$; $b_1, b_2 = const.$

Thus, the general solution corresponding to equations (12) has the form

$$\begin{cases} v_1 = \pm a_1 e^{-\Omega x/\sqrt{2}} \cosh[\theta_1(x)],\\ v_2 = \pm a_2 e^{-\Omega x/\sqrt{2}} \cosh[\theta_2(x)],\\ v_3 = \pm i a_2 e^{-\Omega x/\sqrt{2}} \sinh[\theta_2(x)],\\ v_4 = \pm i a_1 e^{-\Omega x/\sqrt{2}} \sinh[\theta_1(x)], \end{cases} \tag{17}$$

We have obtained exact one–dimensional solutions to the set of field equations (12); the general solution contains four integrating constants: a_1, a_2, b_1, b_2. For the invariant $S = \overset{*}{v}_1 v_1 + \overset{*}{v}_2 v_2 - \overset{*}{v}_3 v_3 - \overset{*}{v}_4 v_4$ we have

$$S = (a_1^2 + a_2^2) e^{-\sqrt{2}\,\Omega x}, \tag{18}$$

which leads to $C_1 = a_1^2 + a_2^2$.

3. ENERGY DENSITY DISTRIBUTION

In this section we investigate the energy density distribution of the nonlinear spinor field along the x axis and determine the properties of the solutions in the Gödel universe. The energy–momentum tensor corresponding to (1) is

$$T^{\mu}_{\nu} = \frac{i}{4}(\overline{\Psi}\gamma^{\mu}\nabla_{\nu}\Psi + \overline{\Psi}\gamma_{\nu}\nabla^{\mu}\Psi - \nabla^{\mu}\overline{\Psi}\gamma_{\nu}\Psi - \nabla_{\nu}\overline{\Psi}\gamma^{\mu}\Psi) - \delta^{\mu}_{\nu}\mathcal{L}. \tag{19}$$

For the zero–component of the energy–momentum tensor we obtain

$$T^0_0 = SF'(S) - F(S). \tag{20}$$

In order to determine whether the solution of the nonlinear spinor field is a soliton-like one, it is necessary to analize the distribution of the energy density per unit of invariant volume, i. e. $\varepsilon = T^0_0\sqrt{|^3g|}$, where ^3g is the determinant of the matrix containing only the spatial components of the metric tensor. Since $\sqrt{|^3g|} = \frac{1}{\sqrt{2}} e^{\sqrt{2}\,\Omega x}$, then for ε we have the following relation

$$\varepsilon = \frac{C_1 S}{\sqrt{2}} \frac{d}{dS}\left(\frac{F}{S}\right). \tag{21}$$

The total energy E_f of the nonlinear spinor field is defined by the formula (by integrating within the unit limits of the y and z axes)

$$E_f = \int\limits_{-\infty}^{+\infty} T^0_0 \sqrt{|^3g|}\, dx = \frac{C_1}{2\Omega}\left(\frac{F}{S}\right)\Bigg|_0^{\infty}. \tag{22}$$

In order to obtain a finite value of E_f it is necessary to choose $F(S)$ in an appropriate way. For example, if

$$F(S) = \frac{\lambda}{n} \frac{(S/C_1)^{n+1}}{[1 + (S/C_1)]^n}, \tag{23}$$

where λ is the nonlinear parameter and $n \geq 1$, then E_f is

$$E_f = \frac{\lambda}{2n\Omega}. \tag{24}$$

In this way we have obtained a nonanalytic dependence of Ω; it means that when $\Omega \to 0$ (transition to Minkowski space–time) the total energy of the nonlinear spinor field tends to infinity: $E_f \to \infty$. An important peculiarity in this expression is that E_f is directly proportional to λ, so when $\lambda = 0$ (the case of the linear spinor field), the total energy is also equal to zero. On the other hand, by susbstituting (23) in (21) we obtain

$$\varepsilon = \frac{\lambda}{\sqrt{2}} \frac{e^{-n\sqrt{2}\Omega x}}{(1 + e^{-\sqrt{2}\Omega x})^{n+1}}. \tag{25}$$

From (25) follows that ε has a maximum when

$$x_{\max} = -\frac{1}{\sqrt{2}\Omega} \ln n$$

and has the following asymptotical behaviour

$$\varepsilon \sim \begin{cases} e^{\sqrt{2}\Omega x} \to 0, & x \to -\infty, \\ e^{-n\sqrt{2}\Omega x} \to 0, & x \to +\infty, \end{cases} \tag{26}$$

i. e., the energy density per unit of invariant volume is regular and localized in the vicinity of its maximum. Thus, from (24) and (26) one can deduce that the solutions of the considered spinor field with $\mathcal{L}_N$ in the form (23) possess localized energy density and finite total energy, i. e. are of soliton–like type. The qualitative dependence of ε along the x coordinate is plotted in Fig. 1.

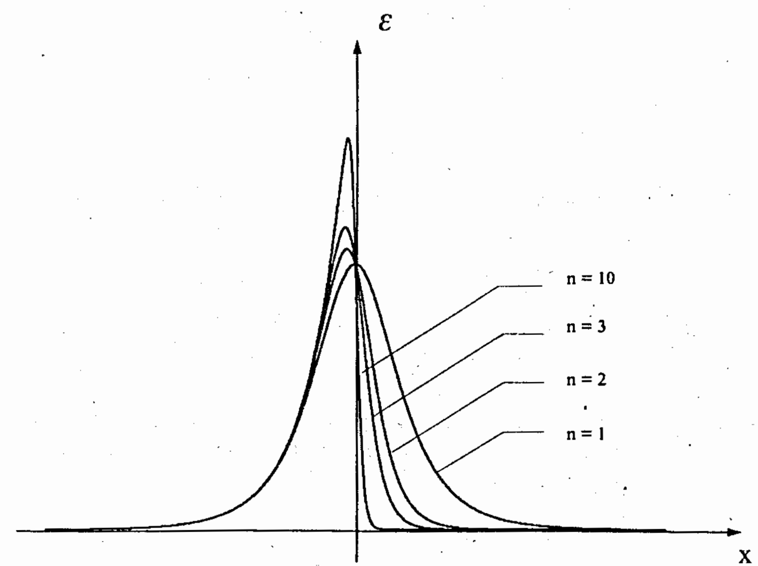

Fig.1 Qualitative distribution of the nonlinear spinor field energy density per unit of invariant volume along the x coordinate.

When $n = 1$, ε is a symmetric function with respect to $x_{\max} = 0$. In this case the rotation axis passes through the maximum of the localized distribution of the energy density (across its center) and for this reason the soliton-like configuration is not deformed. In all the other cases $n > 1$, $x_{\max} \neq 0$, so the rotation axis does not pass through the maximum and this fact leads to the deformation of the soliton-like objects. This means that when $n = 1$ we have an ordinary spinorial field configuration of soliton type [8] and for the other cases ($n > 1$), deformed soliton-like solutions due to the rotation of the universe. From (25) it also follows that when $\Omega \to 0$, $\varepsilon \to$const, i. e. there is no localization of the energy density. It means that in the flat space-time there are no nonlinear spinor field configurations with localized energy density by choosing $F(S)$ in the form (23).

4. NONLINEAR SPINOR FIELD IN FLAT SPACE–TIME

In this section we consider the nonlinear spinor field in the flat space-time as it is interesting to know whether or not, this kind of field leads to soliton-like solutions in it. In this case the corresponding set of field equations is:

$$\begin{cases} v_4' + i(m - \Phi)v_1 = 0, \\ v_3' + i(m - \Phi)v_2 = 0, \\ v_2' - i(m - \Phi)v_3 = 0, \\ v_1' - i(m - \Phi)v_4 = 0. \end{cases} \tag{27}$$

From (27) we obtain an equation for the spinorial invariant S:

$$\frac{dS}{dx} = 0,$$

then S =const. and Φ is also constant. Let us put $M = m - \Phi$=constant. Then from (27) we get the set of equations

$$v_\alpha'' - M^2 v_\alpha = 0, \qquad \alpha = 1, 2, 3, 4. \tag{28}$$

Every equation of the set (28) has the solution

$$v_\alpha = C_{\alpha 1} e^{Mx} + C_{\alpha 2} e^{-Mx},$$

where $C_{\alpha 1}$ and $C_{\alpha 2}$ are integrating constants. Then we substitute v_α in the set (27) and obtain the relation between integrating constants. Finally the solutions of the equations (27) adopt the form:

$$\begin{cases} v_1 = C_{11} e^{Mx} + C_{21} e^{-Mx} \\ v_2 = C_{12} e^{Mx} + C_{22} e^{-Mx} \\ v_3 = -i(C_{12} e^{Mx} - C_{22} e^{-Mx}) \\ v_4 = -i(C_{11} e^{Mx} - C_{21} e^{-Mx}). \end{cases}$$

In this case the energy density T_0^0 is constant in the whole Minkowski space-time (see formulae (20)–(21)), so the total energy of the nonlinear spinor field is infinite for any function $F(S)$. This means that for the equation system (27) there is no soliton-like solution since the energy density is not localized and the total field energy is infinite. This conclusion agrees with the previous result for $\Omega \to 0$. Thus, we conclude that the external gravitational field of the Gödel universe is determinative in the formation of soliton-like objects in the nonlinear spinor field.

5. NONLINEAR SPINOR FIELD WITH $\mathcal{L}_{\rm N} = G(I_P)$

Let us consider the Lagrangian (1) with the nonlinear term $\mathcal{L}_{\rm N} = G(I_P)$, where $G(I_P)$ is an arbitrary function of the spinorial invariant $I_P = P^2 = (i\overline{\Psi}\gamma^5\Psi)^2$. This Lagrangian leads to the following field equations

$$i\gamma^\mu \nabla_\mu \Psi - m\Psi + \frac{\partial G}{\partial \overline{\Psi}} = 0,$$

$$i\nabla_\mu \overline{\Psi}\gamma^\mu + m\overline{\Psi} - \frac{\partial G}{\partial \Psi} = 0. \tag{29}$$

As before we chose $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix}$ and $\Psi_\alpha(x, y, z, t) = v_\alpha(x)$, where $\alpha = 1, 2, 3, 4$; then the first equation of (29) adopts the form

$$\tilde\gamma^1 \partial_x v + \frac{\Omega}{2}(\tilde\gamma^0\tilde\gamma^1\tilde\gamma^2 + \sqrt{2}\,\tilde\gamma^1)v + imv + N(P)\tilde\gamma^5 v = 0, \tag{30}$$

where $N(P) = 2G'P$ and $G' = \frac{dG}{dI_P}$. Consequently, for the $v_\alpha(x)$ one gets the next equation system

$$\begin{cases} v_4' + \frac{\Omega}{\sqrt{2}} v_4 - i\left(\frac{\Omega}{2} - m\right) v_1 - N v_3 = 0, \\ v_3' + \frac{\Omega}{\sqrt{2}} v_3 + i\left(\frac{\Omega}{2} + m\right) v_2 - N v_4 = 0, \\ v_2' + \frac{\Omega}{\sqrt{2}} v_2 - i\left(\frac{\Omega}{2} + m\right) v_3 + N v_1 = 0, \\ v_1' + \frac{\Omega}{\sqrt{2}} v_1 + i\left(\frac{\Omega}{2} - m\right) v_4 + N v_2 = 0. \end{cases} \tag{31}$$

From (31) we obtain the equation system for the bilinear forms P, $R = \overset{*}{v}_2\, v_1 + \overset{*}{v}_1\, v_2 + \overset{*}{v}_4\, v_3 + \overset{*}{v}_3\, v_4$ and $S = \overline{\Psi}\Psi$:

$$\begin{cases} P' + \sqrt{2}\,\Omega\, P + 2\, m\, R = 0, \\ R' + \sqrt{2}\,\Omega\, R + 2\, m\, P + 2\, N\, S = 0, \\ S' + \sqrt{2}\,\Omega\, S + 2\, N\, R = 0. \end{cases} \tag{32}$$

In the Heisenberg's nonlinear unified field theory [9] the mass term does not exist since it does not have the meaning which it has in the linear theory. So, for simplicity we omit m in (1) and consequently in (31) and (32). From the new system (with $m = 0$) we obtain the following equation for P:

$$P' + \sqrt{2}\Omega P = 0,$$

hence we have

$$P = C_2 e^{-\sqrt{2}\Omega x} \tag{33}$$

where C_2 =const. This result means that if one defines $G(I_p)$, then $N(x)$ will be a concrete function of the x variable.

If $v_\alpha(x) = w_\alpha(x) e^{-\Omega x/\sqrt{2}}$, then our equation system becomes:

$$\begin{cases} w_4' - \frac{i\Omega}{2} w_1 - N w_3 = 0, \\ w_3' + \frac{i\Omega}{2} w_2 - N w_4 = 0, \\ w_2' - \frac{i\Omega}{2} w_3 + N w_1 = 0, \\ w_1' + \frac{i\Omega}{2} w_4 + N w_2 = 0. \end{cases} \tag{34}$$

By suming and substracting the corresponding equations, we obtain the next system for the new set of functions $\eta_1 = w_2 + w_1$, $\eta_2 = w_2 - w_1$, $\eta_3 = w_4 + w_3$ and $\eta_4 = w_4 - w_3$:

$$\begin{cases} \eta_3' + \frac{i\Omega}{2} \eta_2 - N \eta_3 = 0, \\ \eta_2' - \frac{i\Omega}{2} \eta_3 - N \eta_2 = 0, \\ \eta_1' + \frac{i\Omega}{2} \eta_4 + N \eta_1 = 0, \\ \eta_4' - \frac{i\Omega}{2} \eta_1 + N \eta_4 = 0. \end{cases} \tag{35}$$

From (35) we get the following pair of equations

$$\begin{cases} \eta_{2,3}'' - 2N\eta_{2,3}' - \left(\frac{\Omega^2}{4} - N^2 + N_x\right) \eta_{2,3} = 0, \\ \eta_{1,4}'' + 2N\eta_{1,4}' - \left(\frac{\Omega^2}{4} - N^2 - N_x\right) \eta_{1,4} = 0. \end{cases} \tag{36}$$

By doing the transformations $\eta_{1,4} = \mu_{1,4} e^{-\int N\, dx}$ and $\eta_{2,3} = \mu_{2,3} e^{\int N\, dx}$, then (36) takes the form

$$\mu_\alpha'' - \frac{\Omega^2}{4} \mu_\alpha = 0, \quad \alpha = 1, 2, 3, 4. \tag{37}$$

The general solution for this system is

$$\mu_\alpha = C_{\alpha 1} e^{\Omega x/2} + C_{\alpha 2} e^{-\Omega x/2}, \tag{38}$$

where $C_{\alpha 1}$ and $C_{\alpha 2}$ are constants.

By substituting these solutions in (35) we obtain the following relation between the integrating constants

$$C_{41} = iC_{11}; \quad C_{42} = -iC_{12}; \quad C_{31} = -iC_{21}; \quad C_{32} = iC_{22}. \tag{39}$$

Thus, the general solution to the equation system (31) (when $m = 0$) is

$$\begin{cases} v_1 = \frac{1}{2} e^{-\Omega x/\sqrt{2}} \left[e^{-\int N\,dx} \left(C_{11} e^{\Omega x/2} + C_{12} e^{-\Omega x/2}\right) - \right. \\ \qquad \left. - e^{\int N\,dx} \left(C_{21} e^{\Omega x/2} + C_{22} e^{-\Omega x/2}\right) \right], \\ v_2 = \frac{1}{2} e^{-\Omega x/\sqrt{2}} \left[e^{-\int N\,dx} \left(C_{11} e^{\Omega x/2} + C_{12} e^{-\Omega x/2}\right) + \right. \\ \qquad \left. + e^{\int N\,dx} \left(C_{21} e^{\Omega x/2} + C_{22} e^{-\Omega x/2}\right) \right], \\ v_3 = -\frac{i}{2} e^{-\Omega x/\sqrt{2}} \left[e^{-\int N\,dx} \left(C_{11} e^{\Omega x/2} - C_{12} e^{-\Omega x/2}\right) + \right. \\ \qquad \left. + e^{\int N\,dx} \left(C_{21} e^{\Omega x/2} - C_{22} e^{-\Omega x/2}\right) \right], \\ v_4 = \frac{i}{2} e^{-\Omega x/\sqrt{2}} \left[e^{-\int N\,dx} \left(C_{11} e^{\Omega x/2} - C_{12} e^{-\Omega x/2}\right) - \right. \\ \qquad \left. - e^{\int N\,dx} \left(C_{21} e^{\Omega x/2} - C_{22} e^{-\Omega x/2}\right) \right]. \end{cases} \tag{40}$$

Thus we have obtained exact solutions to the nonlinear spinor field equations (31) in the Gödel Universe. The general solution contains four integrating constants: $C_{11}, C_{12}, C_{21}, C_{22}$. We substitute the expressions of v_α in $P = i\overline{\Psi}\gamma^5\Psi = i(v_1 \overset{*}{v}_3 - \overset{*}{v}_1 v_3 + v_2 \overset{*}{v}_4 - \overset{*}{v}_2 v_4)$ and then

$$P = 2(C_{11}C_{22} - C_{12}C_{21}) e^{-\sqrt{2}\Omega x}. \tag{41}$$

In correspondence with (33) we have $C_2 = 2(C_{11}C_{22} - C_{12}C_{21})$.

6. ENERGY DENSITY DISTRIBUTION

In this case for the zero–component of the energy–momentum tensor we have

$$T_0^0 = 2G' I_P - G,$$

so, the energy density per unit of invariant volume adopts the form

$$\varepsilon = \sqrt{2} C_2 I_P \frac{d}{dI_P} \left(\frac{G}{\sqrt{I_P}} \right). \tag{42}$$

For E_f (by integrating within the unit limits of the y and z axes) we have

$$E_f = \int_{-\infty}^{+\infty} T_0^0 \sqrt{|^3 g|}\, dx = \frac{C_2}{2\Omega} \left(\frac{G}{\sqrt{I_P}} \right) \Bigg|_0^\infty. \tag{43}$$

If we choose $G(I_P)$ in the form

$$G(I_P) = \frac{\lambda \left(\frac{I_P}{C_2^2}\right)^n \sqrt{\frac{I_P}{C_2}}}{n\left(1+\frac{I_P}{C_2^2}\right)^n}, \quad n = 1, 2, 3 \ldots,$$

where λ is the nonlinear parameter, the total field energy takes finite values

$$E_f = \frac{\lambda}{2n\Omega}. \tag{44}$$

The properties of this expression are analogous to the obtained ones from (24). For the energy density per unit of invariant volume we get the following asymptotical behaviour when x approaches plus and minus infinity:

$$\varepsilon = \sqrt{2}\lambda \frac{e^{-2n\sqrt{2}\Omega x}}{(1+e^{-2\sqrt{2}\Omega x})^{n+1}} \to \begin{cases} 0, & x \to +\infty, \\ 0, & x \to -\infty. \end{cases} \tag{45}$$

Thus, we have obtained again soliton–like solutions since the total field energy is finite and the energy density ε is regular and localized in the vicinity of its maximum $x_{\max} = -\frac{\ln n}{\sqrt{2}\Omega}$. When $n = 1$ the soliton–like configuration is symmetric with respect to $x_{\max} = 0$, whereas $n > 1$ the soliton–like objects are deformed due to the rotation of the universe as in the $\mathcal{L}_N = F(I_S)$ case. The qualitative distribution of ε along the x axis also corresponds to the Fig. 1.

Let us consider the following nonlinear term in the Lagrangian (1):

$$\mathcal{L}_N = G = -\frac{\ln(1+\lambda I_P)^n}{\sqrt{I_P}}, \quad n = 1, 2, 3, \ldots \tag{46}$$

By substituting (46) in (43) one gets the following expression for the total energy

$$E_f = \frac{nC_2\lambda}{2\Omega}. \tag{47}$$

For ε we have

$$\varepsilon = n\sqrt{2}C_2 \left[\frac{\ln(1+\lambda I_P)}{I_P} - \frac{\lambda}{1+\lambda I_P}\right] \to \begin{cases} 0, & x \to +\infty, \\ 0, & x \to -\infty. \end{cases} \tag{48}$$

So, once more we have obtained soliton–like solutions, but in this case all the soliton–like configurations are deformed due to the rotation of the universe. By choosing G in the form

$$G = [\lambda\sqrt{I_P} - \ln(1+\lambda\sqrt{I_P})], \tag{49}$$

we obtain the next expression for E_f

$$E_f = \frac{C_2\lambda}{2\Omega}, \tag{50}$$

whereas the expression for the energy density is the following

$$\varepsilon = \frac{C_2}{\sqrt{2}} \left[\frac{\ln(1+\lambda\sqrt{I_P})}{\sqrt{I_P}} - \frac{\lambda}{1+\lambda\sqrt{I_P}}\right] \to \begin{cases} 0, & x \to +\infty, \\ 0, & x \to -\infty, \end{cases} \tag{51}$$

its asymptotical behaviour is analogous to the (48) one; so we get equivalent results: soliton–like objects since the localization of the energy density and the finite character of the total energy of the nonlinear spinor field are achieved. These configurations are deformed due to the rotation of the Gödel universe.

7. THE FLAT SPACE–TIME CASE

The corresponding field equation system (when $m = 0$) in the flat space–time takes the form

$$\begin{cases} v_4' - Nv_3 = 0, \\ v_3' - Nv_4 = 0, \\ v_2' + Nv_1 = 0, \\ v_1' + Nv_2 = 0. \end{cases} \tag{52}$$

From (52) we obtain

$$\frac{dP}{dx} = 0,$$

which leads to

$$P = \text{const}.$$

Hence N is also a constant: $N = k$. From (52) we also have the following set of equations

$$v_\alpha'' - k^2 v_\alpha = 0, \quad \alpha = 1, 2, 3, 4; \tag{53}$$

which has the solution

$$v_\alpha = C_{\alpha 1}e^{kx} + C_{\alpha 2}e^{-kx}, \tag{54}$$

where $C_{\alpha 1}$ and $C_{\alpha 2}$ are integrating constants. The the following relation exist between them: $C_{11} = iC_{14}$, $C_{21} = iC_{24}$, $C_{12} = iC_{13}$ and $C_{22} = -iC_{23}$.

At the same time the energy density per unit of invariant volume ε in flat space–time is constant everywhere, so the total energy of the nonlinear spinor field tends to infinity for any function $G(I_p)$. From this result it follows that in the Minkowski space–time there are no soliton–like solutions. This agrees with the results which has been obtained in Sec. 4. Thus, we have arrived to an equivalent conclusion about the role of the gravitational field over the nonlinear spinor fields: it localizes their energy density; so, it is determinative in the formation of soliton–like configurations in the nonlinear spinor fields.

8. CONCLUSION AND DISCUSSION

As result of our investigation we have obtained exact static one–dimensional solutions to the nonlinear spinor field equations in the external gravitational field of the Gödel universe. Moreover, it is shown that by chossing $F(I_S)$ and $G(I_P)$ in an adecuate way the nonlinear spinor field (1) has solutions with localized energy density and finite total energy (by integrating within the finite limits of the y and z axes), i. e. soliton–like solutions. These solutions have no flat–space analogues because there the energy density is not localized and the total energy is infinite for any functions F and G. Thus we can conclude that the role of the external gravitational field is determinative in the formation of soliton–like configurations in the nonlinear spinor field. By the way, we have observed that, due to the form of the formulae for the energy density and total field energy (see (43), (45), (48), (51)), for the Lagrangian depending on the spinorial invariant $I_P = P^2$ ($\mathcal{L}_N$ must be a real Lorentz scalar) the total field energy is infinite or the energy density is not localized, whereas the dependences which conduce to finite total field energy and localized energy density contain an odd power of P, so the P–invariance is violated (P is a pseudoscalar). The manifestation of this fact on the classical level reminds of the quantum nature of the spinor fields, because we observe this anomaly in their weak interactions. The gravitational interaction is even weaker and this fact can lead us to deep meditations about the nature of the spinor interactions.

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