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EXACT SOLUTIONS TO THE INTERACTING SPINOR AND SCALAR FIELD EQUATIONS IN THE GÖDEL UNIVERSE

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Точные решения уравнений взаимодействующих спинорного и скалярного полей во Вселенной Гёделя

Получены точные решения уравнений взаимодействующих спинорного и скалярного полей во внешнем гравитационном поле Вселенной Гёделя. Рассмотрены два типа лагранжианов взаимодействия: $\mathcal{L}_{\text{int}} = \frac{1}{2}\varphi_{,\beta}\varphi^{,\beta}F(I_S)$ и $\mathcal{L}_{\text{int}} = \frac{1}{2}\varphi_{,\beta}\varphi^{,\beta}G(I_P)$, где $F(I_S)$ и $G(I_P)$ являются произвольными функциями от спинорных инвариантов $I_S = S = \overline{\Psi}\Psi$ и $I_P = P^2 = (i\overline{\Psi}\gamma^5\Psi)^2$. Установлены условия, при которых имеются одномерные солитоноподобные решения, а также роль гравитационного поля в формировании этих конфигураций.

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Exact Solutions to the Interacting Spinor and Scalar Field Equations in the Gödel Universe

The interacting spinor and scalar field system with two kinds of interaction Lagrangians: $\mathcal{L}_{\text{int}} = \frac{1}{2}\varphi_{,\beta}\varphi^{,\beta}F(I_S)$ and $\mathcal{L}_{\text{int}} = \frac{1}{2}\varphi_{,\beta}\varphi^{,\beta}G(I_P)$ in the external gravitational field of the Gödel universe is considered. Exact solutions to the corresponding field equations are obtained. The conditions under which one-dimensional soliton-like solutions exist are established and the role of gravity in the formation of these objects is determined.

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where e^a_ν form a set of tetradic 4–vectors which is given by the expression [5]

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu \tag{3}$$

and $\tilde{\gamma}^a$ are the flat space–time Dirac matrices; we have chosen them according to Ref. [6]. ∇_μ are the covariant spinor derivatives; they are defined as follows [7]:

$$\nabla_\mu = \partial_\mu - \Gamma_\mu(x), \tag{4}$$

where $\Gamma_\mu(x)$ are the spinorial affine connection matrices of curve space–time, determined by the relation

$$\Gamma_\mu(x) = \frac{1}{4} g_{\rho\delta}(\partial_\mu e^b_\sigma \cdot e^\rho_b - \Gamma^\rho_{\mu\sigma})\gamma^\delta\gamma^\sigma; \tag{5}$$

$\Gamma^\rho_{\mu\sigma}$ being the Christoffel symbols.

The metric of Gödel universe is represented in the following form [8]

$$ds^2 = dt^2 - dx^2 + \frac{1}{2} e^{2\sqrt{2}\Omega x} dy^2 + 2e^{\sqrt{2}\Omega x} dy\, dt - dz^2; \tag{6}$$

here Ω is the rotating angular velocity of the universe.

For the γ^μ and γ_μ we have

$$\gamma_0 = \tilde{\gamma}_0; \quad \gamma_1 = \tilde{\gamma}_1; \quad \gamma_2 = \frac{1}{\sqrt{2}} e^{\sqrt{2}\Omega x}(\sqrt{2}\tilde{\gamma}_0 + \tilde{\gamma}_2); \quad \gamma_3 = \tilde{\gamma}_3; \quad \gamma_5 = \tilde{\gamma}_5;$$

$$\gamma^0 = \tilde{\gamma}^0 - \sqrt{2}\tilde{\gamma}^2; \quad \gamma^1 = \tilde{\gamma}^1; \quad \gamma^2 = \sqrt{2}\, e^{-\sqrt{2}\Omega x}\tilde{\gamma}^2; \quad \gamma^3 = \tilde{\gamma}^3;$$
$$\gamma^5 = -\frac{i}{4} E_{\mu\nu\lambda\tau}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\tau = \tilde{\gamma}^5; \tag{7}$$

where $E_{\mu\nu\lambda\tau} = \sqrt{-g}\varepsilon_{\mu\nu\lambda\tau}$ and $\varepsilon_{0123} = 1$.

Consequently, the spinorial affine connection matrices become

$$\Gamma_0 = \frac{\Omega}{2}\tilde{\gamma}^1\tilde{\gamma}^2; \quad \Gamma_1 = \frac{\Omega}{2}\tilde{\gamma}^0\tilde{\gamma}^2; \quad \Gamma_2 = \frac{\Omega}{2\sqrt{2}} e^{\sqrt{2}\Omega x}\tilde{\gamma}^1\tilde{\gamma}^0; \quad \Gamma_3 = 0. \tag{8}$$

From (1) we obtain the following set of field equations:

$$i\gamma^\mu \nabla_\mu \Psi - m\Psi + \frac{1}{2}\varphi_{,\beta}\varphi^{,\beta}\frac{\partial R}{\partial \overline{\Psi}} = 0,$$

$$i\nabla_\mu \overline{\Psi}\gamma^\mu + m\overline{\Psi} + \frac{1}{2}\varphi_{,\beta}\varphi^{,\beta} - \frac{\partial R}{\partial \Psi} = 0, \tag{9}$$

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}(\sqrt{-g}\, g^{\nu\mu}\varphi_{,\mu}R(S)) = 0,$$

where $R(S) = 1 + F(I_S)$ and $g = \det|g_{\mu\nu}|$. We consider Ψ and φ depending only on x. Then, from the last equation we have

$$\varphi' = \frac{C}{R} e^{-\sqrt{2}\Omega x}; \quad \varphi' = \frac{\partial\varphi}{\partial x}, \quad C = \text{const}. \tag{10}$$

On the other hand, the first equation of (9) takes the form

$$\tilde{\gamma}^1 \partial_x \Psi + \frac{\Omega}{2}(\tilde{\gamma}^0\tilde{\gamma}^1\tilde{\gamma}^2 + \sqrt{2}\tilde{\gamma}^1)\Psi + i(m - \Phi(x))\Psi = 0, \tag{11}$$

where $\Phi(x) = \frac{1}{2}C^2 e^{-2\sqrt{2}\Omega x} \cdot dQ/dS$ and $Q(S) = 1/R$.

Assuming that $\Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\end{pmatrix}$ and $\Psi_\alpha(x, y, z, t) = v_\alpha(x)$, $\alpha = 1, 2, 3, 4$, we get

$$\begin{aligned}
v_4' + \tfrac{\Omega}{\sqrt{2}} v_4 + i(m - \tfrac{\Omega}{2} - \Phi)v_1 = 0,\\
v_3' + \tfrac{\Omega}{\sqrt{2}} v_3 + i(m + \tfrac{\Omega}{2} - \Phi)v_2 = 0,\\
v_2' + \tfrac{\Omega}{\sqrt{2}} v_2 - i(m + \tfrac{\Omega}{2} - \Phi)v_3 = 0,\\
v_1' + \tfrac{\Omega}{\sqrt{2}} v_1 - i(m - \tfrac{\Omega}{2} - \Phi)v_4 = 0.
\end{aligned} \tag{12}$$

From (12) we obtain the following equation for $S = \overline{\Psi}\Psi$:

$$\frac{dS}{dx} + \sqrt{2}\Omega S = 0, \tag{13}$$

which has the solution

$$S = C_1 e^{-\sqrt{2}\Omega x}, \quad C_1 = \text{const}. \tag{14}$$

With the transformation $u_\alpha(x) = v_\alpha(x)e^{-\Omega x/\sqrt{2}}$, the set of equations (12) becomes

$$\begin{aligned}
u_4' + i(m - \tfrac{\Omega}{2} - \Phi)u_1 = 0,\\
u_3' + i(m + \tfrac{\Omega}{2} - \Phi)u_2 = 0,\\
u_2' - i(m + \tfrac{\Omega}{2} - \Phi)u_3 = 0,\\
u_1' - i(m - \tfrac{\Omega}{2} - \Phi)u_4 = 0.
\end{aligned} \tag{15}$$

The solutions which satisfy this system are

$$\begin{aligned}
u_1 = \pm a_1\, \mathrm{ch}[\theta_1(x)],\\
u_2 = \pm a_2\, \mathrm{ch}[\theta_2(x)],\\
u_3 = \pm i a_2\, \mathrm{sh}[\theta_2(x)],\\
u_4 = \pm i a_1\, \mathrm{sh}[\theta_1(x)];
\end{aligned} \tag{16}$$

where $\theta_1 = (\frac{\Omega}{2} - m)\, x + \int \Phi\, dx + b_1$ and $\theta_2 = -\left(\frac{\Omega}{2} + m\right) x + \int \Phi\, dx + b_2$.

The general solution to (12) has the form

$$\begin{aligned} v_1 &= \pm a_1 e^{-\Omega x/\sqrt{2}} \operatorname{ch}[\theta_1(x)], \\ v_2 &= \pm a_2 e^{-\Omega x/\sqrt{2}} \operatorname{ch}[\theta_2(x)], \\ v_3 &= \pm i a_2 e^{-\Omega x/\sqrt{2}} \operatorname{sh}[\theta_2(x)], \\ v_4 &= \pm i a_1 e^{-\Omega x/\sqrt{2}} \operatorname{sh}[\theta_1(x)], \end{aligned} \tag{17}$$

Thus we have obtained exact solutions to the set of field equations (12); the general solution contains four integration constants: a_1, a_2, b_1, b_2. For the invariant $S = \overset{*}{v}_1 v_1 + \overset{*}{v}_2 v_2 - \overset{*}{v}_3 v_3 - \overset{*}{v}_4 v_4$ we have

$$S = (a_1^2 + a_2^2) e^{-\sqrt{2}\Omega x}. \tag{18}$$

3. ENERGY DENSITY DISTRIBUTION

In this section we study the energy density distribution of our system along the x axis and determine the properties of the solutions in the Gödel universe. The energy–momentum tensor (EMT) which corresponds to (1) is

$$T_\nu^\mu = \frac{i}{4}(\overline{\Psi}\gamma^\mu \nabla_\nu \Psi + \overline{\Psi}\gamma_\nu \nabla^\mu \Psi - \nabla^\mu \overline{\Psi}\gamma_\nu \Psi - \nabla_\nu \overline{\Psi}\gamma^\mu \Psi) + \varphi_{,\nu}\varphi^{,\mu} R(S) - \delta_\nu^\mu \mathcal{L}. \tag{19}$$

By using the first two equations from (9), the Lagrangian can be written as follows

$$\mathcal{L} = \frac{1}{2}\varphi_{,\beta}\varphi^{,\beta}\left(-\frac{dR}{dS} S + R\right). \tag{20}$$

Substituting (20) to the zero–component of the EMT, one obtains

$$T_0^0 = \frac{1}{2}\frac{C^2}{(a_1^2 + a_2^2)^2} S^2 \frac{d}{dS}(QS). \tag{21}$$

In order to determine whether the solution is a soliton one, it is necessary to analize the distribution of the energy density per unit invariant volume, i. e., $\varepsilon = T_0^0\sqrt{|{}^3g|}$, where ${}^3g$ is the determinant of the matrix containing only the spatial components of the metric tensor. Since $\sqrt{|{}^3g|} = e^{\sqrt{2}\Omega x}/\sqrt{2} = (a_1^2 + a_2^2)/\sqrt{2}\, S$, for ε we have:

$$\varepsilon = \frac{C^2 S}{2\sqrt{2}(a_1^2 + a_2^2)^2}\frac{d}{dS}(QS). \tag{22}$$

The total energy of the field system is defined by the formula

$$E_f = \int_{-\infty}^{+\infty} T_0^0 \sqrt{|{}^3g|}\, dV. \tag{23}$$

To obtain a finite value of E_f it is necessary to choose $F(S)$ in an appropriate way. For example, if

$$F(S) = \frac{(\lambda S)^{n+1}}{1 + (\lambda S)^n}, \quad n = 1, 2, 3, \ldots \tag{24}$$

where λ is a coupling constant, for E_f (by integrating within the finite limits of the y and z axes) one gets the following expression:

$$E_f = \frac{C^2}{4\Omega(a_1^2 + a_2^2)}(QS)\Big|_0^\infty = \frac{C^2}{4\lambda\Omega(a_1^2 + a_2^2)}. \tag{25}$$

From (25) it follows that when $\lambda = 0$ (absence of interaction), $E_f \to \infty$. The same result we obtain when $\Omega \to 0$ (transition to Minkowski space-time). ε has the following asymptotical behaviour

$$\varepsilon \sim \begin{cases} 1/S \to 0, & x \to -\infty, \\ S \to 0, & x \to +\infty, \end{cases} \tag{26}$$

i. e., the energy density per unit invariant volume is localized. Thus, from (25) and (26) one can deduce that the solutions of the field system under consideration possess a localized energy density and finite total energy, i. e., are soliton-like. The qualitative dependence of ε on the x coordinate is plotted in Fig. 1.

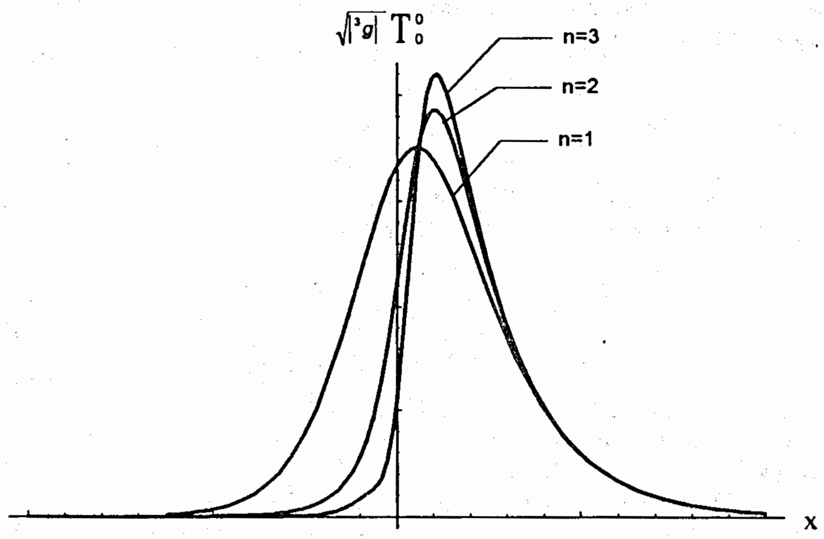

Fig.1 Qualitative dependence of the energy density per unit of invariant volume on the x coordinate: a deformed soliton due to the rotation of the universe.

Thus we have obtain an asymmetric (deformed) soliton due to the rotation of the universe.

4. INTERACTING SPINOR AND SCALAR FIELDS IN FLAT SPACE–TIME

In this section we consider the interacting spinor and scalar fields in flat space–time in order to understand the role of the external gravitational field of the Gödel universe. It is of interest to know, whether or not this interaction in Minkowski space–time leads to soliton-like solutions. With the transition to flat space–time, when $\Omega \to 0$, all Γ matrices vanish. At the same time, ∇_μ reduce to ∂_μ and γ^μ to $\tilde{\gamma}^\mu$. The corresponding set of field equations is

$$\begin{aligned} v_4' + i(m - \tilde{\Phi})v_1 = 0, \\ v_3' + i(m - \tilde{\Phi})v_2 = 0, \\ v_2' - i(m - \tilde{\Phi})v_3 = 0, \\ v_1' - i(m - \tilde{\Phi})v_4 = 0 \end{aligned} \qquad (27)$$

where $\tilde{\Phi} = (C^2/2)\, dQ/dS$. From (27) we get an equation for the spinorial invariant S:

$$S' = 0,$$

then S =const. and $\tilde{\Phi}$ is also constant. Let us put $M = m - \tilde{\Phi}$. Then from (27) we obtain the set of equations

$$v_\alpha'' - M^2 v_\alpha = 0, \qquad \alpha = 1,\, 2,\, 3,\, 4. \qquad (28)$$

with the solution

$$v_\alpha = C_{\alpha 1} e^{Mx} + C_{\alpha 2} e^{-Mx},$$

where $C_{\alpha 1}$ and $C_{\alpha 2}$ are integration constants.

Thus, the solutions of the equation system (27) adopt the form

$$\begin{aligned} v_1 &= C_{11} e^{Mx} + C_{21} e^{-Mx}, \\ v_2 &= C_{12} e^{Mx} + C_{22} e^{-Mx}, \\ v_3 &= -i(C_{12} e^{Mx} - C_{22} e^{-Mx}), \\ v_4 &= -i(C_{11} e^{Mx} - C_{21} e^{-Mx}), \end{aligned} \qquad (29)$$

where C_{11}, C_{12}, C_{21}, C_{22} are constants.

In this case the energy density is constant in the whole Minkowski space–time (see (21)–(22)), so the total energy of the field system is infinite for any function $F(S)$. This means that for the interacting spinor and scalar field system there is no soliton–like solution, since the energy density is not localized and the total field energy is infinite. This conclusion agrees with the previous result for $\Omega = 0$. We conclude that the external gravitational field of the Gödel universe is determinative in the formation of soliton–like objects in the interacting spinor and scalar field system.

5. INTERACTION $\mathcal{L}_{\rm int} = \frac{1}{2}\varphi_{,\beta}\varphi^{,\beta} G(I_P)$ IN THE GÖDEL UNIVERSE

Now we shall consider the Lagrangian

$$\mathcal{L} = \frac{i}{2}(\bar{\Psi}\gamma^\mu \nabla_\mu \Psi - \nabla_\mu \bar{\Psi}\gamma^\mu \Psi) - m\bar{\Psi}\Psi + \frac{1}{2}\varphi_{,\beta}\varphi^{,\beta} + \frac{1}{2}\varphi_{,\beta}\varphi^{,\beta} G(I_P), \qquad (30)$$

where $G(I_P)$ is an arbitrary function of the spinorial invariant $I_P = P^2 = (i\bar{\Psi}\gamma^5\Psi)^2$. This Lagrangian leads to the following field equations:

$$\begin{aligned} i\gamma^\mu \nabla_\mu \Psi - m\Psi + \frac{1}{2}\varphi_{,\beta}\varphi^{,\beta}\frac{\partial J}{\partial \bar{\Psi}} = 0, \\ i\nabla_\mu \bar{\Psi}\gamma^\mu + m\bar{\Psi} - \frac{1}{2}\varphi_{,\beta}\varphi^{,\beta}\frac{\partial J}{\partial \Psi} = 0, \\ \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}(\sqrt{-g}\, g^{\nu\mu}\varphi_{,\mu} J(I_P)) = 0; \end{aligned} \qquad (31)$$

where $J(I_P) = 1 + G$.

We consider φ and Ψ depending only on x; thus, for the scalar field we have

$$\varphi' = \frac{C}{J} e^{-\sqrt{2}\Omega x}; \quad \varphi' = \frac{\partial \varphi}{\partial x}, \quad C = \text{const}. \qquad (32)$$

At the same time the first equation of (31) takes the resultant form

$$\tilde{\gamma}^1 \partial_x \Psi + \frac{\Omega}{2}(\tilde{\gamma}^0\tilde{\gamma}^1\tilde{\gamma}^2 + \sqrt{2}\,\tilde{\gamma}^1)\Psi + im\Psi + N\tilde{\gamma}^5\Psi = 0 \qquad (33)$$

where $N(x) = C^2 e^{-2\sqrt{2}\Omega x} W' P$, $W' = dW/dI_P$ and $W(I_P) = 1/J$. For $\Psi_\alpha(x,\, y,\, z,\, t) = v_\alpha(x)$, $\alpha = 1,\, 2,\, 3,\, 4$ we get

$$\begin{aligned} v_4' + \tfrac{\Omega}{\sqrt{2}} v_4 - i\left(\tfrac{\Omega}{2} - m\right) v_1 - N v_3 = 0, \\ v_3' + \tfrac{\Omega}{\sqrt{2}} v_3 + i\left(\tfrac{\Omega}{2} + m\right) v_2 - N v_4 = 0, \\ v_2' + \tfrac{\Omega}{\sqrt{2}} v_2 - i\left(\tfrac{\Omega}{2} + m\right) v_3 + N v_1 = 0, \\ v_1' + \tfrac{\Omega}{\sqrt{2}} v_1 + i\left(\tfrac{\Omega}{2} - m\right) v_4 + N v_2 = 0. \end{aligned} \qquad (34)$$

From (34) we obtain

$$\begin{aligned} P' + \sqrt{2}\Omega\, P + 2\, m\, H = 0, \\ H' + \sqrt{2}\Omega\, H + 2\, m\, P + 2\, N\, S = 0, \\ S' + \sqrt{2}\Omega\, S + 2\, N\, H = 0, \end{aligned} \qquad (35)$$

where $S = \overset{*}{v}_1 v_1 + \overset{*}{v}_2 v_2 - \overset{*}{v}_3 v_3 - \overset{*}{v}_4 v_4$ and $H = \overset{*}{v}_2 v_1 + \overset{*}{v}_1 v_2 + \overset{*}{v}_4 v_3 + \overset{*}{v}_3 v_4$.

In the Heisenberg's nonlinear unified theory [9] the mass term does not exist since it does not have the meaning which it has in the linear theory where it defines the total energy of the field system. So, for simplicity we shall omit it in (30) and consequently in (34) and (35). From the new system (with $m = 0$) we obtain the following equation for P:

$$P' + \sqrt{2}\,\Omega P = 0,$$

hence we have

$$P = C_2 e^{-\sqrt{2}\Omega x} \tag{36}$$

where C_2 =const. This result means that if one specifies $G(I_P)$, then $N(x)$ will be a concrete function of x.

If $v_\alpha(x) = w_\alpha(x)e^{-\Omega x/\sqrt{2}}$, then our set of equations becomes

$$\begin{aligned} w_4' - (i\Omega/2)\,w_1 - N w_3 &= 0,\\ w_3' + (i\Omega/2)\,w_2 - N w_4 &= 0,\\ w_2' - (i\Omega/2)\,w_3 + N w_1 &= 0,\\ w_1' + (i\Omega/2)\,w_4 + N w_2 &= 0. \end{aligned} \tag{37}$$

By suming and substracting the corresponding equations, one obtains the following equations for the new set of variables $\eta_1 = w_2 + w_1$, $\eta_2 = w_2 - w_1$, $\eta_3 = w_4 + w_3$, $\eta_4 = w_4 - w_3$:

$$\begin{aligned} \eta_3' + \tfrac{i\Omega}{2}\,\eta_2 - N\eta_3 &= 0,\\ \eta_2' - \tfrac{i\Omega}{2}\,\eta_3 - N\eta_2 &= 0,\\ \eta_1' + \tfrac{i\Omega}{2}\,\eta_4 + N\eta_1 &= 0,\\ \eta_4' - \tfrac{i\Omega}{2}\,\eta_1 + N\eta_4 &= 0. \end{aligned} \tag{38}$$

From (38) we get the following pair of equations

$$\begin{aligned} \eta_{2,3}'' - 2N\eta_{2,3}' - \left(\tfrac{1}{4}\Omega^2 - N^2 + N_x\right)\eta_{2,3} &= 0,\\ \eta_{1,4}'' + 2N\eta_{1,4}' - \left(\tfrac{1}{4}\Omega^2 - N^2 - N_x\right)\eta_{1,4} &= 0. \end{aligned} \tag{39}$$

After the transformations $\eta_{1,4} = \mu_{1,4}e^{-\int N\,dx}$ and $\eta_{2,3} = \mu_{2,3}e^{\int N\,dx}$, the equation system (39) takes the form

$$\mu_\alpha'' - \frac{\Omega^2}{4}\mu_\alpha = 0, \qquad \alpha = 1, 2, 3, 4. \tag{40}$$

The general solution to this set of equations is

$$\mu_\alpha = C_{\alpha 1}e^{\Omega x/2} + C_{\alpha 2}e^{-\Omega x/2}, \tag{41}$$

where $C_{\alpha 1}$ and $C_{\alpha 2}$ are constants.

Substituting these solutions to (38), we obtain the following relations between the integration constants

$$C_{41} = iC_{11}, \quad C_{42} = -iC_{12}, \quad C_{31} = -iC_{21}, \qquad C_{32} = iC_{22}. \tag{42}$$

The general solution to the set of equations (34) (when $m = 0$) is

$$\begin{aligned} v_1 &= \tfrac{1}{2}\,e^{-\frac{\Omega x}{\sqrt{2}}}\Big[e^{-\int N\,dx}\left(C_{11}e^{\Omega x/2} + C_{12}e^{-\Omega x/2}\right)\\ &\quad - e^{\int N\,dx}\left(C_{21}e^{\Omega x/2} + C_{22}e^{-\Omega x/2}\right)\Big],\\ v_2 &= \tfrac{1}{2}\,e^{-\frac{\Omega x}{\sqrt{2}}}\Big[e^{-\int N\,dx}\left(C_{11}e^{\Omega x/2} + C_{12}e^{-\Omega x/2}\right)\\ &\quad + e^{\int N\,dx}\left(C_{21}e^{\Omega x/2} + C_{22}e^{-\Omega x/2}\right)\Big],\\ v_3 &= -\tfrac{i}{2}\,e^{-\frac{\Omega x}{\sqrt{2}}}\Big[e^{-\int N\,dx}\left(C_{11}e^{\Omega x/2} - C_{12}e^{-\Omega x/2}\right)\\ &\quad + e^{\int N\,dx}\left(C_{21}e^{\Omega x/2} - C_{22}e^{-\Omega x/2}\right)\Big],\\ v_4 &= \tfrac{i}{2}\,e^{-\frac{\Omega x}{\sqrt{2}}}\Big[e^{-\int N\,dx}\left(C_{11}e^{\Omega x/2} - C_{12}e^{-\Omega x/2}\right)\\ &\quad - e^{\int N\,dx}\left(C_{21}e^{\Omega x/2} - C_{22}e^{-\Omega x/2}\right)\Big]. \end{aligned} \tag{43}$$

Thus we have obtained exact solutions to the interacting spinor and scalar field equations (34) in the Gödel Universe. The solution contains four integration constants: C_{11}, C_{12}, C_{21}, C_{22}. Substituting the expressions of v_α in $P = i\bar{\Psi}\gamma^5\Psi = i(v_1\,\overset{*}{v}_3 - \overset{*}{v}_1\,v_3 + v_2\,\overset{*}{v}_4 - \overset{*}{v}_2\,v_4)$, one gets

$$P = 2(C_{11}C_{22} - C_{12}C_{21})e^{-\sqrt{2}\Omega x}. \tag{44}$$

6. ENERGY DENSITY DISTRIBUTION

By the field equations (31), we obtain the following expression for the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\varphi_{,\beta}\varphi^{,\beta}\left(-\frac{dJ}{dI_P}P + J\right). \tag{45}$$

and hence

$$T_0^0 = \frac{C^2}{2}e^{-2\sqrt{2}\Omega x}(2W'P^2 + W). \tag{46}$$

For the energy density per unit invariant volume we obtain

$$\varepsilon = T_0^0\sqrt{|^3g|} = \frac{C^2P^2}{\sqrt{2}C_2}(WP)', \tag{47}$$

where $(WP)' = \frac{d}{dI_P}(WP)$.

The total field energy takes finite values if we choose $G(I_P)$ in the form

$$G(I_P) = \frac{(\lambda P^2)^n\lambda P}{1 + (\lambda P^2)^n}, \quad n = 1, 2, 3, \ldots \tag{48}$$

where λ is a coupling constant; integrating within finite limits along the y and z axes, we have

$$E_f = \frac{C^2}{4\Omega C_2}(WP)\Big|_0^\infty = \frac{C^2}{4\Omega\lambda C_2}; \quad (49)$$

the properties of this expression are similar to those obtained from (25). The energy density per unit invariant volume has the following asymptotic behaviour:

$$\varepsilon \to \begin{cases} 0, & x \to +\infty, \\ 0, & x \to -\infty. \end{cases} \quad (50)$$

We have again obtained soliton-like solutions since the total energy is finite and the energy density is localized (its qualitative distribution along the x axis also corresponds to Fig. 1).

Quite similar results, including formula (49) for the total field energy and (50) for the asymptotics, are obtained with some other choices of G,such as, e. g.,

$$G = -[\ln(1+\lambda P) - \lambda P] \quad (51)$$

and

$$G = \lambda P. \quad (52)$$

7. THE FLAT SPACE–TIME CASE

The field equations in the Minkowski space-time corresponding to (34) (when $m = 0$) is:

$$\begin{aligned} v_4' - \widetilde{N} v_3 &= 0, \\ v_3' - \widetilde{N} v_4 &= 0, \\ v_2' + \widetilde{N} v_1 &= 0, \\ v_1' + \widetilde{N} v_2 &= 0, \end{aligned} \quad (53)$$

where $\widetilde{N} = C^2 W' P$. From (53) we obtain

$$P' = 0, \quad \text{then} \quad P = const. \quad (54)$$

Hence N is also a constant: $N = k$. From (53) we have

$$v_\alpha'' - k^2 v_\alpha = 0, \quad \alpha = 1, 2, 3, 4; \quad (55)$$

which have the solution

$$v_\alpha = C_{\alpha 1} e^{kx} + C_{\alpha 2} e^{-kx}, \quad (56)$$

where $C_{\alpha 1}$ and $C_{\alpha 2}$ are integration constants. Between them exist the following relations: $C_{11} = iC_{14}$, $C_{21} = iC_{24}$, $C_{12} = iC_{13}$ and $C_{22} = -iC_{23}$.

At the same time the energy density per unit invariant volume ε in flat space–time is constant, so the total energy of the field system is infinite for any function $G(I_P)$. It follows that in Minkowski space–time there are no soliton–like solutions.

Thus we arrive at an equivalent conclusion on the role of the gravitational field in the interaction of elementary particles: it localizes the energy density of the field system; so, it is determinative in the formation of soliton–like configurations for the interaction of spinor and scalar fields.

8. CONCLUSION AND DISCUSSION

As result of our investigation, we have obtained exact static one–dimensional solutions to the interacting spinor and scalar field equations in the external gravitational field of the Gödel universe. Moreover, it is shown that by choosing $F(I_S)$ and $G(I_P)$ in an adecuate way the field systems (1) and (30) have solutions with localized energy density and finite field total energy (within finite limits along the y and z axes), i. e., soliton–like solutions. These solutions have no flat–space analogues because there the energy density is not localized and the total energy is infinite for any functions F and G. Thus we can conclude that the role of the external gravitational field is determinative in the formation of soliton–like configurations for the interacting spinor and scalar fields. By the way, we have observed that,due to the form of the formulae for the energy density and total energy (see (47) and (49)), for the Lagrangian depending on the spinorial invariant $I_P = P^2$ ($\mathcal{L}_{\rm int}$ must be a real Lorentz scalar), the total field energy is infinite or the energy density is not localized, whereas the dependences which conduce to finite total field energy and localized energy density contain an odd power of P, so the P–invariance is violated (P is a pseudoscalar). The manifestation of this fact on the classical level reminds of the quantum nature of spinor fields, because we observe this anomaly in their weak interactions. The gravitational interaction is even weaker and this fact can lead us to deep meditations about the nature of spinor interactions.

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