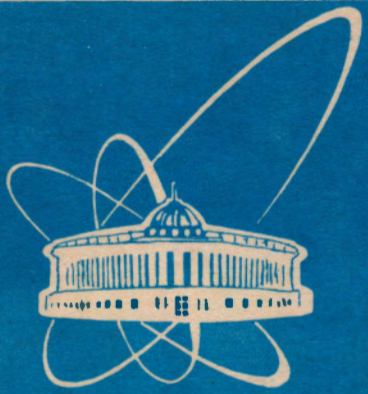

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INTEGRAL TRANSFORM TECHNIQUE FOR MESON WAVE FUNCTIONS

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1 Introduction

An important problem in the theory of strong interactions is to calculate, from the first principles of QCD, hadronic wave functions $\varphi_\pi(x)$, $\varphi_N(x_1, x_2, x_3)$, ... [3] that accumulate necessary information about the non-perturbative long-distance dynamics of the theory. These phenomenological functions naturally appear as a result of applying "factorization theorems" to hard exclusive processes [3], [4], [5]. It seems now that only the QCD Sum Rule (SR) approach [6] and lattice calculations [7] can provide information about the form of hadronic wave functions (WFs).

The most popular set of hadronic wave functions, due to V. L. Chernyak and A. R. Zhitnitsky (CZ) [8], was produced with the help of QCD SR for the first moments $\langle \xi^N \rangle_h$ ($N = 2, 4$) of mesons WFs. But the method they used has three main drawbacks:

1. reducing the initially non-local objects to a few local ones: $\langle \bar{q}(0)q(0) \rangle$, $\langle G(0)G(0) \rangle, \ldots$, – leads to the fast growth with N of the power corrections in the operator product expansion (see discussion in [12]);

2. the reconstruction of the whole function $\varphi_\pi(x)$ from just two non-trivial moments $\langle \xi^2 \rangle_\pi$ and $\langle \xi^4 \rangle_\pi$ is an unreliable procedure;

3. the phenomenological model of the spectral density, based on local quark-hadron duality, is too crude for QCD SRs for the WF.

Now it is known that the hadronic WFs are rather sensitive to the structure of the non-perturbative vacuum [9]. Therefore one should use a non-local condensate like $\langle q(0)E(0,z)q(z) \rangle$ and/or $\langle G(0)\tilde{E}(0,z)G(z) \rangle$ [10] which can reflect the complicated structure of the QCD vacuum. [3]

The modified QCD SR with non-local condensates have been constructed in [9] and it has been demonstrated that the introduction of the correlation length for condensate distributions produces much smaller values for the first moments of the pion WF compared to the CZ values (see also [11], [12]). This leads to a form of the pion WF strongly different from the CZ two-hump form and a little wider than the asymptotic form $\varphi_\pi^{as}(x) = 6x(1-x)$.

In our previous paper [1] we have obtained **directly the forms** of the pion and its first resonance WFs, using the available smooth ansatze for the correlation functions of the non-local condensates and without any suggestion

[3] Here $E(0,z) = P\exp(i\int_0^z dt_\mu A_\mu^a(t)\tau_a)$ is the Schwinger phase factor required for gauge invariance.

about the quark-hadron duality for the spectral density. This program has been suggested and realized in [2], and we have developed alternative methods of extracting WFs from this sum rule for a non-diagonal correlator [1]. It should be mentioned that we do not need the principle of stability over the Borel parameter M^2, and the Borel SR appears only at an intermediate stage.

Here, we test our method using an exactly solvable model as an example (initiated by the two-dimensional quantum-mechanical oscillator). We demonstrate the validity of the method and also suggest a pure algebraic procedure for extracting the masses and WFs of the particle under consideration. We investigate the stability of the procedure to perturbations of the theoretical part of the sum rule and obtain some results to believe that it works well in real QCD SR. In application to the pion case, this results in producing not only the mass and WF of the first resonance (π') but also to provide an estimate of the π''-mass.

Our quantitative results, i.e. the details of the WF shapes and the values of masses, are dependent on the certain form of the ansatz used for the non-local condensates. This form may be obtained in a future theory of the QCD vacuum. In the absence of such a theory the form of the ansatz was chosen as a phenomenological "input" (for details see [1]). From this point of view, our calculational scheme may be considered as a suitable framework for connecting the hadronic properties, on one hand, with the future theory of the non-perturbative vacuum on the other.

2 Non-diagonal QCD SR for pion wave functions and the Method of Integral Transform

2.1 Non-diagonal QCD SR

The sum rule for the "axial" WFs $\varphi_{\pi\cdots}(x)$ of the pseudoscalar mesons, based on the non-diagonal correlator of the axial and pseudoscalar currents

$$\begin{aligned}\varphi_\pi(x) \;+\; \varphi_{\pi'}(x)e^{-m^2_{\pi'}/M^2} + \varphi_{\pi''}(x)e^{-m^2_{\pi''}/M^2} + \ldots &\equiv \Phi(\frac{1}{M^2}, x) \qquad (1)\\ = \;\frac{M^2}{2}\left(1 - x + \frac{\lambda_q^2}{2M^2}\right) f(xM^2) &+ (x \to 1-x),\end{aligned}$$

has been suggested in [2] and possesses some remarkable properties. For the WFs on the l.h.s. of (1) one has

$$\begin{aligned}\langle\xi^N\rangle_M &\equiv \int_0^1 \varphi_M(x)(2x-1)^N\, dx;\\ \langle\xi^{N=0}\rangle_\pi = 1\;, \quad \langle\xi^{N=0}\rangle_{\pi'} &= \langle\xi^{N=0}\rangle_{\pi''} = \ldots = 0. \qquad (2)\end{aligned}$$

On the r.h.s. of (1) there appears a single correlation function $f(\nu)$ parametrizing the z^2-dependence [4] of the non-local quark condensate:

$$\langle : \bar q(0)q(z) : \rangle \equiv \langle : \bar q(0)q(0) : \rangle \int_0^\infty e^{\nu z^2/4} f(\nu)\, d\nu, \qquad (3)$$

$$\int_0^\infty f(\nu)\, d\nu = 1, \; \int_0^\infty \nu f(\nu) d\nu = \frac{1}{2}\left(\frac{\langle : \bar q(0)(D^2)q(0) : \rangle}{\langle : \bar q(0)q(0) : \rangle} = \lambda_q^2\right). \qquad (4)$$

Here D is the covariant derivative, λ_q^2 can be treated as the average virtuality of vacuum quarks. Equation (1) demonstrates in the most explicit manner that the distribution $\varphi_\pi(x)$ of quarks inside the pion over the longitudinal momentum fraction x on the l.h.s. of the equation is directly related to the distribution $f(\nu)$ over the virtuality ν of the vacuum fields on the r.h.s..

It should be noted that the sum rule (1) results from the approximations both in the theoretical (for a detailed discussion, see [2]) and the phenomenological parts (see [1]). These are

1. a reduction of three-point correlators to two-point ones;
2. a reduction of the quark-quark-gluon distribution function to a quark-quark one;
3. neglecting the contributions of higher non-local condensates, like $\langle\bar qGGq\rangle$, $\langle\bar qGGGq\rangle, \ldots$;
4. a representation of the phenomenological spectral density $\rho(s)$ on l.h.s. of the SR as a sum of very narrow resonances

$$\rho(s) = \delta(s)\cdot f_\pi\varphi_\pi(x) + \sum_{i\geq 1}\delta(s-\mu_i)\cdot f_i\varphi_i(x).$$

[4] In deriving these sum rules we can always make a Wick rotation, i.e., we assume that all coordinates are Euclidean, $z^2 < 0$.

In the present work we shall use a specific form of the correlation function $f(\nu)$, given by

$$f(\nu) = N_q \cdot \exp\left(-\Lambda^2/\nu - \sigma\cdot\nu\right); \quad (5)$$
$$N_q = \frac{1}{2\frac{\Lambda}{\sqrt{\sigma}}K_1\left(2\Lambda\sqrt{\sigma}\right)}, \quad K_1(z) \text{ is the modified Bessel function.}$$

As was established in our preceding paper [1], this ansatz (A2-ansatz there) conforms with pion physics. A similar form for the ansatz also appears in the framework of the instanton model for the non-perturbative vacuum [5]. Here, the parameter $\Lambda^2 \approx 0.2\,\mathrm{GeV}^2$ was extracted from the meson QCD SR for the heavy-quark effective theory [2]; the normalization constant N_q and the parameter σ are fixed by Eq.(4). For the average virtuality of vacuum quarks we take the usual QCD sum rule value $\lambda_q^2 \simeq 0.4\,\mathrm{GeV}^2$ [14].

As is clear from the structure of the r.h.s. of (1), the form of the function $\Phi(\frac{1}{M^2}, x)$ on the variable x is given by two humps (one centered at $x_A = s_A/M^2$, where s_A is the point of maximum for the ansatz correlation function $f(\nu)$, and the other at $x_A = 1 - s_A/M^2$) moving as M^2 changes. When M^2 increases, the humps become narrower, higher and more close to the boundary points $x = 0$ or $x = 1$. However, $\Phi(\frac{1}{M^2}, x)$ is not yet the pion WF: the larger M^2, the larger the contamination from higher pseudoscalar states π', π'', ... Due to the properties summarized in (2), the corresponding WFs should oscillate (see, e.g., Fig. 11); therefore the function $\Phi(\frac{1}{M^2}, x)$ does not resemble to $\varphi_\pi(x)$ at not sufficiently low M^2.

At low M^2, the pion WF dominates in the total sum $\Phi(\frac{1}{M^2}, x)$ (however, one cannot take too low M^2 because the operator-product expansion fails for $M^2 < \lambda_q^2$). When $M^2 = 0.4\ GeV^2$, it was observed that $\Phi(\frac{1}{M^2}, x)$ is very close to the asymptotic wave function of the pion (see [2]). But there is no strict criterion for selecting the value of M^2 to determine the WF of the ground state (pion). Our method of extracting the WFs and masses of resonances from (1) allows one to avoid this problem.

2.2 The Method of Integral Transform

In what follows it is convenient to use a new variable $\tau \equiv 1/M^2$ instead of the Borel parameter M^2. The r.h.s. of SR (1), i.e., $\Phi(\tau, x)$ is then defined for $\tau \in [0, \frac{1}{\lambda_q^2}]$. Let us consider this function in the whole complex plane of τ and

[5] A. Dorokhov (private communication)

define the integral transformation (IT) $P(N, \tau_0, \omega)$:

$$P(N,\tau_0,\omega)(\cdots) = \frac{N!}{2\pi i\omega^N}\int_C \frac{\exp(\omega(\tau-\tau_0))}{(\tau-\tau_0)^{N+1}}(\cdots)d\tau\,, \quad (6)$$

where the integration is performed along the vertical line $C = (c - i\infty,\ c + i\infty)$ with c lying to the right with respect to any pole of the integrand, $\omega > 0$ and $\tau_0 \in (0, \frac{1}{\lambda_q^2}]$.

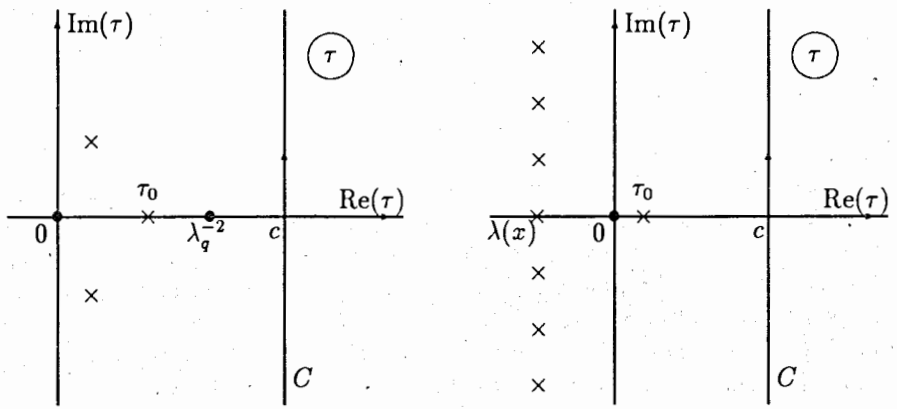

Figure 1 : Here C is the contour of integration in the complex plane of τ; crosses ($\times$) denote the positions (in case A – possible) of the integrand poles.

To obtain the result of this operator action on the initial SR-representation (here $\mu_i = m_i^2$)

$$\varphi(x) + \sum_{i=1}\varphi_i(x)\exp(-\tau\mu_i) = \Phi(\tau, x), \quad (7)$$

consider its action on a simple exponential $e^{-\tau\mu}$. Evidently, if $\omega > \mu$ we can close the contour C to the left and get (due to the well-known residue theorem) the contribution from the pole at $\tau = \tau_0$; if $\mu > \omega$ we can close the contour to the right and, due to the absence of poles in that part of complex τ-plane, obtain 0; therefore

$$P(N,\tau_0,\omega)\exp(-\mu\tau) = \theta(\omega-\mu)\cdot\exp(-\mu\tau_0)\left(1-\frac{\mu}{\omega}\right)^N. \quad (8)$$

Then, for $\omega > 0$, we have

$$\varphi(x) + \sum_{i\geq 1}\theta(\omega-\mu_i)\varphi_i(x)\left(1-\frac{\mu_i}{\omega}\right)^N\exp(-\tau_0\mu_i) = P(N,\tau_0,\omega)\Phi(\tau,x)$$
$$\equiv \Phi_N(\omega, x). \quad (9)$$

We see that by varying ω one can switch on (switch off) successively more and more resonances on the l.h.s. of the SR. So, we have obtained a real step division (smooth at $N > 0$) of the set of resonances by the parameter ω in (9), instead of the exponential suppression of the whole set of resonances with respect to the inverse Borel parameter τ in (7). Accordingly, the hadronic characteristics $\{\mu_i, \varphi_i(x)\}$ are determined by the positions of singularities (and zeros) of the correlation function $f(1/\tau)$ in the complex τ-plane.

Let us assume now that the mass values are known $\mu_0 = 0$, $\mu_1 = \mu_{exp} \approx 1.7\text{GeV}^2$, ... Then, we can obtain an expression for $\varphi(x)$ by employing equation (9) with the parameter $\omega = \omega_1 \leq \mu_1$ that corresponds to the saturation of the ground state $\varphi(x)$

$$\varphi(x) = P(N, \tau_0, \omega_1)\Phi(\tau, x) = \Phi_N(\omega_1, x). \tag{10}$$

The final result should not depend on the parameter N strongly. However, in a real life we don't know the position μ_1 exactly; therefore we may get contamination from the next state. This contamination becomes smaller when N increases due to the power suppression near the threshold μ_1 (see l.h.s. of Eq. (9)). One should also take into account that the WF form is saturated smoothly in the region $\omega < \mu_1$ and its norm reaches unity at the end of the region. The curves corresponding to the function $\varphi(x) = \Phi_N(\omega_1 \approx \mu_1, x)$ are shown in Fig. 2 for different values of N. Their form varies very slightly with $N = 1,\ 2,\ 3$ (the same happens when the parameter τ_0 decreases from 2.5 to 1.5 GeV^2, see the discussion in [1]) and is always somewhat narrower than the asymptotic one. Note that formula (10) may be easily generalized to a state with a finite width (see [1]). The contribution to the pion WF due to finite width effects is not significant numerically.

We now describe the pure algebraic procedure that allows one to extract from the SR (9) both the positions (μ_i) and the WFs ($\varphi_i(x)$). The procedure will be based on the simple step representation appearing on the l.h.s. of (9). Indeed, let us consider the difference

$$D_1\Phi_N(\omega, x) \equiv \Phi_N(\omega, x) - \Phi_{N+1}(\omega, x). \tag{11}$$

We see that the same contributions due to $\varphi(x)$ are cancelled out and this difference becomes non-zero just after $\omega > \mu_1$:

$$D_1\Phi_N(\omega, x) = \sum_{i\geq 1} \theta(\omega - \mu_i)\,\varphi_i(x) \exp(-\tau_0\mu_i) \frac{\mu_i}{\omega}\left(1 - \frac{\mu_i}{\omega}\right)^N. \tag{12}$$

Therefore, this difference is a trigger for the first resonance – the function $D_1\Phi_N(\omega, x)$ must have a root at $\omega = \mu_1$ that fixes the resonance (at fixed x).

Analogous trigger-like differences could be constructed for higher resonances. By induction, let the difference D_n is trigger-like for the n-th resonance (i.e., it has a root at $\omega = \mu_n$) and has the following form

$$D_n\Phi_N(\omega, x) = \sum_{i\geq n} \theta(\omega - \mu_i)\,\varphi_i(x) e^{-\tau_0\mu_i} \frac{\mu_i}{\omega}\frac{\mu_i - \mu_1}{\omega} \ldots \frac{\mu_i - \mu_{n-1}}{\omega}\left(1 - \frac{\mu_i}{\omega}\right)^N. \tag{13}$$

Then the difference D_{n+1} is defined recursively by the recurrence relation

$$D_{n+1}\Phi_N(\omega, x) = \left(1 - \frac{\mu_n}{\omega}\right) D_n\Phi_N(\omega, x) - D_n\Phi_{N+1}(\omega, x). \tag{14}$$

For instance, for the second resonance the corresponding difference is of the form

$$D_2\Phi_N(\omega, x) = \left(1 - \frac{\mu_1}{\omega}\right) D_1\Phi_N(\omega, x) - D_1\Phi_{N+1}(\omega, x), \tag{15}$$

and it yields the value of μ_2 provided the value of μ_1 is known from the previous step. Hence, with the help of these trigger-like differences one can easily determine successively the masses μ_i. After that it is straightforward to determine the WFs $\varphi_i(x)$ itself, i.e.:

$$\varphi_1(x) = \frac{(\Phi_1(\omega, x) - \Phi_1(\mu_1, x))\exp(\tau_0\mu_1)}{\left(1 - \frac{\mu_1}{\omega}\right)} \tag{16}$$

where the value of the parameter $\omega \in (\mu_1, \mu_2]$ corresponds to saturation of the first resonance. One can try the other formula for $\varphi_1(x)$ following from (12), viz.

$$\varphi_1(x) = \frac{D_1\Phi_N(\omega, x)\exp(\tau_0\mu_1)}{\frac{\mu_1}{\omega}\left(1 - \frac{\mu_1}{\omega}\right)^N}. \tag{17}$$

The function $\varphi_1(x)$ in (17) should not depend on the value of N in an appreciable manner when the saturation of the resonance is reached.

3 An exactly solvable toy model

The above formulated procedure is based on the understanding of the initial SR (1) as an exact equality. But in the real case the connection between the non-perturbative vacuum distribution $f(\nu)$ on the r.h.s of SR and the pseudoscalar meson properties reflected on by the spectral expansion $\sum_i \varphi_i(x)\exp(-\mu_i\tau)$ on the r.h.s. of SR is only **an approximation**. To understand the reliability of the suggested technique (IT) as an approximated equality, it should be useful to check it, first, with a model $\Phi(\tau, x)$ having an exact spectral expansion. After

that we deform $\Phi(\tau,x)$ in some special way to destroy the exact equality in the spectral expansion with the initial WFs and masses, and apply our procedure again. Following this way, one may investigate the stability of the procedure to perturbations and its suitability. We hope, despite the fact that the exact equality is broken, that the procedure will "work" and that the new extracted parameters $\{\mu_i', \varphi_i'(x)\}$ will be close to the initial values, if the deformation is not too strong. Only in this case our procedure makes sense.

3.1 Formulating the model

We should like to test our approach for some simple exactly solvable model. We select one which originates from the 2D harmonic oscillator problem in quantum mechanics (for more details see ref.[15]). The Borel transformed Green function $M(\tau,x)$ of the 2D harmonic oscillator (with $\omega_0 = 1$) in the coordinate representation

$$M(\tau,\mathbf{r}) \equiv \sum_{k=0} |\psi_k(\mathbf{r})|^2 e^{-\tau E_k} \tag{18}$$

has for $\mathbf{r} = 0$ the form

$$M(\tau,0) = \frac{m}{2\pi \sinh(\tau)}. \tag{19}$$

The spectrum of this model (as one can easily see from the geometric progression summation formula) is equidistant with a two-fold step

$$E_k = 2k + 1. \tag{20}$$

We can also write down an explicit analytical form for $M(\tau,\mathbf{r})$ with $\mathbf{r} \neq 0$, but there is no need for this, since we shall work, for simplicity, with a modified quantity. To this end, we define the function

$$\Phi(\tau,x;\phi) = \frac{1}{1-\phi(x)\exp(-2\tau)} \tag{21}$$

which obviously has the spectral expansion we are interested in, viz.

$$\Phi(\tau,x;\phi) = \sum_{k=0} \phi(x)^k e^{-2\tau k}. \tag{22}$$

This toy model has a nice feature: all its resonance WFs are defined through one WF, namely the WF of the first resonance $\varphi_1(x) = \phi(x)$ which can be selected at free choice. For our convenience, we take the following form

$$\phi(x) = 4x(1-x), \tag{23}$$

i.e., the asymptotic WF of the pion (albeit with wrong normalization).

3.2 IT-technique for the toy model

For evaluating $\Phi_N(\omega,x)$ in our toy model, we use the well-known residues-technique. Separating residues (see Fig. 1) from the τ_0-pole ($\Delta_{\tau_0}\Phi_N$) and the poles $\tau_n(x) \equiv \lambda(x) + i\pi n$ with $\lambda(x) \equiv \frac{1}{2}\log(\phi(x))$ ($\Delta_\Phi\Phi_N$), we obtain

$$\Phi_N(\omega,x) \equiv \Delta_{\tau_0}\Phi_N + \Delta_\Phi\Phi_N \tag{24}$$

$$\Delta_{\tau_0}\Phi_N = \theta(\omega)\frac{\exp(-\omega\tau_0)}{\omega^N}\frac{d^{(N)}}{d\tau_0^N}\left[\frac{\exp(\omega\tau_0)}{1-\varphi(x)\exp(-2\tau_0)}\right] \tag{25}$$

$$\Delta_\Phi\Phi_N = -\theta(\omega)\Gamma(N+1)\frac{\exp\{-\omega[\tau_0-\lambda(x)]\}}{(-\omega)^N} \times$$

$$\times \left[\frac{1}{2(\tau_0-\lambda(x))^{N+1}} + \sum_{n\geq 1}\frac{\cos[\pi n\omega + (N+1)\phi_n]}{\left[(\tau_0-\lambda(x))^2+\pi^2n^2\right]^{(N+1)/2}}\right] \tag{26}$$

Here $\phi_n \equiv \arctan[\pi n/(\tau_0-\lambda(x))]$.

The l.h.s. of SR (9) in this model has the form

$$1 + \sum_{k\geq 1}\theta(\omega-2k)\,\phi(x)^k\left(1-\frac{2k}{\omega}\right)^N \exp(-2k\tau_0). \tag{27}$$

By this way we obtain the exact SR in the form

$$(24) = (27).$$

Saying "exact" we mean, that this SR is valid for all $x \in [0,1]$ and for all ω. Moreover, it holds also for all choices of the initial WF $\phi(x)$! This behavior is shown explicitly in Fig. 3 , where the step-like structure on the l.h.s. of SR (see (27)) is reproduced by its r.h.s. (see (24)). Our trigger-like differences exactly determine masses $\mu_k = 2k$ and the corresponding WFs $\varphi_k(x) = \phi(x)^k$.

3.3 Deformations of $\Phi(\tau,x)$ and the IT-procedure

All formulas of the preceding subsection are exact (they are in some sense identities). But in real QCD problems we have no such regular behavior of the theoretical part of the SR and one can use the spectral expansion only approximately. Moreover, we cannot be sure about the adequacy of ansatz (5) for the true non-perturbative amplitude in the whole region of variables. Having this in mind, let us inspect the stability of the suggested procedure under perturbations of the r.h.s. of (9).

The first problem here is the following: how can one deform the original function $\Phi(\tau,x)$ while retaining the possibility of controlling the degree of this deformation?

We apply the following method. As we know, the function $\Phi(\tau, x)$ can be expanded in a series of poles over the variable τ:

$$\Phi(\tau, x; \phi) \equiv \frac{1}{1 - \phi(x)\exp(-2\tau)} = \frac{1}{2} + \sum_{n=-\infty}^{\infty} \frac{1}{2(\tau - \tau_n(x))}. \qquad (28)$$

If we exclude in this expansion a set of poles, we obtain some deformed function. It should be stressed that this function does not have the same spectral expansion as the original one. Let us consider our IT-procedure with these deformed functions. The exact result for the difference $D_1\Phi_1$ is reproduced in Fig. 4 . The deformations we have determined **will destroy** the exact step-structure on the l.h.s. of (9) and (13). This procedure proceeds as follows.

1. At first we exclude all the poles $\tau_n(x)$ with $|n| > K$ (the corresponding deformed Φ will be denoted as $\Phi(\omega, x; K)$). Fig. 5 contains the curve for $D_1\Phi_1(\omega, x; K = 6)$. As we can conclude in this case, the situation is fine: the main deviations from $\theta(\omega - 2)$ are located near the origin ($\omega \sim 0.2$). The locations of resonances are well-determined.

2. At the next step we exclude also the last six poles ($|n| = 3, 4$ and 5). The corresponding curve for $D_1\Phi_1(\omega, x; K = 3)$ is reproduced in Fig. 6 . The situation is still good: though deviations from $\theta(\omega - 2)$ are larger relative to $K = 6$, they do not extend over $\omega \sim 0.2$. The locations of resonances are also well-determined and our "triggering" is still valid.

3. The last step is to exclude the same poles as in the first case, but in addition also those with $n = \pm 1$. The curve for $D_1\Phi_1(\omega, x; K = 6 - 1)$ is reproduced in Fig. 7 . Now the pattern is changed: deviations form the initial form are large and they reach $\omega = 1$. Indeed, this new form is rather close to $\theta(\omega - 1)$: there exists the zero for $\omega = 1$ where the original function has no resonances at all. Going through the procedure with this value ($\mu_1 = 1$) for the mass of the first resonance, we obtain the new spectrum $\mu_k = k$ with the new resonance WFs $\{\varphi_k(x)\} \neq \{\phi^k(x)\}$. The crucial indication for the correctness of such an interpretation is the smallness of the deficit of the initial SR

$$\Delta_{SR}\{\Phi(\tau, x; K = 6 - 1)\} \equiv \Phi(\tau, x; K = 6 - 1) - \qquad (29)$$
$$- \varphi_0(x) - \varphi_1(x)\exp(-\mu_1\tau) - \varphi_2(x)\exp(-\mu_2\tau) - \ldots.$$

We evaluate this quantity for the new and for the initial spectra and obtain a clear signal that the new one provides the genuine description of the SR (see Fig. 8 and 9).

As to the extracted wave functions $\varphi_k(x)$, we can characterize their evolution under deformations by the following statements:

- opposed to the exact solution, $\{\varphi_k(x)\}$ become ω-dependent when $\omega \in (\mu_i; \mu_{i+1}]$. They tend to the exact ones with ω more close to the right end of the corresponding interval (see Fig.10);
- the difference between $\varphi_k(x)$ and the exact WF of the k-th resonance becomes stronger for higher k (i.e. for highly exited states; see Fig 10).

To conclude these investigations: the method really works, even for strong deformations of the structure of the initial spectrum as the last example of deformation demonstrates. Moreover, we see that deformations of the analytical structure of the initial $\Phi(\tau, x; \phi)$ in the complex τ-plane far from (near) the main pole $\tau = \lambda(x)$ produce small (large) deformations of the spectrum and the WFs. Therefore, the knowledge of the analytic properties of the function $\Phi(\tau, x)$ in a region in the complex τ-plane when includes the pole $\tau = \lambda(x)$ and a few others appears to be sufficient for obtaining information on the first few resonances and their WFs. We hope that an analogous situation appears in the SR for pion.

4 IT-procedure for the pion case

Applying the algebraic procedures to the IT of SR (1) we arrive at the following results:

1. For $\omega = \omega_* = 1.8\,\mathrm{GeV}^2$, the difference $D_1\Phi_1(\omega_*, 1/2)$ is equal to zero and with increasing ω it becomes negative (this means that $\varphi_1(1/2) < 0$). This value of the first-resonance mass is quite reasonable for the pion case, the experimental one being $\mu_1^{exp} = m_{\pi'}^2 \approx 1.7$ GeV^2 (at the full width $\Gamma_{\pi'} = 0.2 \div 0.6$ GeV). The pion WF $\varphi(x)$, corresponding to this ω_* has been obtained from expression (10) in subsection 2.2 (see Fig. 2). Its width is traditionally characterized by the value of the second moment – $\langle\xi^2\rangle_\pi \approx 0.17$. Another characteristic often appearing in form factor calculations is the integral $\int_0^1 \varphi(x)/x dx \approx 2.8$. Note that the method does not work properly in the neighborhood of the endpoints $x = 0$ and $x = 1$ of the WF. Therefore, to estimate the above values, one should exclude these regions in the integral.

   It should be emphasized that $\Phi_1(\omega, x)$ imitates the l.h.s. of the SR as a function of ω rather well, despite its different origin. Note here that the position of the root ω_* depends also on the value of x: in the region

$0.35 \leq x \leq 0.65$ this dependence is rather weak. We don't see the step structure on the l.h.s. of Eq. (12) – the structure is destroyed due to the employment of too crude approximations in the theoretical part of the SR. Following our experience in the "exactly solvable toy model" considered in subsection 3.3, we may hope, nevertheless, that the roots of the few trigger differences $D_{1,2,...}\Phi_N(\omega_*, 1/2)$ yield reliable approximations for the masses.

2. The saturation by the first resonance is reached near $\omega \approx 3$ GeV^2. The curves corresponding to the resonance $\varphi_1(x)$ (for $\omega \approx 3$ GeV^2) are shown in Fig. 11.

3. The last information we can extract in this case is the position of the second resonance: the equation $D_2\Phi_1(\omega_{**}, x) = 0$ gives us the range of values $\omega_{**} = \mu_2 = 3 \div 4\text{GeV}^2$, which is converted to the mass value $m_2 = 1.86 \pm 0.14$GeV. The experimental parameters for this resonance are still in question in the Particle Data booklet (April 1994), despite that the first evidence was obtained many years ago [16]. However, the recent measurements in [17] certainly provide for the mass $m_{\pi''} \approx 1.78 \pm 0.007$ GeV (at full width $\Gamma_{\pi''} = 0.16$ GeV), which is rather close to our estimate.

5 Conclusion

We have considered a model SR for the pion and the pseudoscalar resonance WFs based on the non-diagonal correlator introduced in [2]. The theoretical side (r.h.s.) of this sum rule depends only on the non-local condensate. We have tested the approach proposed in our recent paper [1], which enables one to extract WFs and masses from this SR, using the exactly solvable toy model as an example. We demonstrated the validity of the method and suggested a pure algebraic procedure for extracting the masses and WFs relating to the case under investigation. We investigated also the stability of the procedure under perturbations of the theoretical part of the sum rule. We obtained as a result that the most crucial domain in the complex plane of $\tau = M^{-2}$ is the neighborhood of the poles lying on the real axes – the perturbations near these points essentially reorganize the spectral expansion under investigation. Applying this method to the pion case and using one of the ansatze given in [1] not only the mass and WF of the first resonance (π') have been derived, but also the mass of π''. Our results confirm the main conclusions about the shapes of the WF of the pion and its resonances, obtained in [2].

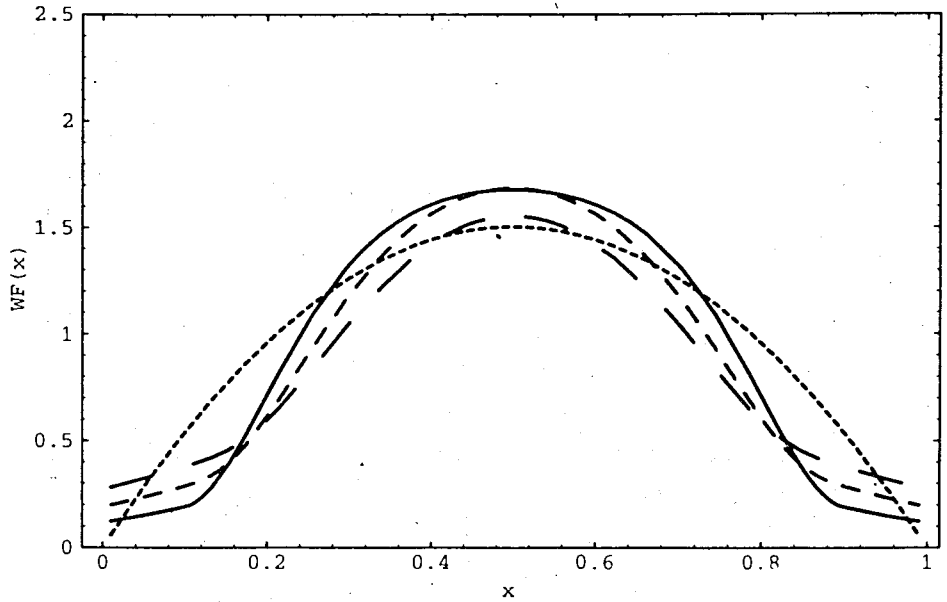

Figure 2: Forms of the WF of pion $\varphi_\pi(x)$ extracted from $\Phi_N(\omega \approx 1.5, x)$ for $N = 1$(line), $N = 2$(dashed) and $N = 3$(dotted)

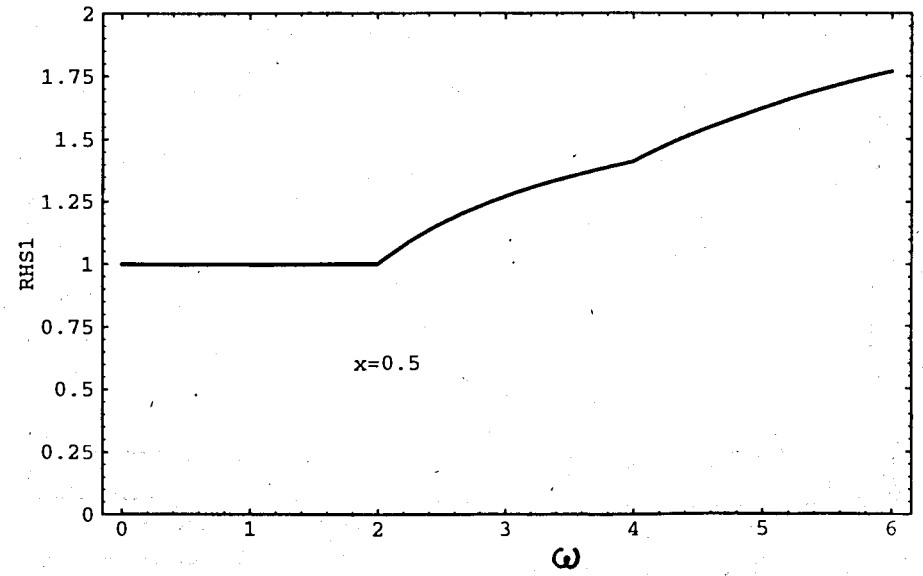

Figure 3: Step-like structure of $\Phi_1(\omega, x)$ with $x =$ fixed $= 0.5$ for the toy model case.

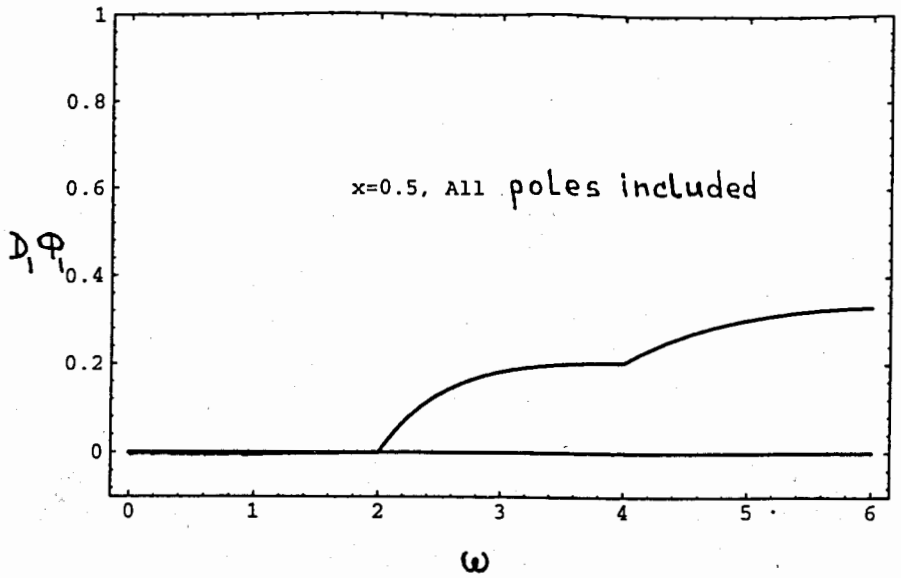

Figure 4: Difference $D_1\Phi_1(\omega, x)$ with x = fixed = 0.5 for the exact r.h.s. of SR for the toy model case.

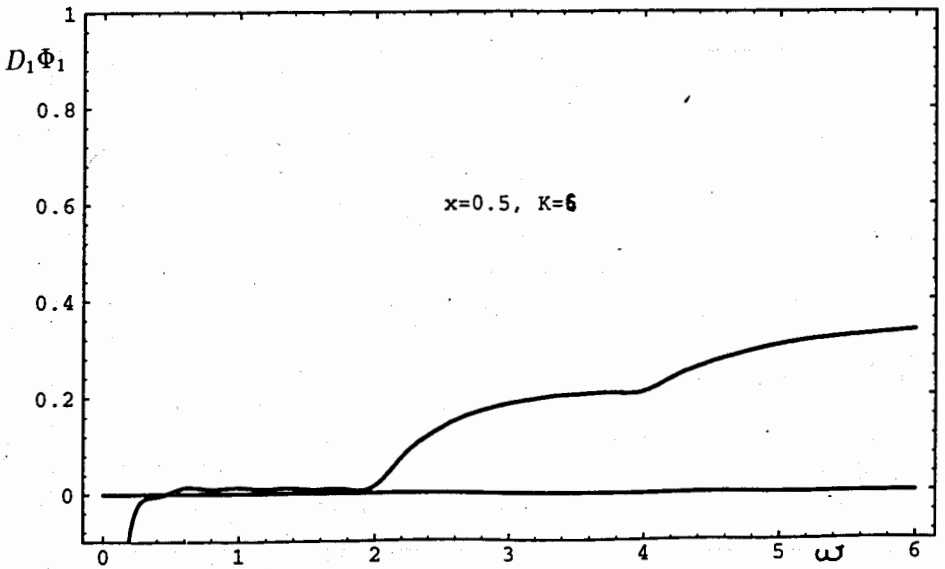

Figure 5: Difference $D_1\Phi_1(\omega, x, K = 6)$ with x = fixed = 0.5 for the deformed $(K = 6)$ r.h.s. of the SR for the toy model case.

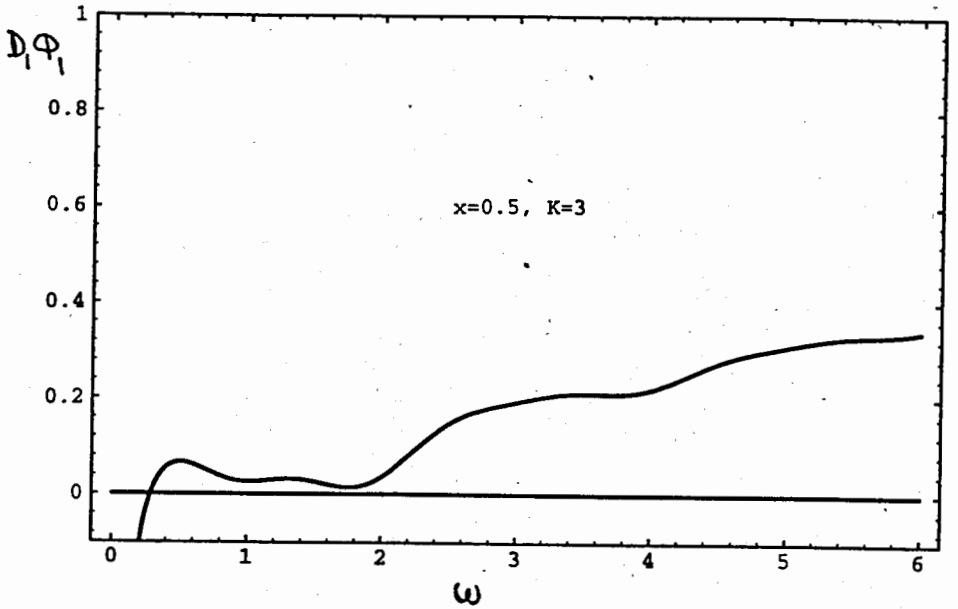

Figure 6: Difference $D_1\Phi_1(\omega, x, K = 3)$ with x = fixed = 0.5 for the deformed $(K = 3)$ r.h.s. of the SR in the toy model case.

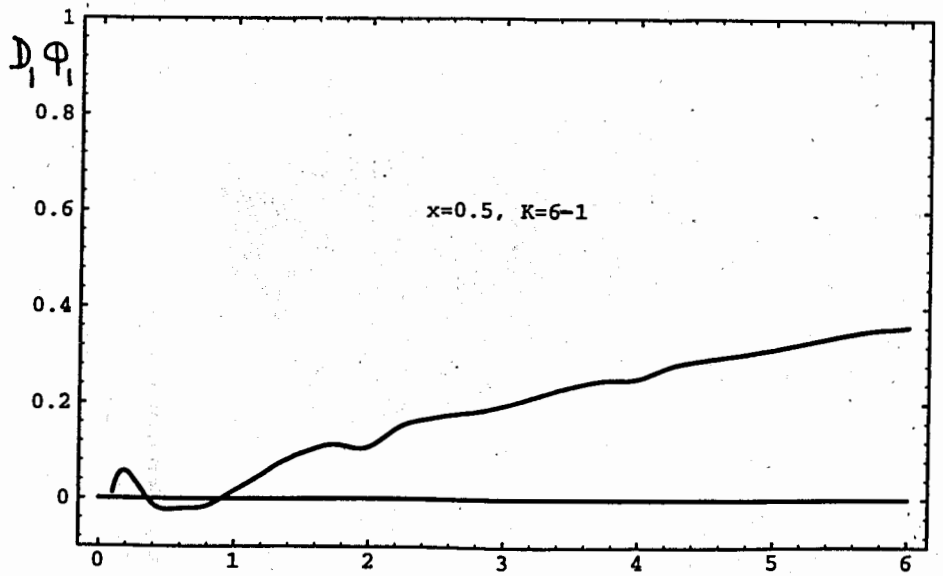

Figure 7: Difference $D_1\Phi_1(\omega, x, K = 6 - 1)$ with x = fixed = 0.5 for the deformed $(K = 6 - 1)$ r.h.s. of the SR in the toy model case.

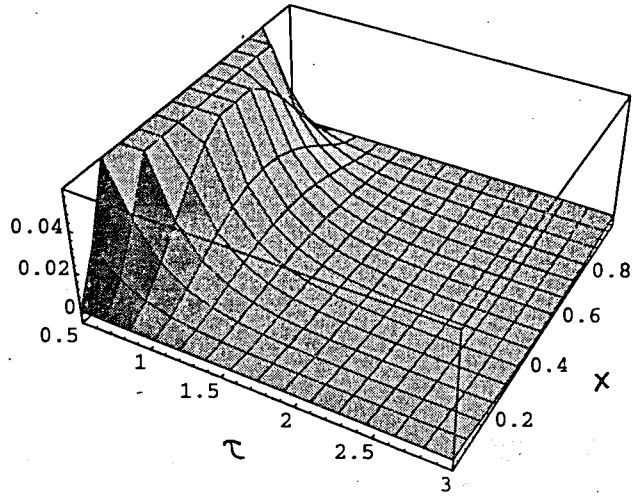

Figure 8: The deficit of the initial SR $\Delta_{SR}\{\Phi(\tau, x; K = 6-1)\}$ for the new spectrum $\mu_k = k$ in the toy model case.

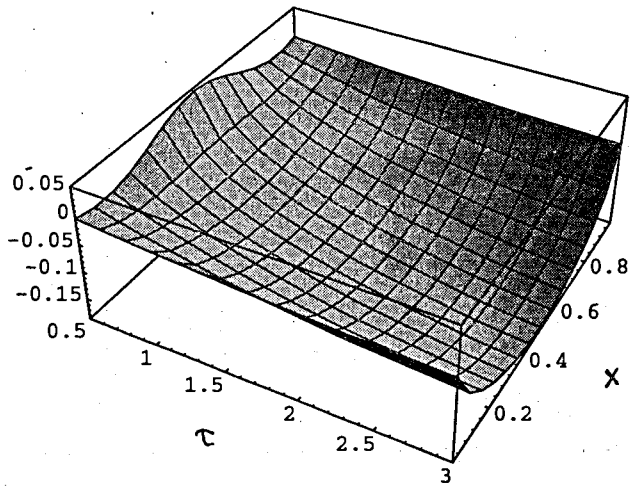

Figure 9: The deficit of the initial SR $\Delta_{SR}\{\Phi(\tau, x; K = 6-1)\}$ for the old spectrum $\mu_k = 2k$ for the toy model case.

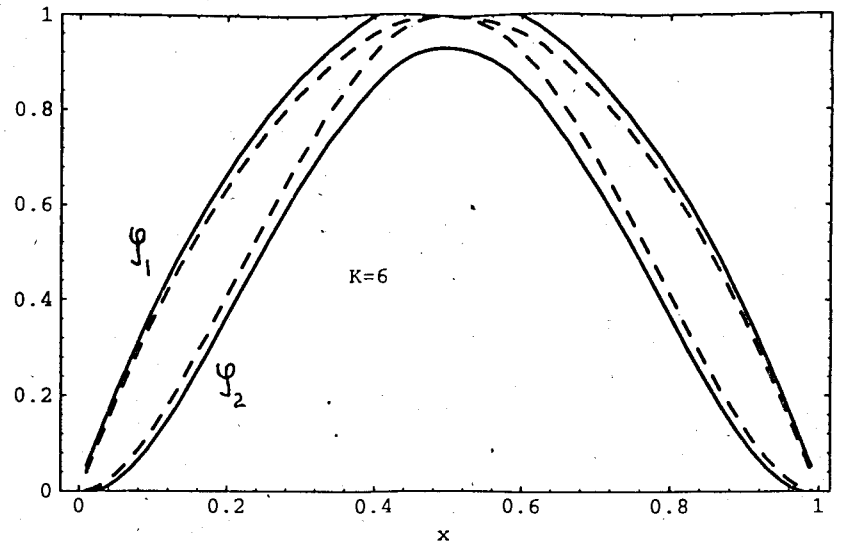

Figure 10: The difference between $\varphi_i(x)$ for the exact WF of the i-th resonance for the $i = 1, 2$ (dashed) and the deformed ($K = 6$) toy model case (lines).

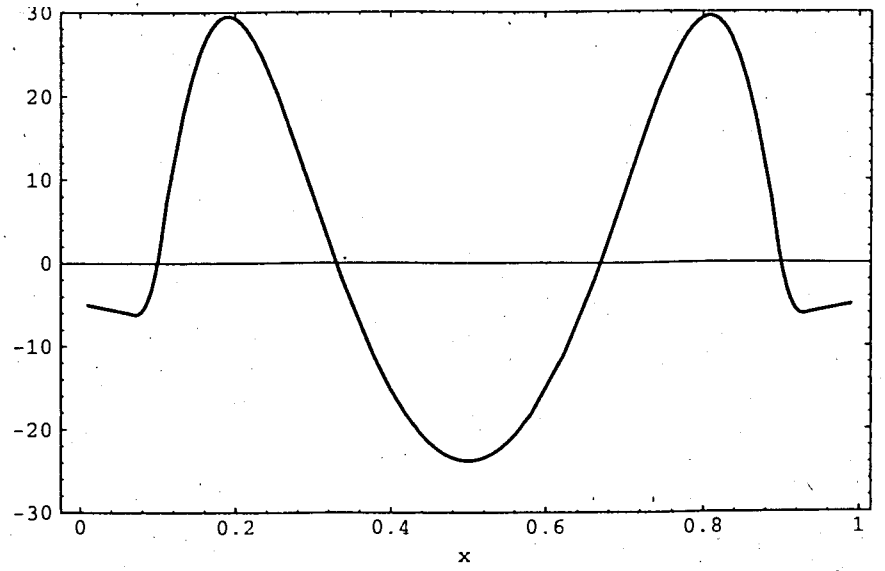

Figure 11: Form of the WF of the first resonance $\varphi_{\pi'}(x)$ extracted from $D_1\Phi_1(\omega \approx 3\ \mathrm{GeV}^2, x)$ for the pion case.

It should be mentioned that the specific form of the ansatz for the correlation function $f(\tilde{\nu})$ plays an important role in determining concrete values of masses and shapes of the WFs corresponding exactly to the pion resonances (on the contrary, the WF of the pion ground state is not too sensitive to the shape of the ansatz). We have chosen our ansatz (5) among others (see e.g. [1]) just due to this reason. So, one may consider it as a first step towards solving the inverse problem, namely to obtain the vacuum condensate properties from hadron phenomenology.

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