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ON THE PHOTON DENSITIES, HEGERFELDT THEOREM AND ALL THAT

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1 Introduction

Attempts to find the probability density for photons have a long and dramatic history. As far as we know, the first attempt was made by Landau and Peierls in 1930 [1]. However, the density obtained by them was not positive definite and, thus, had no physical meaning. This paper has been severely criticized by Ehrenfest [2] and Pauli [3,4]. Further, Newton and Wigner [5][1] have clarified the physical meaning of the particle localizability. Their definition of localizability differs from the usual intuitive one which defines the localizability as the vanishing of the wave function outside the finite region of space. On the contrary, the wave function localized in the Newton-Wigner sense differs from zero everywhere[2]. Later, Hegerfeldt [7] generalized their results by proving an important theorem which states that even if the initial wave function is confined to the finite space region, it instantly fills the whole space at a subsequent time. In an important Zeldovich paper [8] the number of photons was represented as an integral of an expression bilinear in electromagnetic strengths and its relativistic invariance was proved. In two papers by Cook [9,10] two auxiliary functions related to the photon density were introduced. Unfortunately, they are not behaved like tensors under the Lorentz transformations. Akhiezer and Berestetsky [11] and later Bialynicki-Birula [12] considered the complex form of the Maxwell equation. The photon wave function and its density were associated with $\Psi = (\vec{E} + i\vec{H})$ and and $(\vec{E}^2 + \vec{H}^2)/\int(\vec{E}^2 + \vec{H}^2)d^3x$, resp. In the paper by Sipe [13] the photon wave function was identified with the positive-frequency part of the electric strength. Correspondingly, the photon density was normalized to the energy.

In the present paper, we develop and numerically investigate the formalism suggested in refs.[9,10]. The plan of our exposition is as follows. In sect.2, necessary mathematical details are presented. In particular, a number of conservation laws is obtained and their physical interpretation is given. In sect.3, these results are applied to a relatively simple model. We numerically investigate the time evolution of the photon density and other densities corresponding to the conserved quantities. It turns out that the photon wave function (WF) slightly extends over the region where electromagnetic strengths differ from zero. It is tempting to associate this part of the photon WF with the so-called 'empty' wave (a detailed exposition of the empty wave theory accompanied by the thorough analysis of the performed and planned experiments aimed to detect empty waves can be found in book [14]). In sect.4, we turn to the Ehrenfest-Pauli objections against the Landau-Peierls WF that are equally applied to the present function. We believe that at least partly we succeeded in overcoming the Ehrenfest-Pauli objections. In sect. 5, we present our viewpoint on the Hegerfeldt theorem having in mind its application to the time evolution of the photon wave function. In sect. 6, we try to resolve the contradiction between the possible localizability of the classical electromagnetic wave, its absence for photons and the fact that an electromagnetic wave consists of photons. Our resolution of this paradoxical situation differs essentially from that suggested recently by Kim et al. [15]. A brief account of the results obtained is given in sect.7.

2 Preliminaries

Photon densities.
We consider the free electromagnetic field (EMF) described by the Maxwell equations

[1]A nice exposition of their ideas may be found in Bacry's book [6].

[2]In what follows under the localization of the electromagnetic field we mean the possibility of its confinement within the finite region of space, thus, not the localization in the Newton-Wigner sense.

$$\vec{curl}\vec{E} = -\dot{\vec{H}}/c, \quad \vec{curl}\vec{H} = \dot{\vec{E}}/c, \quad div\vec{E} = 0, \quad div\vec{H} = 0 \tag{2.1}$$

(the dot above the letter means a time derivative). The Fourier transformations of $\vec{E}$ and $\vec{H}$ are:

$$\vec{E} = \int e^{i\vec{k}\vec{r}}\vec{E}_k(t)d^3k, \quad \vec{H} = \int e^{i\vec{k}\vec{r}}\vec{H}_k(t)d^3k. \tag{2.2}$$

The reality of $\vec{E}$ and $\vec{H}$ requires that $\vec{E}^*(-\vec{k}) = \vec{E}(\vec{k})$, $\vec{H}^*(-\vec{k}) = \vec{H}(\vec{k})$. These equations are automatically satisfied if $\vec{E}_k$ and $\vec{H}_k$ have the form [16]

$$\vec{E}_k = \frac{\sqrt{\omega}}{2\pi}(\vec{f}_k + \vec{f}^*_{-k}), \quad \vec{H}_k = \frac{c}{2\pi\sqrt{\omega}}\vec{k}\times(\vec{f}_k - \vec{f}^*_{-k}). \tag{2.3}$$

Here $\omega = c|\vec{k}|$. In the Coulomb gauge ($div\vec{A} = 0$, $A_0 = 0$) the vector functions f_k satisfy the equations

$$\vec{k}\vec{f} = 0, \quad i\dot{\vec{f}} = \omega\vec{f}. \tag{2.4}$$

For the free EMF $\vec{f}(\vec{k}) = \vec{f}_0(\vec{k})\exp(-i\omega t)$ where $\vec{f}_0$ is independent of time. The first of Eqs.(2.4) guarantees the transversality of EMF ($div\vec{E} = 0$, $div\vec{H} = 0$). It is essential that $\vec{E}$ and $\vec{H}$ defined by Eq.(2.2) contain both positive and negative frequencies. The energy is equal to

$$\mathcal{E}_{EMF} = \frac{1}{8\pi}\int(E^2 + H^2)d^3x = \int\omega|\vec{f}(k)|^2 d^3k. \tag{2.5}$$

It turns out that $\int E^2 dV$ and $\int H^2 dV$ are not conserved quantities, only their sum does. It follows from (2.5) that

$$\rho_f(\vec{k}) = |\vec{f}(\vec{k})|^2/\hbar \tag{2.6}$$

coincides with the photon density in the momentum space. The number of photons is given by $N = \int\rho_f(k)d^3k$. Thus, $\vec{f}(k)$ may be considered as the photon wave function in the momentum space. Further, the number of photons may be written as an integral over the space variables

$$N = \int\rho_f(\vec{x})d^3x, \quad \rho_f(\vec{x}) = \frac{1}{\hbar(2\pi)^3}|\vec{f}(\vec{x})|^2. \tag{2.7}$$

Here $\vec{f}(\vec{x})$ is the Fourier transform of $\vec{f}(\vec{k})$:

$$\vec{f}(\vec{x}) = \int\vec{f}(\vec{k})\exp(i\vec{k}\vec{x})d^3k. \tag{2.8}$$

It turns out that $\rho_f(\vec{x})$ may be viewed as the photon density in the coordinate space. Since $\rho(\vec{k})$ is independent of time, the number of photons N is a conserved quantity. In what follows we need the representation of the field strengths alternative to (2.3)

$$\vec{H}(\vec{k}) = \frac{\sqrt{\omega}}{2\pi}(\vec{g}(\vec{k}) + \vec{g}^*(-\vec{k})), \quad \vec{E}(\vec{k}) = -\frac{c}{2\pi\sqrt{\omega}}\vec{k}\times(\vec{g}(\vec{k}) - \vec{g}^*(-\vec{k})). \tag{2.9}$$

As before, $\vec{g}(\vec{k}) = \vec{g}_0(\vec{k})\exp(-i\omega t)$. Obviously, $\vec{g}(\vec{k})$ and $\vec{f}(\vec{k})$ may be expressed through $\vec{E}(\vec{k})$ and $\vec{H}(\vec{k})$:

$$\vec{f}(\vec{k}) = \frac{\pi}{\sqrt{\omega}}(\vec{E}(\vec{k}) - \hat{\vec{k}}\times\vec{H}(\vec{k})), \quad \vec{g}(\vec{k}) = \frac{\pi}{\sqrt{\omega}}(\vec{H}(\vec{k}) + \hat{\vec{k}}\times\vec{E}(\vec{k})), \tag{2.10}$$

and through each other:

$$\vec{g}(\vec{k}) = \hat{\vec{k}}\times\vec{f}(\vec{k}), \quad \vec{f}(\vec{k}) = -\hat{\vec{k}}\times\vec{g}(\vec{k}). \tag{2.11}$$

Here $\hat{k} = \vec{k}/k$. Since

$$|\vec{f}(\vec{k})|^2 = |\vec{g}(\vec{k})|^2 = \frac{\pi^2}{\omega}(|\vec{E}(\vec{k})|^2 + |\vec{H}(\vec{k})|^2),$$

the quantities

$$\rho_g(\vec{k}) = |\vec{g}(\vec{k})|^2/\hbar \quad \text{and} \quad \rho_g(\vec{x}) = \frac{1}{\hbar(2\pi)^3}|\vec{g}(\vec{x})|^2, \quad (\vec{g}(\vec{x}) = \int\exp(i\vec{k}\vec{x})\vec{g}(\vec{k})d^3k) \tag{2.12}$$

may be also viewed as photon densities in momentum and coordinate spaces, resp. (on the same footing as $\rho_f(\vec{k})$ and $\rho_f(\vec{x})$). Note that $\rho_g(\vec{x}) \neq \rho_f(\vec{x})$ despite the fact that

$$\int\rho_f(\vec{x})d^3x = \int\rho_g(\vec{x})d^3x = N, \tag{2.13}$$

The relativistic invariance of the photon number N follows from the possibility to represent it in the following bilinear relativistically invariant form [8]

$$N = \frac{1}{16\pi^3\hbar c}\int\frac{\vec{E}(\vec{x})\vec{E}(\vec{y}) + \vec{H}(\vec{x})\vec{H}(\vec{y})}{|\vec{x} - \vec{y}|^2}d^3x d^3y. \tag{2.14}$$

The conservation laws

From the fact that $\vec{f}(\vec{x})$ and $\vec{g}(\vec{x})$ satisfy the same equations as $\vec{E}(\vec{x})$ and $\vec{H}(\vec{x})$:

$$\frac{1}{c}\dot{\vec{f}} = curl\vec{g}, \quad \frac{1}{c}\dot{\vec{g}} = -curl\vec{f}, \quad div\vec{f} = div\vec{g} = 0, \tag{2.15}$$

the following two continuity equations are easily obtained:

$$\dot{\rho} + \frac{c}{2}div(\vec{f}\times\vec{g}^* + \vec{f}^*\times\vec{g}) = 0, \tag{2.16}$$

$$\frac{1}{2c}\frac{\partial}{\partial t}(\vec{f}^2 + \vec{g}^2) + div(\vec{f}\times\vec{g}) = 0. \tag{2.17}$$

Here $\rho(\vec{x}) = (\rho_f + \rho_g)/2$. Equation (2.16) gives the local differential conservation law for the photon density $\rho(\vec{x})$. The densities ρ_f and ρ_g taken separately also satisfy continuity equations but they being highly nonlocal are of no interest for us.

Separating the real and imaginary parts in Eq.(2.17) one gets

$$\frac{1}{2c}\frac{\partial}{\partial t}(\vec{f}_r^2 + \vec{g}_r^2 - \vec{f}_i^2 - \vec{g}_i^2) + div(\vec{f}_r\times\vec{g}_r - \vec{f}_i\times\vec{g}_i) = 0, \tag{2.18}$$

$$\frac{1}{c}\frac{\partial}{\partial t}(\vec{f}_r\vec{f}_i + \vec{g}_r\vec{g}_i) + div(\vec{f}_r\times\vec{g}_i + \vec{f}_i\times\vec{g}_r) = 0. \tag{2.19}$$

Here $\vec{f}_r = Re\vec{f}$, $\vec{f}_i = Im\vec{f}$, $\vec{g}_r = Re\vec{g}$, $\vec{g}_i = Im\vec{g}$. Consider the integral

$$I = \int(\vec{f}^2 + \vec{g}^2)d^3x$$

By expressing $\vec{f}(\vec{x})$ and $\vec{g}(\vec{x})$ through their Fourier transforms one gets $I = 0$. The vanishing of real and imaginary parts of I gives

$$I_r = \int(\vec{f}_r^2 + \vec{g}_r^2 - \vec{f}_i^2 - \vec{g}_i^2)d^3x = 0, \quad I_i = \int(\vec{f}_r\vec{f}_i + \vec{g}_r\vec{g}_i)d^3x = 0. \tag{2.20}$$

Equations (2.18) and (2.19) mean that some quantities composed of vectors $\vec{f}$ and $\vec{g}$ satisfy continuity equations. Their physical interpretation is not clear as the space integrals of the corresponding densities are equal to zero.

Further, we observe that Eqs.(2.15) are equivalent to

$$\frac{\partial}{\partial t}(\vec{f}+i\vec{g}) = -i\,curl(\vec{f}+i\vec{g}), \quad div(\vec{f}+i\vec{g}) = 0. \tag{2.21}$$

From the first of these equations and its complex conjugate one gets

$$\frac{\partial}{\partial t}[|\vec{f}|^2+|\vec{g}|^2+i(\vec{g}\vec{f}^*-\vec{f}\vec{g}^*)] + c\,div[\vec{f}\times\vec{g}^*+\vec{f}^*\times\vec{g}+i(\vec{f}\times\vec{f}^*+\vec{g}\times\vec{g}^*)] = 0.$$

Combining this equation with (2.16) one obtains

$$\frac{\partial}{\partial t}(\vec{g}\vec{f}^*-\vec{f}\vec{g}^*) + c\,div(\vec{f}\times\vec{f}^*+\vec{g}\times\vec{g}^*) = 0. \tag{2.22}$$

The separation of real and imaginary parts gives one relation independent of the previous ones,

$$\frac{\partial}{\partial t}(\vec{f}_i\vec{g}_r-\vec{f}_r\vec{g}_i) + c\,div(\vec{f}_r\times\vec{f}_i+\vec{g}_r\times\vec{g}_i) = 0.$$

To clarify the physical meaning of Eq.(2.22), consider the integral

$$i\int(\vec{g}\vec{f}^*-\vec{f}\vec{g}^*)d^3x.$$

Substituting for $\vec{f}(\vec{x})$ and $\vec{g}(\vec{x})$ their Fourier transformations one gets

$$i\int(\vec{g}\vec{f}^*-\vec{f}\vec{g}^*)d^3x = 2(2\pi)^3\, i\int\frac{d^3k}{k}\vec{k}(\vec{f}(\vec{k})\times\vec{f}^*(\vec{k})). \tag{2.23}$$

Since $\vec{f}(\vec{k})$ is orthogonal to $\vec{k}$, we develop it over the unit vectors of right and left polarizations: $\vec{f} = f_R\vec{e}_R + f_L\vec{e}_L$. Inserting this into Eq.(2.23) and making use of the orthonormal properties of polarization vectors $\vec{e}$ ($\vec{e}^{\,*}_R\vec{e}_R = \vec{e}^{\,*}_L\vec{e}_L = 1$, $\vec{e}^{\,*}_R\times\vec{e}_R = \hat{\vec{k}}$, $\vec{e}^{\,*}_L\times\vec{e}_L = -\hat{\vec{k}}$) one gets:

$$i\int\frac{d^3k}{k}\vec{k}(\vec{f}(\vec{k})\times\vec{f}^*(\vec{k})) = \int(|f_R|^2-|f_L|^2)d^3k. \tag{2.24}$$

On the other hand, the photon number may also be expressed through f_R and f_L,

$$N = \frac{1}{\hbar}\int(|f_R|^2+|f_L|^2)d^3k. \tag{2.25}$$

Comparing this with Eq.(2.24) we conclude that

$$N_- = \frac{1}{\hbar}\int(|f_R|^2-|f_L|^2)d^3k, \tag{2.26}$$

coincides with the number difference of the right- and left- polarized photons. It turns out that Eq.(2.22) describes the conservation of the difference of right and left photons numbers. Likewise the photon number, N_- is a conserved relativistic invariant quantity [17].
As $\vec{f},\vec{g}$ satisfy the same equations as $\vec{E},\vec{H}$, it is possible to write out for them the set of zilch-type invariants similar to those obtained by Lipkin [18] and Ragusa [19] for $\vec{E}$ and $\vec{H}$. For example, take the densities

$$\rho_\epsilon{}^f(\vec{x},t) = \frac{i}{16\pi^3}[\vec{f}^*(\vec{x},t)\dot{\vec{f}}(\vec{x},t)-\vec{f}(\vec{x},t)\dot{\vec{f}}^*(\vec{x},t)] \quad \rho_\epsilon{}^g(\vec{x},t) = \frac{i}{16\pi^3}[\vec{g}^*(\vec{x},t)\dot{\vec{g}}(\vec{x},t)-\vec{g}(\vec{x},t)\dot{\vec{g}}^*(\vec{x},t)], \tag{2.27}$$

satisfying the continuity equations

$$\dot{\rho}_\epsilon{}^f + div\,\vec{J}^f_\epsilon = 0, \quad \dot{\rho}_\epsilon{}^g + div\,\vec{J}^g_\epsilon = 0, \tag{2.28}$$

with

$$\vec{J}^f_\epsilon = \frac{ic^2}{16\pi^3}\sum_i(f_i\vec{\nabla}f_i^* - f_i^*\vec{\nabla}f_i) \quad\text{and}\quad \vec{J}^g_\epsilon = \frac{ic^2}{16\pi^3}\sum_i(g_i\vec{\nabla}g_i^* - g_i^*\vec{\nabla}g_i). \tag{2.29}$$

The space integrals of ρ_ϵ are given by

$$\int\rho_\epsilon{}^f d^3x = \int\rho_\epsilon{}^g d^3x = \int\omega|\vec{f}(\vec{k})|^2d^3k. \tag{2.30}$$

Although these integrals are equal to the electromagnetic energy, the space density $\rho_\epsilon{}^f$ and $\rho_\epsilon{}^g$ do not coincide with the electromagnetic density ρ_{EMF}. In particular, ρ_ϵ may take negative values and may have tails in the regions where $\vec{E}$ and $\vec{H}$ are equal to zero.
Equations (2.15) may also be rewritten in a covariant form. Putting $h^{0i} = f_i$, $h^{ij} = \epsilon_{ijk}g_k$ one gets

$$\frac{\partial h^{\mu\nu}}{\partial x^\nu} = 0, \quad (\frac{\partial h^{\mu\nu}}{\partial x^\rho})_{cycl} = 0. \tag{2.31}$$

From the fact that equations (2.26) have a covariant form it does not follow that $h^{\mu\nu}$ are transformed like tensors (more accurately, they transform according to the nonlocal Lorentz transformation [9,10]) when one compares them in two different reference frames. However, the form of these equations is Lorentz-invariant. This means that if $h^{\mu\nu}$ satisfy Eqs.(2.31) in one particular frame, they satisfy the same equations in any other reference frame, as well. The existence of the invariants different from the energy, momenta and angular momenta of the free EMF is not a new thing at all. In addition to refs.[18,19] mentioned above, we refer to books [20,21] where the history of the findings of EMF invariants is presented in detail. The symmetry properties of the Maxwell equations under the transformations of the E(2)-like little group [22] leaving the four-momentum invariant were discussed recently in [15]. It is the aim of this consideration to investigate some of the afore-mentioned invariants numerically and clarify their physical meaning.
External currents.
Note that when passing from (2.3) to (2.5) we have not used the second of Eqs.(2.4). Thus, we suggest the validity of Eq.(2.3) in the presence of the current source $\vec{j}(\vec{r},t)$. In the momentum space the Maxwell equations are

$$i\vec{k}\times\vec{E}(\vec{k}) = -\frac{1}{c}\dot{\vec{H}}(\vec{k}), \quad i\vec{k}\times\vec{H}(\vec{k}) = \frac{1}{c}\dot{\vec{E}}(\vec{k}) + \frac{4\pi}{c}\vec{j}(\vec{k}), \tag{2.32}$$

Here $\vec{j}(\vec{k},t)$ is the Fourier transform of $\vec{j}(\vec{x},t)$ ($\vec{j}(\vec{x},t) = \int\vec{j}(\vec{k},t)\exp(i\vec{k}\vec{r})d^3k$). Substituting (2.3) and (2.10) into (2.32) one recovers the following equations for $\vec{f}(\vec{k})$ and $\vec{g}(\vec{k})$:

$$\dot{\vec{f}}(\vec{k},t) = -i\omega\vec{f}(\vec{k}) - \frac{4\pi^2}{\sqrt{\omega}}\vec{j}(\vec{k},t), \quad \dot{\vec{g}}(\vec{k},t) = -i\omega\vec{g}(\vec{k}) - \frac{4\pi^2}{\sqrt{\omega}}\hat{\vec{k}}\times\vec{j}(\vec{k},t). \tag{2.33}$$

We rewrite these equations in the coordinate space

$$curl\vec{f} + \frac{1}{c}\dot{\vec{g}} = -\frac{4\pi^2}{c}\vec{J}_2, \quad curl\vec{g} - \frac{1}{c}\dot{\vec{f}} = \frac{4\pi^2}{c}\vec{J}_1, \tag{2.34}$$

with

$$J_1 = \int\frac{d^3k}{\sqrt{\omega}}\exp(i\vec{k}\vec{r})\vec{j}(\vec{k}) \quad\text{and}\quad J_2 = \int\frac{d^3k}{\sqrt{\omega}}\exp(i\vec{k}\vec{r})\hat{\vec{k}}\times\vec{j}(\vec{k}). \tag{2.35}$$

From this one finds the following equations for $\vec{f}$ and $\vec{g}$

$$\Box\vec{f} = \frac{4\pi^2}{c^2}\dot{\vec{J}}_1 + \frac{4\pi^2}{c}curl\vec{J}_2, \quad \Box\vec{g} = \frac{4\pi^2}{c^2}\dot{\vec{J}}_2 - \frac{4\pi^2}{c}curl\vec{J}_1. \tag{2.36}$$

3 Numerical results

We apply the consideration of the previous section to the following simple model. Let at the initial moment $t = 0$ the magnetic field H equals zero everywhere while the electric field E differs from zero only inside the sphere S of radius a:

$$\vec{E}(\vec{r})|_{t=0} = \vec{n}_\phi \sin\theta E_0 \Theta(a - r). \tag{3.1}$$

(It is therefore suggested that only the ϕ component of $\vec{E}$ differs from zero for $t = 0$). Making the Fourier transformation of E one finds from (2.3) the Fourier components of $\vec{f}(\vec{r})$. Only its ϕ component differs from zero:

$$f_0^\phi(k) = \frac{-iE_0}{2\pi\sqrt{(\omega)}} \sin\theta_k \frac{\psi(ka)}{k^3}, \quad \psi(ka) = 2 - 2\cos ka - ka\sin ka. \tag{3.2}$$

Here θ_k is the polar angle in the momentum space ($k_z = k\cos\theta_k$). We normalize the wave function to N photons:

$$\frac{1}{\hbar}\int |\vec{f}(\vec{k})|^2 d^3k = N,$$

which gives

$$E_0 = -\frac{3\sqrt{\pi\hbar c N}}{\sqrt{2(\ln 2 - 1/4)a^2}}.$$

Then, using Eqs.(2.2),(2.3),(2.8) and (2.14) we evaluate $\vec{E}(\vec{x},t)$, $\vec{H}(\vec{x},t)$, $\vec{f}(\vec{x},t)$ and $\vec{g}(\vec{x},t)$. It turns out that only the ϕ components of $\vec{E}$ and $\vec{f}$ differ from zero, while $\vec{H}$ and $\vec{g}$ have nonvanishing r and θ components:

$$E_\phi = -C_0\sqrt{\hbar c}\frac{\sin\theta}{r}I_E^c, \quad H_r = 2C_0\sqrt{\hbar c}\frac{\cos\theta}{r^2}I_{H_r}^s, \quad \vec{H}_\theta = -C_0\sqrt{\hbar c}\frac{\sin\theta}{r}I_{H_\theta}^s,$$
$$f_\phi = -\pi C_0\sqrt{\hbar}\frac{\sin\theta}{r}I_f^\phi, \quad g_r = 2i\pi C_0\frac{\cos\theta}{r}\sqrt{\hbar}I_g^r, \quad g_\theta = i\pi C_0\sqrt{\hbar}\frac{\sin\theta}{r}I_g^\theta. \tag{3.3}$$

Here $C_0 = \frac{3}{a^3}\sqrt{\frac{2N}{\pi(\ln 2 - 1/4)}}$,

$$I_E{}^c = \int dk\left(\frac{\sin kr}{kr} - \cos kr\right)\frac{\psi(ka)}{k^2}\cos\omega t, \quad I_{H_r}^s = \int dk\left(\frac{\sin kr}{kr} - \cos kr\right)\frac{\psi(ka)}{k^3}\sin\omega t, \tag{3.4}$$

$$I_{H_\theta}^s = \int dk\left[\sin kr\left(1 - \frac{1}{k^2r^2}\right) + \frac{\cos kr}{kr}\right]\frac{\psi(ka)}{k^2}\sin\omega t, \quad I_f^\phi = \int dk\left(\frac{\sin kr}{kr} - \cos kr\right)\frac{\psi(ka)}{k^{5/2}}\exp(-i\omega t),$$

$$I_g^r = \int dk\left(\frac{\sin kr}{k^2r^2} - \frac{\cos kr}{kr}\right)\frac{\psi(ka)}{k^{5/2}}e^{-i\omega t}, \quad I_g^\theta = -\int dk\left[\sin kr\left(1 - \frac{1}{k^2r^2}\right) + \frac{\cos kr}{kr}\right]\frac{\psi(ka)}{k^{5/2}}e^{-i\omega t}.$$

All these integrals can be taken in a closed form.
The density of EMF is given by

$$\rho^\epsilon_{EMF} = \frac{1}{8\pi}(\vec{H}^2 + \vec{E}^2). \tag{3.5}$$

The time evolution of this density is shown in the upper parts of figs.1,2. [1] For the definiteness we choose $a = 0.25, N = 1, \sin\theta = \sqrt{2/3}$ (as for this angle ρ_f, ρ_g and ρ_{EMF} coincide with their values averaged over the sphere of an arbitrary radius). As expected, the

[1] In what follows the radial variables are measured in fm, the energy in meV, the energy density in $meV \cdot fm^{-3}$, the photon density in fm^{-3}.

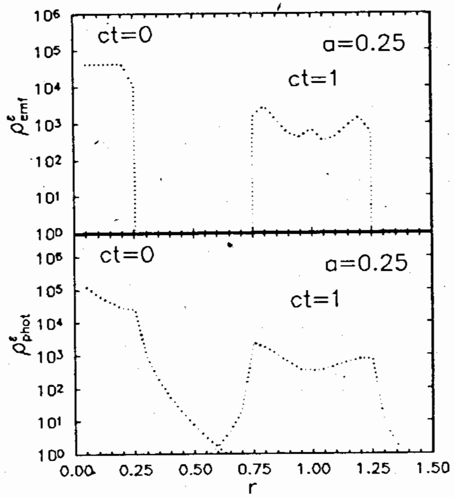

Fig.1. The energy densities for the electromagnetic wave (upper part) and photon (lower part).

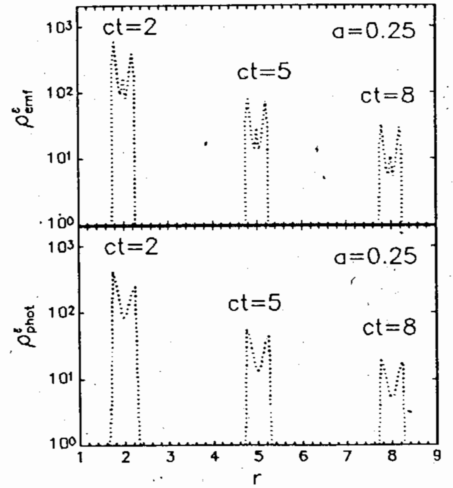

Fig 2. Same as Fig.1 for $ct = 2, 5, 8$.

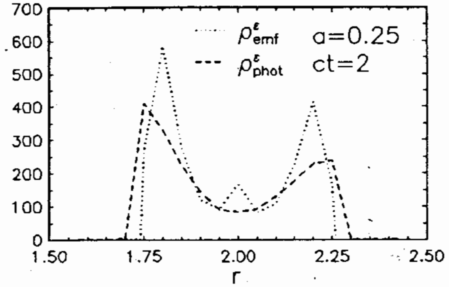

Fig.3. The magnified image of energy density for the electromagnetic wave and photon for $ct = 2$. The first is exactly zero outside the interval $ct - a < r < ct + a$, while the latter has small tail there.

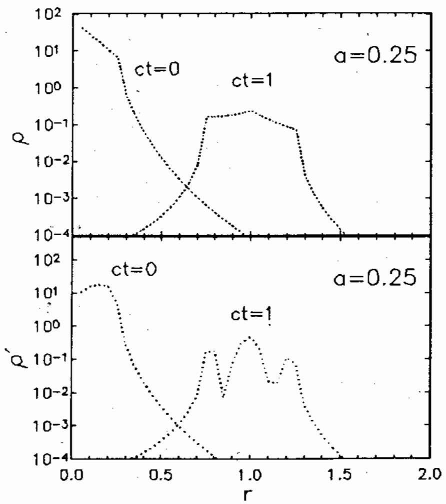

Fig 4. The photon densities corresponding to the positive-frequency wave functions (upper part) and the wave functions containing both positive and negative frequencies (lower part) for $ct = 0, 1$.

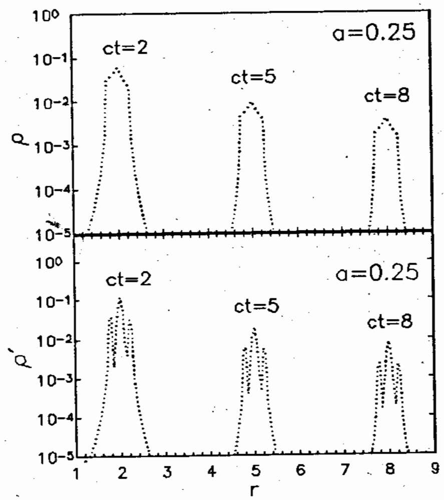

Fig 5. Same as Fig.4 for $ct = 2, 5, 8$.

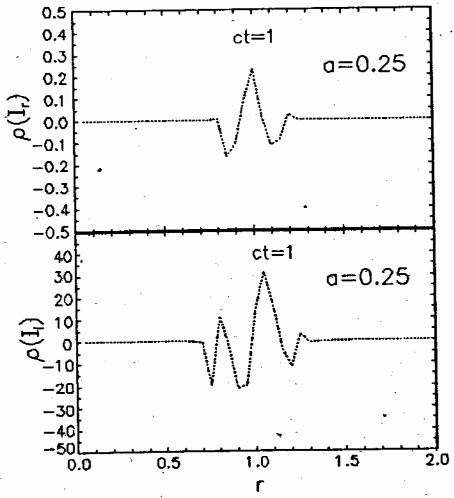

Fig 6. The densities of conserved quantities $\rho(I_r)$ and $\rho(I_i)$ The space integral of them is equal to zero.

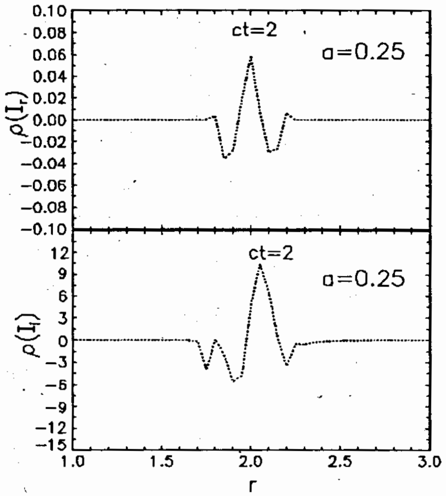

Fig 7. Same as Fig.6 for $ct = 2$.

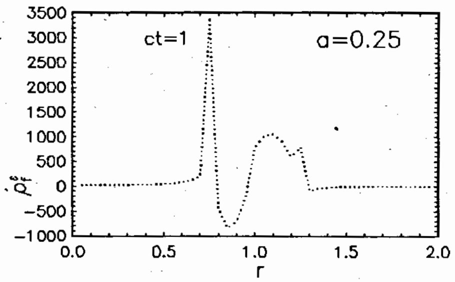

Fig 8. The "pseudo"-energy density for $ct = 1$. The integral of ρ_f^ϵ coincides with the photon energy.

density of EMF energy equals zero for the space points ($r > ct + a$) for which the action has not arrived from the nearest parts of S as well as for those points ($r < ct - a$) for which the action has passed from the most remote part of S. The EMF initially confined to the sphere S propagates outward it with the light velocity c.
Sometimes the photon WF is identified with the positive frequency part of E and H [11,12]:

$$\mathcal{E} = E_+ = -C_0\sqrt{\hbar c}\frac{\sin\theta}{2r}I_B^{(+)}, \quad \mathcal{H}_r = H_+^r = iC_0\sqrt{\hbar c}\frac{\cos\theta}{r^2}I_{H_r}^{(+)},$$

$$\mathcal{H}_\theta = H_+^\theta = -\frac{1}{2}iC_0\sqrt{\hbar c}\frac{\sin\theta}{r}I_{H_\theta}^{(+)},$$

$$I_B^{(+)} = I_B^c - iI_B^s, \quad I_{H_r}^{(+)} = I_{H_r}^c - iI_{H_r}^s, \quad I_{H_\theta}^{(+)} = I_{H_\theta}^c - iI_{H_\theta}^s, \tag{3.6}$$

$$I_B^s = \int dk(\frac{\sin kr}{kr} - \cos kr)\frac{\psi(ka)}{k^2}\sin\omega t, \quad I_{H_r}^c = \int dk(\frac{\sin kr}{kr} - \cos kr)\frac{\psi(ka)}{k^3}\cos\omega t,$$

$$I_{H_\theta}^c = \int dk[\sin kr(1 - \frac{1}{k^2r^2}) + \frac{\cos kr}{kr}]\frac{\psi(ka)}{k^2}\cos\omega t. \tag{3.7}$$

The following equations are valid:

$$\int \rho(E_+)dV = \int \rho(H_+)dV = \int \rho_{phot}^\epsilon dV = \int \omega|\vec{f}(\vec{k})|^2 d^3k.$$

Here

$$\rho(E_+) = \frac{1}{2\pi}E_+E_+^*, \quad \rho(H_+) = \frac{1}{2\pi}H_+H_+^*, \quad \rho_{phot}^\epsilon = \frac{1}{2}[\rho(E_+) + \rho(H_+)]. \tag{3.8}$$

These quantities may be considered as photon energy densities. Contrary to ρ_{BMF} given by (3.5) the densities (3.8) cannot be localized (see lower parts of figs.1,2). Closely following ρ_{BMF}^ϵ they have small tails in the region where $\rho_{BMF}^\epsilon = 0$. The magnified representation of ρ_{BMF}^ϵ and ρ_{phot}^ϵ densities for particular value $ct = 2$ is shown in fig.3.

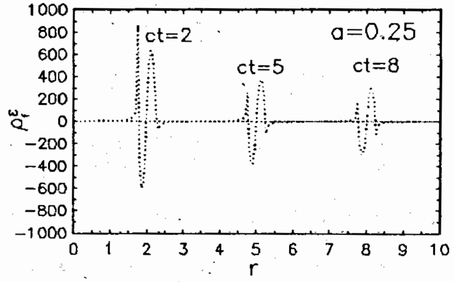

Fig 9. Same as Fig.8 for $ct = 2, 5, 8$.

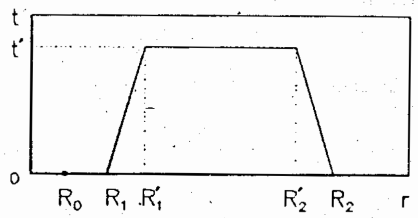

Fig 10. Demonstrates the absence of the superluminal spreading for the particle positive-definite density initially confined to the region $r < R_0$ and satisfying the continuity equation.

Turning to the time evolution of the photon densities ρ_f, ρ_g and ρ we observe that they, being maximal in those space regions where ρ_{BMF}^ϵ differ from zero, have small tails outside them (these tails are in fact so small as indistinguishable on the most of figures). The typical behavior of $\rho = (\rho_f + \rho_g)/2$ is shown in the upper parts of figs 4,5.

The local differential continuity equations (2.18) and (2.19) suggest the existence of the conserved densities

$$\rho(I_r) = \vec{f}_r^2 + \vec{g}_r^2 - \vec{g}_i^2 - \vec{f}_i^2 \quad \text{and} \quad \rho(I_i) = \vec{f}_r\vec{f}_i + \vec{g}_r\vec{g}_i, \tag{3.9}$$

the space integral of which is zero (figs.6,7). Likewise ρ_f, ρ_g and ρ they follow closely the electromagnetic density ρ_{EMF} and practically are equal to zero outside the region where $\vec{E}$ and $\vec{H}$ differ from zero. So far the physical meaning of these densities remains unclear to us. Probably, they characterize the internal structure of the photon.
Now we turn to the density ρ_e^f defined by Eqs.(2.27). Its typical behavior is shown in figs.8,9. The quantity ρ_e^f has the following properties: it has small tails outside the region where $\vec{E}$ and $\vec{H}$ differ from zero. Inside that region ρ_e^f exhibits one-two oscillations. The spatial integral of ρ_e^f is equal to the electromagnetic energy. The physical meaning of the densities ρ_e^f and ρ_e^g is also a mystery for us.
It would be interesting to look at the spatial distribution

$$\rho(N_-) = \frac{i}{16\pi^3}(\vec{g}\vec{f}^* - \vec{f}\vec{g}^*) \tag{3.10}$$

of the difference of right and left photons numbers N_- (see Eqs.(2.23)-(2.25)). Yet, this density identically vanishes for the treated photon configuration defined by Eqs. (3.2) and (3.3).
We evaluate now the photon current densities $\vec{J}_1$ and $\vec{J}_2$ introduced at the end of sect.2 for the particular current density. We choose it in the form

$$\vec{j}(\vec{x},t) = f_0(t)\,\vec{curl}(n_z\delta^3(\vec{x})),$$

simulating the infinitesimal magnetic dipole moment placed at the origin and directed along the z axis. Its time dependence is governed by the function $f_0(t)$. At first we find the Fourier transform of $\vec{j}$:

$$j_x(k) = \frac{ik_y}{8\pi^3}f_0, \quad j_y(k) = -\frac{ik_x}{8\pi^3}f_0, \quad j_z(k) = 0.$$

The components of the photon current density entering into Eqs. (2.33) and (2.35) are given by

$$J_{1x} = -5\dot{j}_0\frac{\sin\theta\sin\phi}{r^{7/2}}, \quad J_{1y} = 5\dot{j}_0\frac{\sin\theta\cos\phi}{r^{7/2}}, \quad J_{1z} = 0, \quad j_0 = \frac{f_0}{8\pi^{3/2}}\frac{1}{\sqrt{2c}}$$

$$J_{2x} = -21i\ddot{j}_0\frac{\sin\theta\cos\theta\cos\phi}{r^{7/2}}, \quad J_{2x} = -21i\ddot{j}_0\frac{\sin\theta\cos\theta\sin\phi}{r^{7/2}}, \quad J_{2z} = -12i\ddot{j}_0\frac{1-\frac{7}{4}\sin^2\theta}{r^{7/2}}.$$

4 On the Ehrenfest-Pauli objections

Although the Pauli objections were primarily concerned the photon densities suggested by Landau and Peierls, they, in fact, are equally applied to the photon densities discussed in the previous section.
The first objection by Pauli is that at the point where $\vec{E}, \vec{H} \neq 0$ but $\rho = 0$ the photon density looses its sense as it is not clear what means the absence of photons at the point where $\vec{E}, \vec{H} \neq 0$ (this contradicts the generally accepted identification of photons as carriers of $\vec{E}$ and $\vec{H}$). It follows from the numerical results of the previous section that there are no space regions where photon density equals zero but $\vec{E}, \vec{H} \neq 0$. This is also confirmed by the Hegerfeldt theorem discussed in the next section.
On the other hand, the nonvanishing of the photon density in the space regions where $\vec{E} = \vec{H} = 0$ means that the role played by photons is not limited by that of $\vec{E}, \vec{H}$ carriers. This claim is supported by the existence of the invariants discussed in sect.2 as well as the new helicity invariant obtained recently in [17].
The second objection is due to the complicated behavior of the photon density under the Lorentz transformation and to the absence of the covariant 4-current satisfying the continuity equation. In our case we have the 4-vector $J^\mu_- = (\rho, \frac{c}{2}(\vec{f}\times\vec{g}^* + \vec{f}^*\times\vec{g}))$ satisfying the continuity equation and having a positive definite density. The components of J^μ exhibit the nonlocal Lorentz transformation when one passes from one reference frame to another. Yet, the complicated nature of this transformation does not destroy the Lorentz invariance of the integral $\int \rho d^3x$ (see the next section). Thus, this Pauli objection is considerably weakened. After all, one may disregard the complicated nature of the above transformation if the following procedure is adopted. It is known how $\vec{E}$ and $\vec{H}$ behave under the Lorentz transformation. Having $\vec{E}, \vec{H}$ in one particular frame we evaluate $\vec{E}(\vec{k}), \vec{H}(\vec{k}), \vec{f}(\vec{K}), \vec{g}(\vec{k}), \vec{f}(\vec{x})$ and $\vec{g}(\vec{x})$. Using the Lorentz transformation to obtain $\vec{E}, \vec{H}$ in another reference frame and performing the same procedure as above we evaluate $\vec{f}, \vec{g}$ and, finally, J^μ in a new reference frame.

5 On the Hegerfeldt theorem and all that

Klein-Gordon equation
This theorem states [7] that a wave function being originally confined to the space region S and, thus, giving zero probability to find a particle outside S, at a later time gives a finite probability to find the particle at a distance $r > ct$ away from S. At first glance this implies that a particle travels with a superluminal velocity. As an illustration, consider the Klein-Gordon equation

$$(\Box + \mu^2)\Psi = 0, \quad \mu = mc/\hbar, \tag{5.1}$$

We seek its solution in the form

$$\Psi(\vec{x},t) = \int \exp[i(\vec{k}\vec{x} - \omega_\mu t)]\Psi(\vec{k})d^3k, \quad \omega_\mu = ck_\mu, \quad k_\mu = \sqrt{k^2+\mu^2}. \tag{5.2}$$

We express $\Psi(\vec{k})$ through the initial value of $\Psi(\vec{x},t)$

$$\Psi(\vec{x},t) = \int G_\mu(\vec{x}-\vec{x}',t)\Psi_0(\vec{x}')d^3x', \quad G_\mu = \frac{1}{(2\pi)^3}\int \exp[i\vec{k}(\vec{x}-\vec{x}') - i\omega_\mu t]d^3k. \tag{5.3}$$

Let $\Psi_0(\vec{x}) = \Psi(\vec{x},0)$ be zero outside the space region S. Then, the Hegerfeldt theorem states that $\Psi(\vec{x},t)$ will be everywhere nonzero for $t > 0$. To see the reason for this strange behavior, we observe that (5.2) is a very special solution of Eq.(5.1). Its general solution is completely determined by the initial values of Ψ and its time derivative [23],

$$\Psi(\vec{x},t) = \int d^3x'[\frac{\partial D_\mu(\vec{x}-\vec{x}',t)}{\partial t}\Psi_0(\vec{x}') + D_\mu(\vec{x}-\vec{x}',t)\dot{\Psi}_0(\vec{x}')], \tag{5.4}$$

$$D_\mu(\vec{x},t) = \frac{1}{(2\pi)^3}\int\frac{d^3k}{k_\mu}\exp(i\vec{k}\vec{x})\sin\omega_\mu t = \frac{1}{2\pi^2 r}\int\frac{k}{k_\mu}dk\sin kr\sin k_\mu ct.$$

The following properties of D_μ,

$$D_\mu(\vec{x},0) = 0, \quad \frac{\partial D_\mu(\vec{x},t)}{\partial t}|_{t=0} = c\,\delta^3(\vec{x}),$$

guarantee the fulfillment of the initial conditions. Further, $D_\mu(\vec{x},t) = 0$ outside the light cone. Equation (5.4) tells us that the disappearance of the initial wave function outside the space region S is not enough for the disappearance of $\Psi(\vec{x},t)$ outside the light cone. The latter takes place only if the initial values of both Ψ and $\dot{\Psi}$ equal zero outside S.

Turning again to Ψ defined by Eq.(5.3) we evaluate the initial value of its time derivative

$$\dot{\Psi}_0 = \frac{\partial \Psi(\vec{x},t)}{\partial t}|_{t=0} = -\frac{i}{(2\pi)^3}\int \omega_\mu exp[i\vec{k}(\vec{x}-\vec{x}')]\Psi(\vec{x}',0)d^3kd^3x'. \tag{5.5}$$

It then follows from this that $\dot{\Psi}_0(\vec{x})$ differs from zero everywhere even if $\Psi_0(\vec{x})$ disappears outside S. Substituting initial values Ψ_0 and $\dot{\Psi}_0$ given by (5.5) into (5.4) we arrive at (5.3). Thus, we obtain an alternative interpretation of the Hegerfeldt theorem. Consider the solution (5.3) of Eq.(5.1) corresponding to $\Psi_0 = 0$ outside S. Then, nonzero values of Ψ outside the light cone are due to the nonvanishing of $\dot{\Psi}_0$ outside S.

To clarify the situation, consider the energy density of the scalar field

$$|grad\Psi|^2 + \frac{1}{c^2}|\dot{\Psi}|^2 + \mu^2|\Psi|^2.$$

Now, if $\Psi_0 = 0$, $\dot{\Psi}_0 \neq 0$ outside S, the initial energy density also differs from zero outside S. This points out on the unphysical nature of the afore-mentioned initial condition (as it is not clear what means the absence of the particle probability density and the presence of its energy density in the same space region) leading to the superluminal spreading of the probability density.

As far as we know, the first estimations of the effects arising from the superluminal spreading of the Klein-Gordon wave function as well as the possibility of their experimental verification were performed by Shirokov [24].

Wave equation.

All these considerations remain valid for the wave equation if one puts $m = 0$ in Eqs.(5.1)-(5.5). The general solution is completely determined by the initial values of Ψ and its time derivative

$$\Psi(\vec{x},t) = \int d^3x'[\frac{\partial D_0(\vec{x}-\vec{x}',t)}{\partial t}\Psi_0(\vec{x}') + D_0(\vec{x}-\vec{x}',t)\dot{\Psi}_0(\vec{x}')], \tag{5.6}$$

$$D_0(\vec{x},t) = \frac{1}{(2\pi)^3}\int\frac{d^3k}{k}\exp(i\vec{k}\vec{x})\sin\omega_t = \frac{1}{2\pi^2 r}\int dk \sin kr \sin kct = \frac{1}{4\pi r}[\delta(r-ct) - \delta(r+ct)].$$

The function $\vec{f}(\vec{x})$ defined by Eq.(2.8) and its time derivative may be rewritten as

$$\vec{f}(\vec{x},t) = \frac{1}{(2\pi)^3}\int \exp[i\vec{k}(\vec{x}-\vec{x}') - i\omega t]\vec{f}(\vec{x}',0)d^3kd^3x', \tag{5.7a}$$

$$\dot{\vec{f}}(\vec{x},t) = -\frac{i}{(2\pi)^3}\int \omega\exp[i\vec{k}(\vec{x}-\vec{x}') - i\omega t]\vec{f}(\vec{x}',0)d^3kd^3x'. \tag{5.7b}$$

Let the initial value of $\vec{f}$ be zero outside S. It follows from the Hegerfeldt theorem that $\vec{f}$ will be different from zero everywhere for $t > 0$. It turns out that the total density $\rho = (|\vec{f}|^2 + |\vec{g}|^2)$ will be different from zero everywhere for every moment of time even if $\vec{f} = 0$ for this particular moment. To prove, this we put $t = 0$ in (5.7),

$$\dot{\vec{f}}(\vec{x},0) = -\frac{i}{(2\pi)^3}\int \omega\exp[i\vec{k}(\vec{x}-\vec{x}')]\vec{f}(\vec{x}',0)d^3kd^3x'.$$

Obviously, this function differs from zero everywhere. It follows from Eqs.(2.15) that $\vec{g} \neq 0$ everywhere for this particular moment of time. This completes the proof.

For the wave equation (in addition to the expression similar to (5.4)) there is known another formula (the Poisson one, see,e.g., Smirnov [25] or Courant [26] treatises):

$$\Psi(x,y,z,t) = \frac{t}{4\pi}\int d\Omega'\dot{\Psi}_0(\alpha,\beta,\gamma) + \frac{\partial}{\partial t}[\frac{t}{4\pi}\int d\Omega'\Psi_0(\alpha,\beta,\gamma)], \tag{5.8}$$

where Ψ_0 and $\dot{\Psi}_0$ are the initial values of Ψ and $\dot{\Psi}$. Further, $d\Omega' = \sin\theta' d\theta' d\phi'$, $\alpha = x + ct\sin\theta'\cos\phi'$, $\beta = y + ct\sin\theta'\sin\phi'$, $\gamma = z + ct\cos\theta'$. It would be useful to have a similar closed expression for the solutions of Klein-Gordon equations.

We observe that Ψ given by (5.2) has the constant (i.e., independent of time) norm. Being initially confined to the space region S, it fills the whole space for a later time. On the other hand, if for Ψ given by (5.4) the initial values of Ψ and $\dot{\Psi}$ lie inside S, then Ψ propagates with the light velocity, but its norm $\int|\Psi|^2dV$ changes with time.

Maxwell equations.

The following qualitative considerations show that it is impossible to localize the positive frequency solutions of the Maxwell equations. For this we put

$$\Psi_+ = \vec{\mathcal{E}} + i\vec{\mathcal{H}},$$

where $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ were defined by Eqs.(3.6). Evidently, $\vec{\Psi}$ satisfies the equations

$$\frac{i}{c}\dot{\vec{\Psi}} = curl\vec{\Psi}, \quad div\vec{\Psi} = 0. \tag{5.9}$$

According to our assumption $\vec{\Psi}$ contains only the positive frequencies,

$$\Psi(\vec{x},t) = \int \vec{\Psi}(\vec{k})e^{i(\vec{k}\vec{x}-\omega t)}d^3k. \tag{5.10}$$

It turns out that $\vec{\Psi}(\vec{k})$ satisfies the equation

$$\frac{\omega}{c}\vec{\Psi}(\vec{k}) = i\vec{k}\times\vec{\Psi}(\vec{k}), \quad \omega = \sqrt{k_x^2 + k_y^2 + k_z^2}. \tag{5.11}$$

Using (5.9) we express $\vec{\Psi}(\vec{k})$ through the initial value of $\vec{\Psi}(\vec{x},t) = \vec{\Psi}_0(\vec{x})$:

$$\vec{\Psi}(\vec{k}) = \frac{1}{(2\pi)^3}\int\vec{\Psi}_0(\vec{x})e^{-i\vec{k}\vec{x}}d^3x. \tag{5.12}$$

Let $\vec{\Psi}_0$ be localized inside a finite space region around the origin. Then, expanding the exponential factor and integrating over x one concludes that $\vec{\Psi}(\vec{k})$ and therefore RHS of (5.11) are entire functions of k_x, k_y, k_z. On the other hand, the LHS of the same equation is not an entire function due to the factor $\omega = \sqrt{k_x^2 + k_y^2 + k_z^2}$ in it.

This contradiction means that the positive frequency solutions of Maxwell equations cannot be localized exactly.

On the other hand, let the solution of (5.9) be a superposition of the positive- and negative-frequency solutions:

$$\vec{\Psi}(\vec{x},t) = \int e^{i\vec{k}\vec{x}}[\vec{\Psi}_+(\vec{k})e^{-i\omega t} + \vec{\Psi}_-(\vec{k})e^{i\omega t}].$$

Substituting this Eq. into (5.9) one gets

$$\omega(\vec{\Psi}_+ - \vec{\Psi}_-) = i\vec{k}\times(\vec{\Psi}_+ + \vec{\Psi}_-). \tag{5.13}$$

Now, if $\vec{\Psi}(\vec{x},0)$ is localized in the space region including the origin, the same reasoning as above shows that RHS of (5.13) is an entire function of k_x, k_y, k_z. However, no contradiction arises since different functions $(\bar{\Psi}_+ + \bar{\Psi}_-)$ and $(\bar{\Psi}_+ - \bar{\Psi}_-)$ enter into right and left hand sides of (5.13). Thus, the solutions of Maxwell equations containing both positive and negative frequencies can in principle be localized. The appearance of positive and negative frequencies is a necessary but not sufficient condition for the localization. To illustrate this, consider two vectors,

$$\vec{f}_B(x) = \frac{1}{2\pi}\int e^{i\vec{k}\vec{r}}(\vec{f}_k + \vec{f}^*_{-k})d^3k \quad \text{and} \quad \vec{f}_H(x) = \frac{c}{2\pi}\int e^{i\vec{k}\vec{r}}\frac{1}{\omega}\vec{k}\times(\vec{f}_k - \vec{f}^*_{-k})d^3k.$$

It is easy to check that

$$\int \rho' d^3x = \frac{1}{\hbar}\int |\vec{f}(\vec{k})|^2 d^3k,$$

coincides with the photon number N. However, the space density

$$\rho'(\vec{x}) = \frac{1}{8\pi\hbar}[(\vec{f}_B(x))^2 + (\vec{f}_H(x))^2]$$

is not localizable for $\vec{f}(k)$ defined by Eq.(3.2) although ρ'_f contains both positive and negative frequencies. The typical behavior of ρ' is shown in lower parts of figs. 4,5. A detailed analysis shows that it is impossible to localize $\vec{E}$, $\vec{H}$ and $\vec{f}_B$, $\vec{f}_H$ (and, therefore,ρ') simultaneously. The initial condition (3.1) corresponds to the localization of $\vec{E}$ and $\vec{H}$ at the moment $t = 0$. The initial conditions may be chosen so as to localize $\vec{f}_B$ and $\vec{f}_H$. Yet, in this case $\vec{E}$ and $\vec{H}$ will be distributed over the whole space and, therefore, nonlocalizable. Due to the greater physical meaning of $\vec{E}$ and $\vec{H}$ we have preferred the first possibility.

We turn now to the Sipe paper [13]. The wave function used there was chosen to be the positive definite part of $\vec{E}$, i.e.,

$$\vec{\psi}_s(\vec{k}) = \frac{\sqrt{\omega}}{2\pi}\vec{f}(\vec{k}), \quad \vec{\psi}_s(\vec{r}) = \int d^3k \exp(i\vec{k}\vec{x})\vec{\psi}_s(\vec{k}). \tag{5.14}$$

The value of $\vec{\psi}_s$ at the moment t is related to its initial value by the expression similar to (5.7)

$$\vec{\psi}_s(\vec{x},t) = \frac{1}{(2\pi)^3}\int \exp[i\vec{k}(\vec{x}-\vec{x}') - i\omega t]\vec{\psi}_s(\vec{x}',0)d^3k d^3x'. \tag{5.15}$$

It follows from this that $\vec{\psi}_s(\vec{x},t)$ differs from zero everywhere even if the initial value differs from zero only inside the space region S. For example, we choose $\vec{\psi}_s(\vec{x}, t=0) = \vec{a}\Theta(R-r)$ ($\vec{a}$ is a constant vector, Θ is a step function) and evaluate the initial derivative of $\vec{\psi}$:

$$\frac{\partial}{\partial t}\vec{\psi}_s(\vec{x},t)|_{t=0} = -\vec{a}\frac{ic}{2\pi^2 r}[\frac{1}{4}\ln(\frac{r+R}{r-R})^2 - \Phi(r,R)],$$

where $\Phi(r,R) = \pi/2$ for $r > R$, $= \pi/4$, for $r = R$ and $= 0$ for $r < R$. This derivative differs from zero everywhere. Then, turning to (5.6) we conclude that $\psi_s(\vec{r},t)$ will be different from zero everywhere for $t > 0$. Obviously, this conclusion may be obtained without any calculations by applying the Hegerfeldt theorem to the Sipe wave function (5.14).

On the transformations of photon densities.
Consider the definition of $\vec{f}(\vec{x})$ given by (2.8):

$$\vec{f}(\vec{x}) = \int e^{i\vec{k}\vec{x}}\vec{f}(\vec{k})d^3k.$$

Let $\vec{f}(\vec{k})$ be transformed like a tensor. Then, $\vec{f}(\vec{x})$ is not a tensor as d^3k is not an invariant volume. For $\vec{f}(\vec{x})$ and $\vec{f}(\vec{k})$ having the same tensor properties, the transformation law should be as follows:

$$\vec{f}(\vec{x}) = \int \exp[i(\vec{k}\vec{x} - \omega t)]\vec{f}(\vec{k})\frac{d^3k}{k}. \tag{5.16}$$

The densities in x and k spaces used in section 2 were related by the condition

$$\frac{1}{(2\pi)^3}\int |\vec{f}(\vec{x})|^2 d^3x = \int |\vec{f}(\vec{k})|^2 d^3k. \tag{5.17}$$

Now, if $|\vec{f}(\vec{k})|^2$ in the RHS is a scalar, the LHS is not still a scalar (due to the same non-invariance of d^3k). For the scalar product in k space being invariant, it should have the form

$$\int |\vec{f}(\vec{k})|^2 \frac{d^3k}{k}. \tag{5.18}$$

We extract $\vec{f}(\vec{k})$ from (5.16)

$$\vec{f}(\vec{k}) = \frac{k}{8\pi^3}\exp(i\omega t)\int \vec{f}(\vec{x})\exp(-i\vec{k}\vec{x})d^3x$$

and insert it into (5.18):

$$\int |\vec{f}(\vec{k})|^2 \frac{d^3k}{k} = \int \rho_{rel}(\vec{x})d^3x. \tag{5.19}$$

Here

$$\rho_{rel} = \vec{f}(\vec{x})\vec{F}^*(\vec{x}), \quad \vec{F}^*(\vec{x}) = \int G(\vec{x}-\vec{x}')f^*(\vec{x}')d^3x', \quad G(\vec{x}-\vec{x}') = \frac{1}{(2\pi)^6}\int k d^3k \exp[-i\vec{k}(\vec{x}-\vec{x}')]. \tag{5.20}$$

Although the integral in (5.19) is a positive definite quantity, the relativistic density $\rho_{rel}(\vec{x})$ may take negative values. This invalidates its physical meaning. The numerical investigations of these densities have been reported in ref. [27].

Despite the fact that $\vec{f}(x)$ in (5.17) does not behave properly under the Lorentz transformation, the integral in the RHS of (5.17) is a Lorentz invariant quantity. To prove this, we consider the complex tensor $\mathcal{F}_{\mu\nu}(x)$ satisfying Maxwell equations. $\mathcal{E}$ and $\mathcal{H}$ forming this tensor may be taken as positive frequency parts of $\vec{E}(x)$ and $\vec{H}(x)$ (the positive and negative frequency parts of a tensor are again tensors). $\mathcal{F}_{\mu\nu}(x)$ can be expressed as an integral over the plane waves

$$\mathcal{F}_{\mu\nu}(x) = \int \mathcal{F}_{\mu\nu}(k)\frac{d^3k}{\omega}, \quad \mathcal{F}_{\mu\nu}(k) = a(k)(e_\mu k_\nu - e_\nu k_\mu)e^{i(\vec{k}\vec{x}-\omega t)}. \tag{5.21}$$

Here e_μ is the polarization vector of the plane wave satisfying the conditions

$$(ke) = \vec{k}\vec{e} - k_0 e_0 = 0, \quad ee^* = \vec{e}\vec{e}^* - e_0 e_0^* = 1, \tag{5.22}$$

$a(k)$ is the Lorentz scalar. The explicit form of $\mathcal{F}_{\mu\nu}$ is

$$\vec{\mathcal{E}}(k) = a(k)(\vec{e}\omega - \vec{k}e_0), \quad \vec{\mathcal{H}}(k) = a(k)\vec{e}\times\vec{k}.$$

It then follows that under the Lorentz transformation $\mathcal{F}_{\mu\nu}(k)$, $\mathcal{E}(k)$ and $\mathcal{H}(k)$ are transformed like $\mathcal{F}_{\mu\nu}(x)$, $\mathcal{E}(x)$ and $\mathcal{H}(x)$, resp. As Eqs.(5.21),(5.22) are invariant WRT the gauge transformation $e_\mu \to e_\mu + \alpha k_\mu$, it is possible to put $e_0 = 0$ in them (the consideration following below does not depend on this fact). Elementary calculations show that

$$\frac{|\mathcal{E}(k)|^2}{\omega^2} = \frac{|\mathcal{H}(k)|^2}{\omega^2} = |a(k)|^2, \tag{5.23}$$

i.e., $\mathcal{E}(k)/\omega^2$ and $\mathcal{H}(k)/\omega^2$ are the Lorentz scalars. It follows from Eqs. (2.2),(2.3) and (2.9) that under the Lorentz boosts the quantities $\omega^{3/2}\vec{f}_k e^{i\omega t}$ and $\omega^{3/2}\vec{g}_k e^{i\omega t}$ are transformed like $\vec{\mathcal{E}}(x)$ and $\vec{\mathcal{H}}(x)$ and, therefore, like $\vec{\mathcal{E}}(k)$ and $\vec{\mathcal{H}}(k)$ (as, being multiplied by d^3k/ω and integrated they coincide with the positive frequency parts of $\vec{E}(x)$ and $\vec{H}(x)$). It follows from (5.23) that $\omega|\vec{f}_k|^2$ and $\omega|\vec{g}_k|^2$ are the Lorentz scalars. Thus, the quantities $\int|\vec{f}_k|^2\omega\frac{d^3k}{\omega} = \int|\vec{f}_k|^2d^3k$ and $\int|\vec{g}_k|^2\omega\frac{d^3k}{\omega} = \int|\vec{g}_k|^2d^3k$ are the relativistic invariants. This completes the proof. The moral of these considerations is that the seemingly noncovariant form of the integrand does not necessary mean the noncovariance of the integral itself.

Causality and positive definiteness of the probability density.

The following considerations point to close relationship between the causality and positive definiteness of the probability density. The idea was put forward by Wigner [22]. Let at the initial moment $t = 0$ the particle be localized inside the sphere S of the radius R_0 (fig.10). We surround S by two spheres of radii R_1 and R_2, $(R_0 < R_1 < R_2)$. Let sphere S_1 expand with the light velocity c up to the radius $R_1' = R_1 + ct$, while sphere S_2 contracts to the radius $R_2' = R_2 - ct$, $R_1' < R_2'$. Consider the 4-volume V surrounded by the 3-surfaces:

$$\begin{aligned} &t = 0, \quad R_1 < r < R_2, \\ &t = t', \quad R_1' < r < R_2', \\ &0 < t < t', \quad r = R_1 + ct, \\ &0 < t < t', \quad r = R_2 - ct \end{aligned} \tag{5.24}$$

Due to the Gauss Theorem

$$\int \frac{\partial J_\mu}{\partial x_\mu} d^4x = \int J_\mu d\sigma_\mu$$

the flux of the 4-current through the closed hypersurface equals zero if the continuity equation $\partial J_\mu/\partial x_\mu = 0$ is fulfilled. Being applied to hypersurface (5.24) this gives:

$$\int_{R_1}^{R_1'} (\rho - J_r)|_{t=(r-R_1)/c} dV + \int_{R_1'}^{R_2'} \rho(r,t')dV + \int_{R_2'}^{R_2} (\rho + J_r)|_{t=(R_2-r)/c} dV - \int_{R_1}^{R_2} \rho(r,0)dV = 0. \tag{5.25}$$

The last integral equals zero as at the initial moment $t = 0$ the probability density equals zero for $R_1 < r < R_2$. Thus, the sum of the remaining three integrals is zero. The positivity of the integrands in (5.25)[1] leads to the disappearance of each integral and integrand entering this equation. In particular, this gives $\rho(t) = 0$ for $r > R_1 + ct$. This means that the causality is not violated for the conserved 4-current with positive definite density.

We prove now that for spin 1/2 the causality is not violated for rather general interactions. Consider the Dirac equation

$$(\gamma_\mu \frac{\partial}{\partial_\mu} + Q)\Psi = 0, \tag{5.26}$$

where Q is the operator independent of coordinates with the property that the wave function $\bar{\Psi} = \Psi^*\gamma_4$ satisfies the equation

$$\bar{\Psi}(\gamma_\mu \frac{\partial}{\partial_\mu} - Q) = 0. \tag{5.27}$$

It follows from these equations that the continuity equation is fulfilled:

$$\frac{\partial}{\partial x_\mu} i(\bar{\Psi}\gamma_\mu\Psi) = \frac{\partial J_\mu}{\partial x_\mu} = 0.$$

[1] From the positive definiteness of ρ and the time likeness of J_μ it follows immediately that $\rho \geq |\vec{J}|$.

For the plane wave

$$\Psi(x) = \exp(i(\vec{p}\vec{x} - \epsilon t))u, \tag{5.28}$$

one gets

$$(-\epsilon\gamma_4 + i\vec{p}\vec{\gamma} + Q)u = 0, \quad \bar{u}(-\epsilon\gamma_4 + i\vec{p}\vec{\gamma} + Q) = 0.$$

Differentiate the first of these equations with respect to $\vec{p}$:

$$(-\vec{v}\gamma_4 + i\vec{\gamma})u + (-\epsilon\gamma_4 + i\vec{p}\vec{\gamma} + Q)\frac{\partial u}{\partial \vec{p}} = 0. \tag{5.29}$$

Multiplying (5.29) by $\bar{u}$ one gets

$$-\vec{v}\bar{u}\gamma_4 u + i(\bar{u}\vec{\gamma}u) = 0 \quad \text{or} \quad \vec{v} = \frac{i(\bar{u}\vec{\gamma}u)}{|u|^2} = \frac{\vec{J}}{\rho}.$$

Due to the positive definiteness of ρ and time likeness of J_μ the velocity of $|\vec{v}|$ is always smaller than c.

Another example is the motion of a neutral particle with spin 1/2 and anomalous magnetic moment (e.g., neutron) in a constant electromagnetic field. The corresponding Dirac equation $(\hbar = c = \mu = 1)$ is

$$(\gamma_\mu \frac{\partial}{\partial x_\mu} - \frac{1}{2}F_{\mu\nu}\sigma_{\mu\nu} + m)\Psi = 0.$$

For the plane wave (5.28) this gives

$$(ip_\mu\gamma_\mu - \frac{1}{2}F_{\mu\nu}\sigma_{\mu\nu} + m)u = 0.$$

The corresponding dispersion equation

$$det[ip_\mu\gamma_\mu - \frac{1}{2}F_{\mu\nu}\sigma_{\mu\nu} + mI] = 0$$

defines the energy as a function of the momentum $(\epsilon = \epsilon(\vec{p}))$ and this in turn allows one to obtain the group velocity

$$\vec{v} = \frac{\partial \epsilon}{\partial \vec{p}}. \tag{5.30}$$

The analysis of (5.30) shows [28] that $|\vec{v}|$ is always smaller than c. For the magnetic field equal to zero and $\vec{p}$ directed along $\vec{E}$ the indeterminacy of the form 0/0 arises. Being resolved it again gives $|\vec{v}| < c$. This case is equivalent to the conical refraction in optics predicted by Hamilton [29].

6 The electromagnetic waves versus photons

Consider the complex form of Maxwell equations:

$$\frac{i}{c}\frac{\partial \vec{\Psi}_1}{\partial t} = curl\vec{\Psi}_1, \quad div\vec{\Psi}_1 = 0. \tag{6.1}$$

Here $\vec{\Psi}_1 = \vec{E} + i\vec{H}$. Consider the positive- frequency plane wave

$$\vec{\Psi}_1^R(x,t) = \vec{e}(k)e^{i(\vec{k}\vec{x} - \omega t)}. \tag{6.2}$$

Substituting (6.2) into (6.1) one gets

$$\frac{\omega}{c}\vec{e} = i(\vec{k} \times \vec{e}), \quad \vec{e}\vec{k} = 0. \tag{6.3}$$

Let vector $\vec{k} = (0, 0, \frac{\omega}{c})$. Then, it follows from (6.3) that $\vec{e} = (1, i, 0)/\sqrt{2}$ and

$$\vec{E}_R = \vec{e}_x \cos(kz - \omega t) - \vec{e}_y \sin(kz - \omega t), \quad \vec{H}_R = \vec{e}_x \sin(kz - \omega t) + \vec{e}_y \cos(kz - \omega t). \tag{6.4}$$

It turns out that $\vec{E}_R$ and $\vec{H}_R$ are rotated in the clockwise direction for an observer looking along the $\vec{k}$ direction. Thus, the plane wave (6.2) is a right-polarized one. The negative-frequency solution of (6.1)

$$\vec{\Psi}_1^L = \vec{e}(k)e^{-i(\vec{k}\vec{x}-\omega t)} \tag{6.5}$$

(the vector $\vec{e}(k)$ is the same as in (6.2)) describes the left-polarized electromagnetic wave. In general, the superposition of (6.2) and (6.5)

$$\vec{\Psi}(\vec{x}, t) = \vec{e}(\vec{k})[C_R e^{i(\vec{k}\vec{x}-\omega t)} + C_L e^{-i(\vec{k}\vec{x}-\omega t)}] \tag{6.6}$$

corresponds to the classical elliptically polarized wave. Particular cases $C_R = 0$ (or $C_L = 0$) and $|C_R| = |C_L|$ correspond to the circular and linear polarizations, resp.
Now we interpret (6.1) as equation describing photons. In quantum field theory only the positive-frequency solutions are admissible. This means that Eq.(6.1) describes the photons with right-hand polarization. To describe the photons with left-hand polarization, consider the function $\vec{\Psi}_2 = \vec{E} - i\vec{H}$ which satisfies the equation

$$-\frac{i}{c}\frac{\partial \Psi_2}{\partial t} = curl\vec{\Psi}_2. \tag{6.7}$$

This equation also has positive- and negative- frequency solutions corresponding to the left-hand and right-hand polarizations, resp. The positive-frequency solution of (6.7)

$$\vec{E}_L = \vec{e}_x \cos(kz - \omega t) + \vec{e}_y \sin(kz - \omega t), \quad \vec{H}_L = \vec{e}_x \sin(kz - \omega t) - \vec{e}_y \cos(kz - \omega t) \tag{6.8}$$

corresponds to the left-hand- polarized photon, while the negative-frequency solution describes the right-hand polarized entity (not a photon, as it corresponds to the positive-frequency solution). The negative-frequency solution of Eqs.(6.7) and (6.1) are complex conjugated to the positive-frequency solution of (6.1) and (6.7),resp. As negative-frequency solutions are discarded, the positive-frequency solutions of (6.1) and (6.7) are no more complex conjugated. We refer to them as to $\vec{\Psi}_R$ and $\vec{\Psi}_L$. We conclude: positive-frequency solutions corresponding to the right-hand-($\vec{\Psi}_R$) and left-hand-($\vec{\Psi}_L$) polarized photons satisfy the following equations:

$$\frac{i}{c}\frac{\partial \vec{\Psi}_R}{\partial t} = curl\vec{\Psi}_R, \quad div\vec{\Psi}_R = 0, \quad \frac{-i}{c}\frac{\partial \Psi_L}{\partial t} = curl\vec{\Psi}_L, \quad div\vec{\Psi}_L = 0 \tag{6.9}$$

The pay for discarding the negative-frequency solutions and the necessity to have right- and left- polarized photons is the doubling of the number of equations . This doubling is not needed for the classical electromagnetic wave as both positive and negative frequencies are allowable for its description. Under the Lorentz transformation $\vec{E}_{R,L}$ and $\vec{H}_{R,L}$ defined by Eqs.(6.4) and (6.8) behave as usual field strengths $\vec{E}$ and $\vec{H}$.
Now the following dramatic situation arises:
1) Single photons as positive-frequency solutions of Maxwell equations are not localizable. This fact is confirmed by numerous experiments [30].

2) Classical electromagnetic waves are localizable. Mention ,e.g., clystrons, waveguides, laser beams propagating in vacuum without spreading, etc.
3) It is generally believed that the electromagnetic wave consists of photons. Experiments seem to confirm this viewpoint (for example, a photomultiplier being placed into the electromagnetic wave detects particular photons).
The appearance of negative frequencies in the classical electromagnetic field may be understood in the framework of quantum electrodynamics. In it, the quantized EMF is described by equations of the same form as the classical ones (6.1) in which the function $\vec{\Psi}$ should be changed by the operator. In quantum mechanics the time derivative of the operator $\vec{\Psi}$ is expressed through the commutator of $\vec{\Psi}$ with the Hamilton operator $\hat{\mathcal{H}}$:

$$i\hbar\dot{\vec{\Psi}} = [\vec{\Psi}, \hat{\mathcal{H}}]. \tag{6.10}$$

It is suggested that $\hat{\mathcal{H}}$ has the same form as its classical counterpart $\mathcal{H}$:

$$\hat{\mathcal{H}} = \frac{1}{8\pi}\int \vec{\Psi}^+\vec{\Psi}d^3x. \tag{6.11}$$

Equations (6.10) and (6.11) are reconciled if $\vec{\Psi}$ satisfy the following commutation relations

$$[\vec{\Psi}_m(\vec{x}, t), \vec{\Psi}_n^+(\vec{x}', t)] = 8i\pi\hbar\epsilon_{mnl}\frac{\partial}{\partial x_l}\delta(\vec{x} - \vec{x}') \tag{6.12}$$

(other commutators are zero). These commutation relations are satisfied if we take the usual second-quantized expression for the 4-vector potential A_μ. Then , evaluating $\vec{E}$, $\vec{H}$ and $\vec{\Psi} = \vec{E} + i\vec{H}$ we arrive at commutation relations (6.12). The expressions for A_μ, $\vec{E}$, $\vec{H}$ and $\vec{\Psi}$ operators contain the terms with positive and negative frequencies (of the creation and annihilation operators). According to the prescription of quantum electrodynamics the classical electromagnetic field is obtained by averaging the quantum operators $\vec{E}$, $\vec{H}$ over the so- called coherent states. As a result, terms with positive and negative frequencies arise on the same footing. The obtained $\vec{E}_{clas}$ and $\vec{H}_{clas}$ are reduced to the sum of plane waves with positive and negative frequencies. Summing is performed over all possible wave vectors and polarizations. Changing the sum over k by the integration $((1/V)\sum_k = (2\pi)^{-3}\int d^3k)$ and averaging over the polarizations one obtains for $\vec{E}_{clas}$ and $\vec{H}_{clas}$ expressions exactly coinciding with (2.2) and (2.9) in which $|f_k|^2$ and $|g_k|^2$ mean the average number of photons with the wave vector k. Thus, representations (2.2) and (2.9) arise in a natural way. They, in fact, are the consequence of averaging over the coherent states. This procedure is justified by the fact that photon states generated by the classical current coincide with the coherent ones [11,31]. Obviously, the photon states are not exhausted by the coherent ones. As an example, mention the black body radiation and other states used in quantum optics ([31,32]).

The main result of this section is the fact that the classical electromagnetic wave contains both positive- and negative- frequency solutions of the Maxwell equations, while only positive frequency solutions (if we do not abandon the standard interpretation of particle in quantum field theory) are permissible for the description of photons. The availability of positive and negative frequencies makes the localization of the electromagnetic wave to be possible. On the other hand, in the interpretation by Kim et al [15] both the electromagnetic waves and photons are superpositions of positive- frequency solutions. The sole difference between them is that photons require the covariant description, while electromagnetic waves do not [1]. According to the Hegerfeldt theorem the photon and electromagnetic wave thus defined cannot be localized

[1] We have seen in sect.5 that the illusive noncovariance of the integrand does not mean, in general, the noncovariance of the integral itself.

in the sense to be confined within a finite volume V. However, the positive- frequency photons cannot be localized in a Newton-Wigner sense either (as particles with mass zero can be localized only for spins 0 and 1/2 [5]). As far as we can understand, Kim et al [15] tried to achieve approximate localization of the photon wave function in the Newtion-Wigner sense. The photon wave function localized in the Newton-Wigner sense differs from zero everywhere. As claims of ref.[15] and of the present consideration are referred to different definitions of localizability, there is no contradiction between them.

7 Discussion

The main question to be answered is whether the functions f and g introduced in sect.2 have a physical meaning. Consider one particular photon. Its wave function, density and energy density are distributed over the whole space and cannot be localized. We have seen that major parts of the photon density and energy density are confined to a small region of space with small tails outside it. Let the detector D (e.g., photomultiplier) be placed into the photon field. How much should the photon or energy density be overlapped with detector in order to be registered? The same question concerns the electromagnetic wave in which $\vec{E}$ and $\vec{H}$ equal zero outside a finite volume V but the photon density $\rho = |\vec{f}|^2 + |\vec{g}|^2$ differs from zero everywhere. Can this density produce any physical effect? In the momentum space there is only one vector function $\vec{f}(k)$ defining the evolution of EMF. For the choice (3.2) the electromagnetic field initially confined to the sphere of radius a expands radially with the light velocity c. Can the initial conditions be chosen so, that the subsequent motion of the EMF bunch would be in one particular direction (e.g., along the z axis)? Consider the impenetrable sphere S with a small hole in it. Let the emitter of electromagnetic waves (e.g., oscillating electric dipole) be placed at the center of S. Then, outside S a thin nondivergent electromagnetic wave beam will be observed. Now let inside S (instead of the afore-mentioned electromagnetic wave emitter) the source of photons (e.g.,radioactive atom) is imbedded. Sometimes the particular photon will pass through the hole in S and the isolated photons should be observed outside S. As photons cannot be localized, their density differs from zero everywhere and, this can, in principle, be observed. It is tempting to associate the afore-mentioned tails of photon and energy densities with the so-called empty waves [14]. Their existence was predicted by the founders of quantum mechanics (Gespensterfelder (or ghost fields) according to Einstein, virtual waves according to Bohr, etc.). According to the modern viewpoint on empty waves they are needed for the correct evaluation of quantum probabilities as well as for the preparation of a quantum system to the subsequent arrival of the EMF wave. We quote two citations from book [14]:

"How can one ever hope to reveal the presence of a wave which does not carry energy or momentum? This problem can have an answer if it is noticed that one does not only measure energy changing processes but probabilities as well: the wave could therefore reveal its presence by modifying decay probabilities for an unstable system"(p.137).

And further:

"...the associated wave packet, though devoid of energy and momentum, has a chance to reveal its existence by generating a zero-energy transfer simulated emission" (p.138).

The authors of the present consideration although being not the adherents of the empty wave existence, should make emphasis on the following properties of the functions $\vec{f}$ ang $\vec{g}$ (see set.2 and 3) resembling the empty-wave ones:

i) For the electromagnetic wave the photon densities differ from zero everywhere and, in particular, in those space regions where $\vec{E} = \vec{H} = 0$. This takes place at each instant of time including the initial one. Such a space distribution of densities is needed to obtain the correct value of the photon number N.

ii) There are the energy-like carrying densities ρ_ϵ^f and ρ_ϵ^g (see sects. 2 and 3) distributed over the whole space (contrary to the electromagnetic energy density). Under the term 'energy carrying densities' we mean that the space integral from them coincides with the electromagnetic energy. Among the followers of the empty wave concept there is no overall agreement whether empty waves carry the energy and momentum or not (for example, de Broglie suggested that an empty wave carries a tiny part of them). The present consideration shows that the photon wave carries the entity that strongly resembles the energy. Yet, it is not known how this entity affects a charged matter.

Among the three densities ρ_f, ρ_g and ρ introduced in sect.2 the most promising seems to be ρ as it satisfies the local differential conservation law (2.16).

A few words should be added on the photon localization. It is not localizable in the ordinary meaning (photon confinement within a finite region of space) if by the photon wave function one understands the positive-frequency functions $\vec{f}$ and $\vec{g}$ or positive-frequency parts of $\vec{E}$ and $\vec{H}$. According to the Hegerfeldt theorem all of them are distributed over the whole space. On the other hand, the electromagnetic wave can be localized in the same sense as it contains both positive and negative frequencies. So far we have identified photons with positive-frequency solutions of the free Maxwell equations. The situation changes for the photon placed into the cavity with absolutely reflective boundaries. As a result of reflections the standing photonic wave arises inside the cavity and this makes the photon localization to be possible.

Another drawback of this consideration is that we have not concretized the process of creation and detection of photons. The importance of this effect was demonstrated by Sipe [13], Shirokov [33] and Kaloyerou [34].

To the end, we see that photon has a number of intriguing features. The appearance of the first volume of the book 'The Enigmatic Photon' [35] is also an argument confirming the inexhaustibility of photon properties.

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Афанасьев Г.Н. и др. E2-96-278

О плотностях фотонов, теореме Хегерфельдта и прочем

Выполнены численные исследования различных вариантов плотностей фотонов и соответствующих им законов сохранения. Проанализированы и частично сняты возражения Паули-Эренфеста против использования подобных плотностей. Показано, почему нелокализуемость отдельного фотона не противоречит возможности локализации классической электромагнитной волны.

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Afanasiev G.N. et al. E2-96-278

On the Photon Densities, Hegerfeldt Theorem and All That

Numerical investigations of different photon densities and corresponding to them conservation laws are presented. The Ehrenfest-Pauli objections against the using of such densities are analyzed and partly removed. It is shown how the nonlocalizability of the single photon can be reconciled with the localizability of the classical electromagnetic wave.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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