ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E2-96-250

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TWO CONNECTIONS IN THE GRAVITY THEORY

Submitted to «Гравитация»

1996

Introduction

In considering the transition to the nonrelativistic limit in the general theory of relativity [1] I was convinced that in addition to the field connection Γ^a_{mn} one should include in the Einstein theory of gravity also the background connection $\mathring{\Gamma}^a_{mn}$. I have never thought the statement that in this limit the Euclidean geometry should necessary appear in the visible world to be well-grounded. Indeed, why would not the Lobachevsky geometry appear in it [2] instead of the Euclidean one? It turns out that this opportunity can be realised if the background connection is chosen properly. It is true that the connection thus chosen takes us beyond the Einstein theory whereas the background connection with the zero curvature tensor does not, thus allowing us to solve the energy problem without making a conclusion about its nonlocalizability. This absurd conclusion (about nonlocalizability of energy) is the result of illegitimate application of the Gauss integral theorem to the pseudo-vector field (rather than vector, as it is required by the theorem).

Having introduced the background connection, given the theory of gravity a totally tensor nature and abolished the conclusion about nonlocalizability of energy, the discussion of the energy pseudo-tensor may be thought to be completed.

In the theory of gravity with two affine connections the Einstein theory comes out as a particular case when the background connection is primitive [3]. Another case in which the background connection is given by the Lobachevsky geometry has been considered in [4] - [6].

I shall start the review of the results concerning this theme with the summary of the theory of affine connection following book [7].

Affine Connections and Tensor Fields

Consider affine connections on the N-dimensional manifold and attribute them to the basis $d^a = dx^a$ and the dual basis $\partial_a = \partial/\partial x^a$, where x^a are the coordinates forming the map x of manifold. We proceed in the same manner with tensor fields.

The tensor field of type (A/B) is given by its components with upper indices $a_1, ..., a_A$ and lower indices $b_1, ... b_B$.

The affine connection Γ is introduced so that the covariant derivative ∇T of any tensor field of the type (A/B) taken with its help be a tensor field of the type $(A/B+1)$. It is given by its components Γ^a_{mn}.

For the covector field it is assumed that

$$\nabla_m T_n = \partial_m T_n - \Gamma^a_{mn} T_a \,. \tag{1}$$

Hence it follows that passing from the map x to the map $\tilde{x}$ the connection components transform according to the rule

$$\tilde{\Gamma}^a_{mn} = \left(\Gamma^a_{pq} \frac{\partial x^p}{\partial \tilde{x}^m} \frac{\partial x^q}{\partial \tilde{x}^n} + \frac{\partial^2 x^s}{\partial \tilde{x}^m \partial \tilde{x}^n} \right) \frac{\partial \tilde{x}^a}{\partial x^s} \,. \tag{2}$$

In its turn, it follows from (2) that for any vector field the combinations

$$\nabla_m T^a = \partial_m T^a + \Gamma^a_{mn} T^n \tag{3}$$

compose a tensor field of the type (1/1) which is a covariant derivative of the vector field.

A covariant derivative of any tensor field is formed according to the rule of differentiation of the product: $\nabla(AB) = (\nabla A)\, B + A\, (\nabla B)$. In particular, for the scalar field T the covariant derivative equals the partial derivative:

$$\nabla_m T = \partial_m T \,. \tag{4}$$

Torsion Tensor

As it follows from (2), the difference

$$S^a_{mn} = \Gamma^a_{mn} - \Gamma^a_{nm} \tag{5}$$

is a tensor field of the type (1/2).

Tensor (5) is called the torsion tensor.

Curvature Tensor

Like the operator

$$\nabla_k - \partial_k$$

the operator

$$\nabla_{kl} = \nabla_k \nabla_l - \nabla_l \nabla_k + S^p_{kl} \nabla_p \tag{6}$$

as applied to the scalar field T gives zero

$$\nabla_{kl}\, T = 0\,. \tag{7}$$

The same operator as applied to the covector field T_n gives

$$\nabla_{kl}\, T_n = -\, R^a_{kln}\, T_a\,, \tag{8}$$

whereas being applied to the vector field T^a it gives

$$\nabla_{kl}\, T^a = R^a_{kln}\, T^n\,, \tag{9}$$

where

$$R^a_{kln} = \partial_k\, \Gamma^a_{ln} - \partial_l\, \Gamma^a_{kn} + \Gamma^a_{ks}\, \Gamma^s_{ln} - \Gamma^a_{ls}\, \Gamma^s_{kn}\,. \tag{10}$$

It follows from (8) and (9) that the combination (10) is the tensor of type (1/3). It is called the curvature tensor or the Riemann-Christoffel tensor.

It is obviosly that

$$R^a_{kln} + R^a_{lkn} = 0\,. \tag{11}$$

It is to be noted that if in the formula

$$(\nabla_k - \partial_k)\, T = \ldots$$

for the tensor field T of any type (A/B) the combination of letters Γ_k is changed by R_{kl} , one gets the formula

$$\nabla_{kl}\, T = \ldots$$

These combinations should not be confused with contractions given below.

Two Contractions of the Curvature Tensor and the Contracted Connection

Owing to antisymmetry (11) of the Riemann-Christoffel tensor, only two of its contractions are of interest. One of them,

$$R^a_{aln} = R_{ln}\,, \tag{12}$$

is called the Ricci tensor. The other,

$$R^a_{kla} = \Omega_{kl} = \partial_k\, \Gamma_l - \partial_l\, \Gamma_k, \tag{13}$$

is called the curvature tensor of the contracted connection Γ_m equal to

$$\Gamma_m = \Gamma^a_{ma}\,. \tag{14}$$

According to (5) the second contraction of the connection differs from the first one by the torsion covector

$$S_n = S^a_{an}\,.$$

According to (2), in passing from a map to a map the components of the contracted connection (14) are transformed according to the rule

$$\tilde{\Gamma}_m = \left(\Gamma_p + \frac{\partial}{\partial x^p} \right) \frac{\partial x^p}{\partial \tilde{x}^m} . \tag{15}$$

The last formula can be converted into

$$\tilde{\Gamma}_m = \frac{\partial x^p}{\partial \tilde{x}^m} \left(\Gamma_p + \frac{\partial}{\partial x^p} \ln J \right) , \tag{16}$$

where J is the Jacobian of the transformation

$$J = \partial(x^1 \ldots x^N) \, / \, \partial(\tilde{x}^1 \ldots \tilde{x}^N) . \tag{17}$$

Proof. The differential of determinant A of any matrix (A^p_q) is equal to

$$dA = B^q_p \, d \, A^p_q ,$$

where B^q_p is the cofactor of the element A^p_q . Therefore,

$$\frac{\partial x^p}{\partial \tilde{x}^m} \frac{\partial}{\partial x^p} \ln J = \frac{1}{J} \frac{\partial J}{\partial \tilde{x}^m} =$$

$$= \frac{\partial \tilde{x}^q}{\partial x^p} \frac{\partial}{\partial \tilde{x}^m} \frac{\partial x^p}{\partial \tilde{x}^q} = \frac{\partial \tilde{x}^q}{\partial x^p} \frac{\partial}{\partial \tilde{x}^q} \frac{\partial x^p}{\partial \tilde{x}^m} = \frac{\partial}{\partial x^p} \frac{\partial x^p}{\partial \tilde{x}^m}$$

and formula (16) is proved.

Let us apply the operator $\nabla - \partial$ to the tensor E of type $(0/N)$. We have

$$(\nabla_m - \partial_m) \, E_{k_1 \ldots k_N} = - \Gamma^a_{m k_1} \, E_{a k_2 \ldots k_N} - \ldots - \Gamma^a_{m k_N} \, E_{k_1 \ldots k_{N-1} a} .$$

If the tensor E is antisymmetric in any pair of indices $k_1 \ldots k_N$, the combination

$$\Gamma^b_{ma} \, E_{k_1 \ldots k_N} - \Gamma^a_{m k_1} \, E_{a k_2 \ldots k_N} - \ldots - \Gamma^a_{m k_N} \, E_{k_1 \ldots k_{N-1} a}$$

is antisymmetric in any pair of indices $a k_1 \ldots k_N$. Consequently, this combination equals zero and

$$\nabla_m \, E_{k_1 \ldots k_N} = (\partial_m - \Gamma_m) \, E_{k_1 \ldots k_N} . \tag{18}$$

In the same way one can prove that

$$\nabla_{mn} \, E_{k_1 \ldots k_N} = - \Omega_{mn} \, E_{k_1 \ldots k_N} . \tag{19}$$

Two Connections

Now let on the same manifold be given two connections Γ and $\check{\Gamma}$ with the components Γ^a_{mn} and $\check{\Gamma}^a_{mn}$. As it follows from (2), the difference

$$P^a_{mn} = \check{\Gamma}^a_{mn} - \Gamma^a_{mn} \tag{20}$$

is the tensor field of type (1/2).

Tensor (20) is called the affine deformation tensor.

Substituting into (10) the expression for $\check{\Gamma}^a_{mn}$ taken from (20) we get the law of changing the curvature tensor while passing from the first connection to the second one

$$\check{R}^a_{kln} = R^a_{kln} + S^m_{kl} \, P^a_{mn} + \nabla_k \, P^a_{ln} - \nabla_l \, P^a_{kn} + P^a_{ks} \, P^s_{ln} - P^a_{ls} \, P^s_{kn} . \tag{21}$$

The contraction

$$P_m = \check{\Gamma}_m - \Gamma_m = P^a_{ma} \tag{22}$$

is called the affine deformation covector. From (21) we get the law of changing the curvature tensor of the contracted connection

$$\check{\Omega}_{kl} = \Omega_{kl} + \nabla_k \, P_l - \nabla_l \, P_k + S^m_{kl} \, P_m = \Omega_{kl} + \partial_k \, P_l - \partial_l \, P_k . \tag{23}$$

By interchanging the position of the connections Γ and $\check{\Gamma}$ the affine deformation tensor, according to (20), changes its sign

$$\check{P}^a_{mn} = - P^a_{mn} . \tag{24}$$

According to (21) the law of changing the tensor will take the form

$$R^a_{kln} = \check{R}^a_{kln} + \check{S}^m_{kl} \, P^a_{mn} + \check{\nabla}_k \, P^a_{ln} - \check{\nabla}_l \, P^a_{kn} - P^a_{ks} \, P^s_{ln} + P^a_{ls} \, P^s_{kn} . \tag{25}$$

Connection without Torsion

If the torsion tensor (5) equals zero,

$$\Gamma^a_{mn} = \Gamma^a_{nm} \, . \tag{26}$$

This connection is called symmetric or connection without torsion. In this case, alongside with (11) the curvature tensor satisfies one more algebraic condition

$$R^a_{kln} + R^a_{nkl} + R^a_{lnk} = 0 \tag{27}$$

and one more differential condition

$$\nabla_i \, R^a_{kln} + \nabla_l \, R^a_{ikn} + \nabla_k \, R^a_{lin} = 0 \, . \tag{28}$$

The latter is called the Bianchi - Padova identity.

Hence, as a result of contraction we get

$$\Omega_{kl} + R_{kl} - R_{lk} = 0 \, , \tag{29}$$

$$\nabla_i \, \Omega_{kl} + \nabla_l \, \Omega_{ik} + \nabla_k \, \Omega_{li} = 0 \, . \tag{30}$$

For the connection without torsion it follows from (30) that

$$\partial_i \, \Omega_{kl} + \partial_l \, \Omega_{ik} + \partial_k \, \Omega_{li} = 0 \, .$$

For each point of the manifold one can determine such a system of coordinates in which values of all the components of the connection without torsion vanish at this point.

Equiaffine Connection

Symmetric connection is called the equiaffine one if there exists a tensor of the type $(0/N)$ antisymmetric in any pair of its indices whose covariant derivative (18) equals zero. In this case

$$\Omega_{kl} = 0 \, , \qquad R_{kl} = R_{lk} \, . \tag{31}$$

A Pair of Symmetric Connections

For symmetric connections the affine deformation tensor (20) is also symmetric

$$P^a_{mn} = P^a_{nm} \, . \tag{32}$$

Like in ref. [8], in this case the curvature tensor is written in the following two forms:

$$\breve{R}^a_{mnb} - R^a_{mnb} = \nabla_m \, P^a_{nb} - \nabla_n \, P^a_{mb} - P^a_{mnb} \, , \tag{33}$$

$$\breve{R}^a_{kln} - R^a_{kln} = \breve{\nabla}_m \, P^a_{nb} - \breve{\nabla}_n \, P^a_{mb} + P^a_{mnb} \, , \tag{34}$$

where

$$P^a_{mnb} = P^s_{mb} \, P^a_{sn} - P^s_{nb} \, P^a_{sm} \, . \tag{35}$$

Tensor (35) satisfies the conditions

$$P^s_{mns} = 0 \, , \tag{36}$$

$$P^a_{mnb} + P^a_{nmb} = 0 \, , \tag{37}$$

$$P^a_{mnb} + P^a_{bmn} + P^a_{nbm} = 0 \, , \tag{38}$$

which result in the symmetry of the next tensor

$$P_{mn} = P^a_{amn} = P^a_{mb} \, P^b_{an} - P_s \, P^s_{mn} \, . \tag{39}$$

The symmetry of tensor (39) can be verified immediately.

Christoffel's Connection

If the covariant derivative of the symmetric nondegenerate tensor field g_{mn} equals zero, the connection Γ^a_{mn} equals the Christoffel brackets $\left\{ {a \atop mn} \right\}$:

$$\Gamma^a_{mn} = \left\{ {a \atop mn} \right\} = \frac{1}{2} \, g^{as} \left(\partial_m \, g_{sn} + \partial_n \, g_{sm} - \partial_s \, g_{mn} \right) , \tag{40}$$

where g^{as} is the tensor inverse to the tensor g_{sn} so that

$$g^{as}\, g_{sb} = \delta^a_b\,. \tag{41}$$

The equality $\nabla_k\, g_{mn}$ results in the equality $\nabla_k\, g^{mn}$ so that the operation of lowering and raising tensor indices with the help of the tensors g_{mn} and g^{mn} is commutative with the operation ∇_k of covariant differentiation with respect to Christoffel's connection (40).

It is to be noted that Christoffel's connection is symmetric and its contraction equals

$$\Gamma_m = \left\{{a \atop ma}\right\} = \frac{1}{2}\, g^{ab}\, \partial_m\, g_{ab} = \frac{1}{2g}\, \partial_m\, g\,, \tag{42}$$

where g is the determinant of the matrix g_{mn}. Therefore, in formula (18) we assume that

$$E_{1\ldots N} = \sqrt{|g|}\,, \tag{43}$$

we will get zero in the right-hand side. Consequently, Christoffel's connection is equiaffine.

Einstein's Tensor

In the case of Christoffel's connection (40), besides the Ricci tensor (12) one also considers the curvature scalar

$$R = g^{mn}\, \dot{R}_{mn} \tag{44}$$

and Einstein's tensor

$$G_{ab} = R_{ab} - \frac{1}{2}\, R\, g_{ab}\,. \tag{45}$$

Concerning Christoffel's connection the Einstein tensor satisfies the condition

$$\nabla_s\, G_{ab}\, g^{sb} = 0\,. \tag{46}$$

To prove this, let us contract the Bianchi - Padova identity (28) with the tensor $\delta^k_a\, g^{in}$ and get

$$(\nabla_i\, R^a_{\cdot aln} + \nabla_l\, R^a_{\cdot ian} + \nabla_a\, R^a_{\cdot lin})\, g^{in} = 0\,.$$

By definition (12) the first term in these brackets gives

$$\nabla_i\, R^a_{\cdot aln}\, g^{in} = \nabla_i\, R_{ln}\, g^{in}\,.$$

By property (11), definition (12) and definition (44) the second term in these brackets gives

$$\nabla_l\, R^a_{\cdot ian}\, g^{in} = -\nabla_l\, R^a_{\cdot ain}\, g^{in} = -\nabla_l\, R_{in}\, g^{in} = -\nabla_l\, R\,.$$

Running a few steps forward we can say that the third term in these brackets gives the same as the first term

$$\nabla_a\, R^a_{\cdot lin}\, g^{in} = \nabla_a\, R^{ai}_{\cdot\cdot li} = \nabla_a\, R^{ia}_{\cdot\cdot il} = \nabla_a\, R^i_{\cdot iln}\, g^{in} = \nabla_a\, R_{ln}\, g^{na}\,.$$

Summing these results we get (46), but to prove the third result one has to know the properties of the curvature tensor to be discussed below.

Algebraic Identities for the Curvature Tensor in the Case of Chistoffel's Connection

Having applied to the tensor field g^{ab} the operator (6) we obtain

$$\nabla_{kl}\; g^{ab} = R^a_{\cdot kls}\, g^{sb} + R^b_{\cdot kls}\, g^{as} = 0\,.$$

It is convenient to write this result down in a compact form

$$R^{ab}_{kl} + R^{ba}_{kl} = 0\,. \tag{47}$$

by denoting

$$R^{ab}_{kl} = R^a_{\cdot kls}\, g^{sb}\,. \tag{48}$$

Taking into account (11) we get

$$R^{ab}_{kl} + R^{ab}_{lk} = 0\,. \tag{49}$$

and, consequently,

$$R^{ab}_{kl} = R^{ba}_{lk} . \qquad (50)$$

These data are sufficient to fulfil the proof of theorem (46) on Einstein's tensor which was undertaken earlier.

The result, equivalent to (47), can be derived in a different way by applying the operator (6) to the tensor field g_{mn}. We have

$$\nabla_{kl}\ g_{mn} = -\ R^{a}_{klm}\, g^{an} - R^{a}_{kln}\, g^{ma} = 0 ,$$

which is convenient to be written down in a compact form

$$R_{klmn} + R_{klnm} = 0 , \qquad (51)$$

by denoting

$$R_{klmn} = -\ R^{a}_{klm}\, g_{an} = R^{ab}_{kl}\, g_{am}\, g_{bn} . \qquad (52)$$

It is evident that the result (51) is equivalent to (47).

Taking account of (11) we get

$$R_{klmn} + R_{lkmn} = 0 , \qquad (53)$$

and consequently

$$R_{klmn} = R_{lknm} . \qquad (54)$$

The result (53) is equivalent to (49); and the result (54), to (50).

Now let us prove that

$$R_{klmn} = R_{mnkl} . \qquad (55)$$

According to (27) the tensor (52) satisfies the condition

$$R_{klmn} + R_{mkln} + R_{lmkn} = 0 . \qquad (56)$$

Let us make in this identity all cyclic permutations of indices

$$R_{klmn} + R_{mkln} + R_{lmkn} = 0 . \qquad R_{mnkl} + R_{kmnl} + R_{nkml} = 0 .$$

$$R_{nklm} + R_{lnkm} + R_{klnm} = 0 . \qquad R_{lmnk} + R_{nlmk} + R_{mnlk} = 0 .$$

Subtracting from the upper equations the lower ones we get

$$R_{klmn} - R_{klnm} = (R_{nklm} + R_{lnkm}) - (R_{mkln} + R_{lmkn}) .$$

$$R_{mnkl} - R_{mnlk} = (R_{lmnk} + R_{nlmk}) - (R_{kmnl} + R_{nkml}) .$$

Changing the second and the sixth columns according to (51) we obtain

$$R_{klmn} + R_{klmn} = R_{nklm} + R_{lnkm} - R_{mkln} + R_{lmnk}) .$$

$$R_{mnkl} + R_{mnkl} = R_{lmnk} + R_{nlmk} - R_{kmnl} + R_{nklm} .$$

Hence, according to (54) we get (55).

Primitive and Semiprimitive Connections

The affine connection is called primitive if it is symmetric and its Riemann-Christoffel tensor equals zero. Thus, primitive connection satisfies the system of equations

$$S^{a}_{mn} = 0, \qquad R^{a}_{kln} = 0 . \qquad (57)$$

Connection satisfying a much weaker system of equations

$$S^{a}_{mn} = 0, \qquad R_{mn} + R_{nm} = 0 , \qquad (58)$$

is called semiprimitive. Remember that connection satisfying the system of equations

$$S^{a}_{mn} = 0, \qquad R_{mn} - R_{nm} = 0 ,$$

is called equiaffine. Semiprimitive equiaffine connection satisfies the system of equations

$$S^{a}_{mn} = 0, \qquad R_{mn} = 0 . \qquad (59)$$

Coordinate Connection

The affine connection is called coordinate if there is a map y on the manifold such that all the component of the connection equal zero. At any map x connected with y the components of such connection equal

$$\Gamma^a_{mn} = \frac{\partial x^a}{\partial y^s} \frac{\partial^2 y^s}{\partial x^m \partial x^n} . \tag{60}$$

More precisely, connection (60) is called the connection given by map y .

One can easily verify that connection (60) satisfies the system (57). Consequently, the coordinate connection is primitive.

It can be proved that any solution of the system (57) can be represented as a coordinate connection (60). In other words, primitive connection is the coordinate one.

But the coördinate connection does not define the map y unambiguously: if one substitutes into (60) the affine transformation

$$y^s = A^s_r z^r + B^s , \tag{61}$$

the solution of the system (57) will not change. On the contrary, if passing from the map y to the map z the solution of the system (57) does not change, this transition is affine. Thus, any coordinate connection defines the affine geometry.

It is important to realize that affine geometry is not a metric one, because the connection (60) does not define any metric. As one can see, in formula (60) any metric is absent.

Metric Admitted by the Coordinate Connection

The coordinate connection (60) does not define any metric but admits it in the form

$$C_{mn} \frac{\partial y^m}{\partial x^a} \frac{\partial y^n}{\partial x^b} d\,x^a\, d\,x^b = C_{mn}\, d\,y^m\, d\,y^n , \tag{62}$$

where about the matrix (C_{mn}) we know only that it is symmetric, is not degenerate and is independent of the coordinates y .

One can easily check up that Christoffel's brackets for such a metric equal (60). Indeed, in this case

$$g_{ab} = C_{mn} \frac{\partial y^m}{\partial x^a} \frac{\partial y^n}{\partial x^b} , \quad g^{mn} = C^{ab} \frac{\partial x^m}{\partial y^a} \frac{\partial x^n}{\partial y^b} ,$$

where (C^{ab}) is the matrix inverse to (C_{mn}) . Consequently

$$\left\{ {a \atop mn} \right\} = g^{as} C_{pq} \frac{\partial y^p}{\partial x^s} \frac{\partial^2 y^q}{\partial x^m \partial x^n} =$$

$$= C^{rt} \frac{\partial x^a}{\partial y^r} \frac{\partial x^s}{\partial y^t} C_{pq} \frac{\partial y^p}{\partial x^s} \frac{\partial^2 y^q}{\partial x^m \partial x^n} =$$

$$= C^{rp} \frac{\partial x^a}{\partial y^r} C_{pq} \frac{\partial^2 y^q}{\partial x^m \partial x^n} = \frac{\partial x^a}{\partial y^r} \frac{\partial^2 y^r}{\partial x^m \partial x^n} .$$

Changing the summation index we get (60).

Thus, the connection (60) does not contain any information about the matrix (C_{mn}) .

A Pair of Primitive Connections

Let us consider a pair of solutions Γ and $\check{\Gamma}$ of the system of equations (57): one as the coordinate connection (60) and the other as the coordinate connection

$$\check{\Gamma}^a_{mn} = \frac{\partial x^a}{\partial z^s} \frac{\partial^2 z^s}{\partial x^m \partial x^n} . \tag{63}$$

In this case, the affine deformation tensor (20) equals

$$P^a_{mn} = \frac{\partial x^a}{\partial z^s} \frac{\partial y^p}{\partial x^m} \frac{\partial y^q}{\partial x^n} \frac{\partial^2 z^s}{\partial y^p \partial y^q} . \tag{64}$$

Indeed, we have

$$\frac{\partial z^s}{\partial x^n} = \frac{\partial z^s}{\partial y^q} \frac{\partial y^q}{\partial x^n} .$$

Consequently,

$$\frac{\partial^2 z^s}{\partial x^m \partial x^n} \cdot = \frac{\partial z^s}{\partial y^q} \frac{\partial^2 y^q}{\partial x^m \partial x^n} + \frac{\partial y^q}{\partial x^n} \frac{\partial}{\partial x^m} \frac{\partial z^s}{\partial y^q}$$

and

$$P^a_{mn} = \frac{\partial x^a}{\partial z^s} \frac{\partial}{\partial x^m} \frac{\partial z^s}{\partial y^q} \,.$$

This equality immediately results in (64). In its turn, it follows from (64) that a non-affine transition from the map y to the map z changes the primitive connection and gives a new solution of the system (57); on the contrary, the affine transition from y to z does not change this connection.

The Number of Essential Components of the Curvature Tensor

Let us count the number C of essential (i.e. linearly independent) components of the tensor satisfying the conditions (11) and (12).

It is obvious that $C = N\,P$ where N is the manifold dimension, P is the number of essential components of the tensor P_{kln} satisfying the conditions

$$P_{kln} + P_{lkn} = 0\,, \qquad (S)$$
$$P_{kln} + P_{nkl} + P_{lnk} = 0\,. \quad (A)$$

As the tensor

$$S_{kln} = P_{kln} + P_{lkn}$$

is symmetric in the first two indices, the number of linearly independent conditions of the type (S) equals

$$S = N\,\frac{N(N+1)}{2} \,.$$

If the conditions (S) are fulfilled, the tensor P_{kln} is antisymmetric in the first two indices, and the tensor

$$A_{kln} = P_{kln} + P_{nkl} + P_{lnk}$$

is antisymmetric in any pair of indices. Consequently, the number S is added by the number

$$A = \frac{N(N-1)(N-2)}{6}$$

of linearly independent conditions of the type (A). Therefore,

$$P = N^3 - S - A = \frac{1}{3}\,N\,(N^2-1)$$

and, consequently,

$$C = \frac{1}{3}\,N^2\,(N^2-1)\,. \qquad (65)$$

In particular, in the two-dimensional case $C = 4$. The Ricci tensor has the same number of components. Therefore, in the two-dimensional case the following equality is possible and is indeed fulfilled (see [7, p. 291]):

$$R^a_{klm} = \delta^a_k\,R_{lm} - \delta^a_l\,R_{km}\,. \qquad (66)$$

For the Christoffel connection, as is shown in [9, p. 201], the number of essential components of the curvature tensor (52) equals

$$\frac{C}{4} = \frac{1}{12}\,N^2\,(N^2-1)\,. \qquad (67)$$

In the two-dimensional case $\frac{C}{4} = 1$, the Einstein tensor (45) is zero whereas the Riemann–Christoffel tensor equals

$$R_{klmn} = \frac{1}{2}\,R\,(g_{km}\,g_{ln} - g_{kn}\,g_{lm})\,. \qquad (68)$$

In the three-dimensional case $\frac{C}{4} = 6$. In this case, the Ricci tensor has the same number of components. Therefore, in the three-dimensional case the following equality is possible and indeed fulfilled (see [10, p. 366]):

$$R_{klmn} = L_{km}\,g_{ln} - L_{kn}\,g_{lm} + L_{ln}\,g_{km} - L_{lm}\,g_{kn}\,, \qquad (69)$$

where

$$L_{mn} = R_{mn} - \frac{1}{4} R g_{mn} . \qquad (70)$$

Consequently, in the three-dimensional case, semiprimitive connection given by the Christoffel bracket is primitive.

By virtue of (47) and (49), the tensor R^{ab}_{kl} can be considered as a linear operator acting in the space of bivectors X^{kl}, i.e. anti-symmetric tensors of the type (2/0). It seems to be interesting to examine the equation

$$\frac{1}{2} R^{ab}_{kl} X^{kl} = \lambda X^{ab}$$

for eigenvalues of this operator, as well as the tensors

$$\lambda (\delta^a_k \delta^b_l - \delta^b_k \delta^a_l) - R^{ab}_{kl} ,$$

$$\lambda (g_{km} g_{ln} - g_{kn} g_{lm}) - R_{klmn}$$

and a symmetric bilinear form

$$X^{kl} R_{klmn} Y^{mn} .$$

Gauss Curvature

Inserting the definition (10) and formula (40) into (52), it is not difficult to obtain the following expression for the Riemann-Christoffel tensor:

$$R_{klmn} = g_{pq} (\Gamma^p_{kn} \Gamma^q_{lm} - \Gamma^p_{ln} \Gamma^q_{km}) +$$

$$+ \frac{1}{2} (\partial^2_{kn} g_{lm} + \partial^2_{lm} g_{kn} - \partial^2_{ln} g_{km} - \partial^2_{km} g_{in}) . \qquad (71)$$

Specifically, at $N = 2$ in the Gauss notation we write

$$g_{11} = E, \quad g_{12} = g_{21} = F, \quad g_{22} = G, \qquad (72)$$

$$g = E G - F F .$$

Hence, we get

$$g^{11} = \frac{G}{g}, \quad g^{12} = g^{21} = -\frac{F}{g}, \quad g^{22} = \frac{E}{g}. \qquad (73)$$

If then we denote

$$\Gamma_{a,mn} = g_{as} \Gamma^a_{mn} , \qquad (74)$$

we obtain

$$\begin{aligned} 2\Gamma_{1,11} &= \partial_1 E, & 2\Gamma_{2,11} &= 2\partial_1 F - \partial_2 E, \\ 2\Gamma_{1,12} &= \partial_2 E, & 2\Gamma_{2,12} &= \partial_1 G, \\ 2\Gamma_{1,22} &= 2\partial_2 F - \partial_1 G, & 2\Gamma_{2,22} &= \partial_2 G. \end{aligned} \qquad (75)$$

Making use of the equality

$$g_{pq} (\Gamma^p_{kn} \Gamma^q_{lm} - \Gamma^p_{ln} \Gamma^q_{km}) = g^{pq} (\Gamma_{p,kn} \Gamma_{q,lm} - \Gamma_{p,ln} \Gamma_{q,km}) , \qquad (76)$$

and utilizing (68), (71) and (75) we find

$$R_{1212} = K g = g^{pq} (\Gamma_{p,12} \Gamma_{q,12} - \Gamma_{p,11} \Gamma_{q,22}) -$$

$$- \frac{1}{2} \partial^2_{22} E + \partial^2_{12} F - \frac{1}{2} \partial^2_{11} G , \qquad (77)$$

where

$$K = \frac{1}{2} R \qquad (78)$$

is the Gauss curvature for which Gauss [11, p. 139] derived the following formula

$$4 g g K = E X + F Y + G Z - 2 g (\partial^2_{22} E - 2 \partial^2_{12} F + \partial^2_{11} G) . \qquad (79)$$

Here

$$X = \partial_2 E \cdot \partial_2 G - 2 \partial_1 F \cdot \partial_2 G + \partial_1 G \cdot \partial_1 G ,$$

$$Y = \partial_1 E \cdot \partial_2 G - \partial_2 E \cdot \partial_1 G -$$

$$- 2 \partial_2 E \cdot \partial_2 F + 4 \partial_1 F \cdot \partial_2 F - 2 \partial_1 F \cdot \partial_2 G ,$$

$$Z = \partial_1 E \cdot \partial_1 G - 2 \partial_1 E \cdot \partial_2 F + \partial_2 E \cdot \partial_2 E .$$

Distinguished Gauss Theorem

Consider a two-dimensional manifold represented as a surface

$$\mathbf{r} = \mathbf{r}(u, v)$$

in the three-dimensional Euclidean space. In this case the components of the metric tensor are scalar products

$$g_{mn} = \mathbf{r}_m \, \mathbf{r}_n \, . \tag{80}$$

Consequently, the components (74) are also scalar products

$$\Gamma_{a,mn} = \frac{1}{2} (\partial_m \, g_{an} + \partial_n \, g_{am} - \partial_a \, g_{mn}) = \mathbf{r}_a \, \mathbf{r}_{mn} \, . \tag{81}$$

Then it follows that the scalar products

$$\mathbf{r}_a \, (\mathbf{r}_{mn} - \Gamma^s_{mn} \, \mathbf{r}_s)$$

vanish, and thus,

$$\mathbf{r}_{mn} = \Gamma^s_{mn} \, \mathbf{r}_s + B_{mn} \, \mathbf{P} \, , \tag{82}$$

where $\mathbf{P}$ is a unit vector perpendicular to the surface at the point of application, the components B_{mn} are scalar products

$$B_{mn} = \mathbf{P} \, \mathbf{r}_{mn} \, . \tag{83}$$

Differentiating (81) we obtain the equality

$$\mathbf{r}_{ab} \, \mathbf{r}_{mn} + \mathbf{r}_a \, \mathbf{r}_{bmn} = \frac{1}{2} (\partial^2_{bm} \, g_{an} + \partial^2_{bn} \, g_{am} - \partial^2_{ab} \, g_{mn}) \, .$$

Rearranging here indices m and b we obtain another equality

$$\mathbf{r}_{am} \, \mathbf{r}_{bn} + \mathbf{r}_a \, \mathbf{r}_{mbn} = \frac{1}{2} (\partial^2_{mb} \, g_{an} + \partial^2_{mn} \, g_{ab} - \partial^2_{am} \, g_{bn}) \, .$$

Subtracting the second from the first equality results in the third equality

$$\mathbf{r}_{ab} \, \mathbf{r}_{mn} - \mathbf{r}_{am} \, \mathbf{r}_{bn} = \frac{1}{2} (\partial^2_{bn} \, g_{am} + \partial^2_{am} \, g_{bn} - \partial^2_{ab} \, g_{mn} - \partial^2_{mn} \, g_{ab}) \, .$$

Comparing the latter with equality (71) we, on the one hand, find

$$\mathbf{r}_{ab} \, \mathbf{r}_{mn} - \mathbf{r}_{am} \, \mathbf{r}_{bn} = R_{bman} - g_{pq} \, (\Gamma^p_{bn} \, \Gamma^q_{am} - \Gamma^p_{mn} \, \Gamma^q_{ab}) \, .$$

On the other hand, according to (82),

$$\mathbf{r}_{ab} \, \mathbf{r}_{mn} = B_{ab} \, B_{mn} + g_{pq} \, \Gamma^p_{mn} \, \Gamma^q_{ab} \, .$$

From the two last equalities it follows that

$$R_{bman} = B_{ab} \, B_{mn} - B_{am} \, B_{bn} \, . \tag{84}$$

Comparing this formula with (68) and (78) we obtain

$$B_{ab} \, B_{mn} - B_{am} \, B_{bn} = K \, (g_{ab} \, g_{mn} - g_{am} \, g_{bn}) \, . \tag{85}$$

In particular,

$$B_{11} \, B_{22} - B_{21} \, B_{12} = K \, (g_{11} \, g_{22} - g_{21} \, g_{12}) \, . \tag{86}$$

Gauss proceeded in the opposite way: having called as the measure of curvature the ratio K of determinants of $|B_{mn}|$ to $|g_{mn}|$, he deduced formula (79) for K [11, p. 139], that itself leads to the following distinguished theorem. The Gauss curvature is invariant under isometric transformations of surfaces [11, p. 140].

Integral Stokes Theorem

In the theory of integration on manifolds (I have composed a summary of this theory and published in part in [12]), especially distinguished are tensor fields without upper indices with components antisymmetric in all lower indices

$$\Omega_{a_1 \ldots a_K} \, . \tag{87}$$

The tensor field with antisymmetric components (87) will be called the field of type $[K]$. When $K < 2$, the condition of antisymmetry loses its meaning and is removed in this case, however, the concept of type $[K]$ is retained, and so

$$[1] = (0/1)\,, \qquad [0] = (0/0)\,.$$

In other words, a tensor field of the type [1] is a covector field Ω_a, whereas a tensor field of the type [0] is a scalar field Ω. However, when $K > N$, all fields of the type $[K]$ vanish, and therefore, in the bracket $[K]$, it makes sense to consider only integer K from the interval $0 \leq K \leq N$. Within these limits, the dimension of region of integration changes, respectively. The number of linearly independent components (87) equals

$$C_N^K = \frac{N!}{K!\,(N-K)!} \tag{88}$$

Integral over the K-dimensional region of integration D is taken of a tensor field of the type $[K]$. If this region is given in the form

$$x^a = x^a(u^1, ..., u^K), \qquad a = 1, ..., N\,, \tag{89}$$

the integral of tensor field (87) of the type $[K]$ over the region D is equal to the K-multiple integral

$$\Omega\,(D) = \int \cdots \int \Omega_{a_1 \dots a_K} \frac{\partial x^{a_1}}{\partial u^1} \cdots \frac{\partial x^{a_K}}{\partial u^K}\, d\,u^1 \dots d\,u^K\,. \tag{90}$$

The one-dimensional region L is a curve. If it is given in the form

$$x^a = x^a\,(u)\,, \qquad a = 1, ..., N\,, \tag{91}$$

the integral of the covector field Ω_a equals

$$\Omega\,(L) = \int \Omega_a \frac{dx^a}{du}\, d\,u\,. \tag{92}$$

The zero-dimensional region is point P. The integral of the scalar field Ω over this region equals the value of $\Omega\,(P)$ of that field at point P.

Under the conditions formulated above one can guarantee that the integrals depend neither on the choice of coordinates x, nor on the choice of parameters u, but if an integral does depend on the choice of coordinates, it occurs neither in the theory of manifolds, not in general relativity.

A principal part in the theory of integration on manifolds belongs to the Stokes theorem according to which an integral of the tensor field Ω of type $[K]$ along the boundary D', restricting a $(K+1)$-dimensional region D equals an integral over the region D of the external derivative Ω' of the field Ω, i.e.

$$\Omega\,(D') = \Omega'\,(D)\,. \tag{93}$$

The external derivative is defined as follows. If the component (87) is antisymmetric in all K indices, then the combination

$$\begin{aligned} \Omega'_{a a_1 \dots a_K} = \partial_a\, \Omega_{a_1 \dots a_K} - \partial_{a_1}\, \Omega_{a a_2 \dots a_K} - \\ - \partial_{a_2}\, \Omega_{a_1 a a_3 \dots a_K} - \dots - \partial_{a_K}\, \Omega_{a_1 \dots a_{K-1} a} \end{aligned} \tag{94}$$

of its partial derivatives is antisymmetric in all its $(K+1)$ indices. If, besides, (87) is a component of a tensor field of the type $[K]$, the combination (94) is a component of a tensor field of the type $[K+1]$. The tensor field Ω' of type $[K+1]$ thus obtained is called the external derivative of the initial field Ω of type $[K]$. At $K = 0$ formula (94) loses its meaning and it should be supplemented with the definition

$$\Omega'_a = \partial_a\, \Omega\,.$$

Let us make the following highly important remark: Unlike the covariant derivative, the definition of the external derivative did not require the affine connection.

Let us notice also that the second external derivative equals zero.

Integral Gauss Theorem

If the manifold X is orientable, there exists a tensor field E of the type $[N]$ such that everywhere

$$e = E_{1 \dots N} > 0 . \tag{95}$$

We shall call it the measure of volume. Along with E, consider a vector field F^a and compose a tensor field (FE) of the type $[N-1]$ with components being contractions

$$(F\, E)_{a_1 \dots a_{N-1}} = F^b\, E_{ba_1 \dots a_{n-1}} . \tag{96}$$

The external derivative $(F\, E)'$ of the field $(F\, E)$ is a tensor field of the type $[N]$. Therefore, it differs from the field E only by a scalar factor f:

$$(F\, E)' = f\, E . \tag{97}$$

Substituting (96) into (94) we find this factor to be

$$f = e^{-1}\, \partial_b\, (e\, F^b) . \tag{98}$$

The scalar field f is called the divergence of the vector field F^b (with respect to the measure of volume E).

The Stokes theorem as applied to the tensor field (96) of the type [N-1] is called the Gauss theorem.

Integral Gauss Theorem in the Riemann World

In the Riemann world with the metric

$$ds^2 = g_{ab}\, dx^a\, dx^b \tag{99}$$

the affine connection Γ^a_{mn} is given by the Christoffel brackets (40). The volume element is defined as follows:

$$d\,X = e\, dx^1 \dots dx^N , \tag{100}$$

where the component e determined by formula (95) equals (43). The divergence (98) is then equal to

$$f = e^{-1}\, \partial_b\, (e\, F^b) = \nabla_b\, F^b . \tag{101}$$

The Gauss theorem in the Riemann world is written in the form

$$\int_{D'} F^a\, g^{ab}\, d\,\Sigma^b = \int_D \nabla_b\, F^b\, d\,X , \tag{102}$$

where $d\,\Sigma^b$ is the vector of an area element on the boundary D' of the region D . Note that

$$\nabla_b\, F^b = (\overset{\circ}{\nabla}_b - P^a_{ab})\, F^b , \tag{103}$$

where P^a_{mn} is the tensor of affine deformation (20).

It is highly important to realize that the Gauss theorem is applicable only to the vector field. A widely spread but incorrect conclusion that the energy of gravitational field is nonlocalizable results from application of the Gauss theorem to an object that is not a vector field.

The expression "energy is nonlocalizable" means that the conception of local energy (i.e. energy belonging to the given place) is ambiguous. For instance, according to [13, p. 436]:

"The energy of the gravitational field is nonlocalizable, i.e. no unambiguously definite density of energy exists."

But we must imagine that we are speaking about the concept playing the central role in modern theoretical physics! Just with this statement the paper [13] opens:

"The concept of energy plays the central role in modern theoretical physics."

It is obvious, that this assertions contradict one another. How do you find this pseudo-tensor dislocation?

The introduction of the background conection gives the gravity theory a totally tensor nature and abolish incorrect conclusion that energy is nonlocalizable.

Equations of Geodesics

The concept of affine connection can be obtained by considering the equations of geodesics in the form

$$\frac{dx^a}{d\tau} = p^a , \qquad \frac{dp^a}{d\tau} = - \Gamma^a_{mn} \, p^m \, p^n . \tag{104}$$

The right-hand sides of these equations will be considered to be components .

$$F^a(x,p) = p^a , \qquad F^{N+a}(x,p) = - \Gamma^a_{mn}(x) \, p^m \, p^n \tag{105}$$

of the vector field on the 2 N-dimensional manifold with coordinates

$$x^1 , \dots , x^N , \qquad x^{N+1} = p^1 , \dots , x^{2N} = p^N . \tag{106}$$

In accordance with this condition, passing to the new coordinates

$$y^k = y^k (x,p) , \qquad k = 1 , \dots , 2 N , \tag{107}$$

we transform Eqs. (104) to the form

$$\frac{dy^k}{d\tau} = H^k = \frac{\partial y^k}{\partial x^a} \, p^a - \frac{\partial y^k}{\partial p^a} \, \Gamma^a_{mn} \, p^m \, p^n , \qquad k = 1 , \dots 2 N . \tag{108}$$

In particular, for

$$y^a = y^a (x) , \qquad y^{N+a} = q^a = \frac{\partial y^a}{\partial x^m} \, p^m \tag{109}$$

we obtain

$$H^a = \frac{\partial y^a}{\partial x^m} \, p^m , \qquad H^{N+a} = \left(\frac{\partial^2 y^a}{\partial x^m \partial x^n} - \frac{\partial y^a}{\partial x^s} \, \Gamma^s_{mn} \right) p^m \, p^n , \tag{110}$$

so that in coordinates (109) Eqs. (104) retain their form:

$$\frac{dy^a}{d\tau} = q^a , \qquad \frac{dq^a}{d\tau} = - H^a_{mn} \, q^m \, q^n , \tag{111}$$

where the components q^a are connected with components p^a by the vector rule

$$q^a = \frac{\partial y^a}{\partial x^m} \, p^m \tag{112}$$

and the components H^a_{mn} are connected with the components Γ^a_{mn} by the rule

$$- H^a_{mn} = \left(\frac{\partial^2 y^a}{\partial x^i \partial x^j} - \frac{\partial y^a}{\partial x^s} \, \Gamma^s_{ij} \right) \frac{\partial x^i}{\partial y^m} \, \frac{\partial x^j}{\partial y^n} , \tag{113}$$

equivalent to the rule (2) that defines the affine connection.

Separation of transformations (109) among all coordinate transformations of $2N$-dimensional manifold preserves its structure of the tangent bundle $P(X)$ of vectors of the initial manifold X. We have proved the theorem: if the functions (105) of x , p constitute a vector field on $P(X)$, the functions Γ^a_{mn} of x compose a symmetric affine connection on X. It is not difficult to observe that the inverse theorem is also true: if the functions Γ^a_{mn} of x constitute a symmetric affine connection on X, the functions (105) of x , p make up a vector field on $P(X)$.

It is interesting that $P(X)$ is an orientable manifold since the Jacobian of transformation (109) is larger than zero. Indeed, as the derivatives $\partial y^k / \partial p^b$ are zero if $k \le N$ and are equal to $\partial y^k / \partial x^b$ if $k > N$, this Jacobian equals

$$\frac{\partial \left(y^1, \dots , y^N ; \, q^1, \dots , q^N\right)}{\partial \left(x^1, \dots , x^N ; \, p^1, \dots , p^N\right)} = \left(\frac{\partial \left(y^1, \dots , y^N\right)}{\partial \left(x^1, \dots , x^N\right)} \right)^2 > 0 . \tag{114}$$

Since it is positive and independent of p, one can on the manifold $P(X)$ define the measure of volume with the use of a tensor field of the type $2N$ whose principal component is independent of p:

$$E_{1 \dots N \dots 2N} = E = E(X^1 \dots , x^N) > 0. \tag{115}$$

The divergence of the vector field (105) with respect to that measure on $P(X)$ equals the scalar field

$$f(x, p) = E^{-1} \, p^a \left(\partial_a E - 2 \, \Gamma^m_{am} \, E\right) . \tag{116}$$

If this divergence is zero, the connection Γ^a_{mn} is equiaffine. In this case

$$\Gamma_a == \Gamma^m_{am} = \frac{1}{2E}\,\partial_a E\,. \tag{117}$$

If the connection Γ^a_{mn} equals the Christoffel bracket (40), then according to (42), $E = |g|$ and the divergence (116) is zero.

This approach to the concept of affine connection has been considered in paper [14] on the kinetic theory of gases in general relativity. The statement that the divergence (116) in this theory is zero is analogous to the Liouville theorem known in statistical physics.

Fundamentals of the Tensor Theory of Gravitation

We assume that there exist two symmetric tensor fields: g^{ab} and P^a_{mn}. The condition of symmetry means that $g^{ab} = g^{ba}$ and $P^a_{mn} = P^a_{nm}$.

We also suppose that the tensor g^{ab} at the point of its application defines a tangent space of velocities endowing it with the Lobachevsky geometry [15]. Consequently, there exists an inverse field g_{ab} and the metric (99) of a normal hyperbolic type.

We introduce the Christoffel connection (40) and background connection

$$\check{\Gamma}^a_{mn} = \Gamma^a_{mn} + P^a_{mn}\,. \tag{118}$$

We assume that the action of matter is independent of the tensor P^a_{mn} and the action of the gravitational field is given by the contraction

$$L = g^{ab}\,P_{ab}\,, \tag{119}$$

where the tensor P_{ab} is defined by (39). In view of the first condition, the energy-momentum tensor of matter M_{ab} defined according to Hilbert in the expounded theory is exactly the same as in the Einstein theory, and thus,

$$\nabla_a\,g^{an}\,M_{nb} = 0\,. \tag{120}$$

By virtue of the second condition, the tensor R_{ab} in the field gravitational Einstein equation is replaced by the tensor

$$S_{ab} = R_{ab} - \frac{1}{2}\,(\check{R}_{ab} + \check{R}_{ab})\,, \tag{121}$$

and the connection Γ^a_{mn} in the Einstein pseudotensor is changed by the tensor $(-\,P^a_{mn})$. As a result, we arrive at the gravitational equation

$$S_{ab} - \frac{1}{2}\,S\,g_{ab} = \frac{8\pi\gamma}{c^4}\,M_{ab}\,, \tag{122}$$

where

$$S = g^{mn}\,S_{mn}\,, \tag{123}$$

and at the energy-momentum tensor

$$E^a_b = \Phi^{mn}_b\,(P^a_{mn} - P_m\,\delta^a_n) - L\,\delta^a_b\,, \tag{124}$$

where

$$\Phi^{mn}_b = (\check{\nabla}_b - P_b)\,g^{mn} = g^{ms}\,P^n_{sb} + g^{ns}\,P^m_{sb} - g^{mn}\,P_b\,. \tag{125}$$

In deriving the field equation (122), we can replace the Lagrangian L by S according to the Gauss theorem (102) since (33) results in the equality

$$L = S + \nabla_a(\Phi^a - P^a)\,, \tag{126}$$

where

$$\Phi^a = \Phi^{an}_n = g^{mn}\,P^a_{mn}\,, \tag{127}$$

$$P^a = g^{ab}\,P_b = g^{ab}\,P^n_{bn}\,. \tag{128}$$

In connection with the tensor (121), of interest are the following equalities:

$$\nabla_a\,g^{an}\,(2\,S_{nb} - S\,g_{nb}) = \nabla_a\,g^{an}\,(\check{R}_{nb} + \check{R}_{bn}) - \nabla_b\,g^{mn}\,\check{R}_{mn} = \tag{129}$$
$$= (\check{\nabla}_a - P_a)\,[g^{an}\,(\check{R}_{nb} + \check{R}_{bn})] - g^{mn}\,\check{\nabla}_b\,\check{R}_{mn}\,,$$

$$g^{an}(2S_{nb} - S g_{nb}) - \dot{E}^a_b = (\breve{\nabla}_n - P_n)[U^{an}_b + \Phi^a \delta^n_b - \Phi^{na}_b], \tag{130}$$

where

$$U^{an}_b = g^{ns} P^a_{bs} - g^{as} P^n_{bs} + \delta^a_b(\Phi^n - P^n) - \delta^n_b(\Phi^a - P^a). \tag{131}$$

The vector Φ^a will be called the anharmonicity vector of the background connection (118) since in the Einstein theory of gravity the condition

$$\Phi^a = 0 \tag{132}$$

determines harmonic coordinates. Therefore, the tensor (125) will also be called the anharmonicity tensor of the background connection.

Provided that

$$\breve{\nabla}_a(\breve{R}_{nb} + \breve{R}_{bn}) = 0, \tag{133}$$

from (120), (122) and (129) it follows that

$$(\breve{R}_{nb} + \breve{R}_{bn})\Phi^n = 0. \tag{134}$$

Einstein-Rosen Theory as a Special Case

If the background connection (118) is semiprimitive, then

$$S_{ab} = R_{ab} \tag{135}$$

and equation (122) coincides with the Einstein-Hilbert equation. However, the condition (135) is too weak for separating the part of the Einstein theory that is historically related to the pseudotensor subculture. The necessary and sufficient condition for this is the following condition

$$\breve{R}^a_{kln} = 0 \tag{136}$$

of primitivity of the background connection (118). Under this condition, there is such a coordinate map, in which

$$\breve{\Gamma}^a_{mn} = 0 \tag{137}$$

throughout. The tensor (124) in this map coincides with the Einstein pseudotensor.

In time, N.Rosen has put things in order in the pseudotensor part of the Einstein theory [16], by introducing, besides the field metric tensor g_{ab}, the background tensor $\breve{g}_{ab}$ defining the metric of the Minkowski space-time

$$d\breve{s}^2 = \breve{g}_{ab}\, dx^a\, dx^b, \tag{138}$$

written in arbitrary coordinates. The Christoffel brackets for the metric (138) constitute the background connection $\breve{\Gamma}^a_{mn}$ obeying the condition (137). The method developed by Rosen was called the two-metric formalism.

The background tensor itself does not enter into the final Rosen formulae, the latter contain only the Christoffel brackets. This can be verified, e.g., for the condition of harmonicity (132). However, we already know that the metric (62) contains more information than the connection (60). This information, excessive as compared to the condition (136), has been taken into account in the Logunov relativistic theory of gravity [17].

Logunov Theory

The Logunov theory goes beyond the scope of the consideration made until now for the tensor theory of gravity because in the Logunov theory, the action of matter is added with the sum

$$g^{ab} P_{ab} + m^2\left(\frac{1}{2} g^{ab}\breve{g}_{ab} - \sqrt{\breve{g}\, g^{-1}} - 1\right), \tag{139}$$

rather than with the contraction (119). In this case, equation (122) is replaced by the equation

$$R_{ab} - \breve{R}_{ab} = \frac{8\pi\gamma}{c^4}(M_{ab} - \frac{1}{2} M g_{ab}) + \frac{m^2}{2}(g_{ab} - \breve{g}_{ab}). \tag{140}$$

In the given case, the background connection is the Christoffel brackets for the background metric, and consequently,

$$\breve{R}_{ab} = \breve{R}_{ba}. \tag{141}$$

If the background connection is semiprimitive, i.e. if in the given case

$$\breve{R}_{ab} = 0\,, \tag{142}$$

equation (140) coincides with the Logunov equation of gravitation that was originally written in [17, p. 40]. In the Logunov theory, the condition (136), more strong than (142), is fulfilled. The gravitational equation in another form is written in [17 p.28] and in our notation it looks as follows:

$$G_{ab} + \frac{m^2}{2}\left(g_{ab} + \breve{g}_{ab} - \frac{1}{2}\breve{g}_{rs}\, g^{rs}\, g_{ab}\right) = \frac{8\,\pi\,\gamma}{c^4}\, M_{ab}\,. \tag{143}$$

The Logunov gravitational equation (143), in view of (46) and (120), results in the remarkable corollary

$$m^2\, \nabla_b \left(\breve{g}_{as}\, g^{sb} - \frac{1}{2}\breve{g}_{rs}\, g^{rs}\, \delta^b_a\right) = 0\,, \tag{144}$$

where

$$\nabla_b \left(\breve{g}_{as}\, g^{sb} - \frac{1}{2}\breve{g}_{rs}\, g^{rs}\, \delta^b_a\right) = \breve{g}_{as}\, \Phi^s \tag{145}$$

with the vector of anharmonicity Φ^a given by (127). Therefore, as in the Logunov theory, $m \neq 0$ and the tensor $\breve{g}_{as}$ is nondegenerate, the condition of harmonicity (132) is a consequence of the field equation (143). However, when $m = 0$, the condition of harmonicity (132) does not follow from equation (143) and the Logunov equation (143) coincides with the Einstein equation.

Introduction of the background metric into the Lagrangian density (139) is, in our opinion, the central point in the Logunov theory of gravity.

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Received by Publishing Department
on July 9, 1996.

Черников Н.А. E2-96-250
Две связности в теории тяготения

Конспективно изложена теория аффинной связности. Построена теория тяготения с двумя аффинными связностями без нелокализуемых физических объектов. Эта теория включает теорию Эйнштейна — Розена как частный случай.

Работа выполнена в Лаборатории теоретической физики им. Н.Н.Боголюбова ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 1996

Chernikov N.A. E2-96-250
Two Connections in the Gravity Theory

A review of the theory of affine connection is made. The theory of gravitation with two affine connections without nonlocalizable physical objects is devepoled. This theory includes the Einstein — Rosen theory as a special case.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 1996