СООБЩЕНИЯ
ОБЪЕДИНЕННОГО
ИНСТИТУТА
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E2-96-244

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QUASIGROUP OF LOCAL-SYMMETRY TRANSFORMATIONS IN CONSTRAINED THEORIES

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1996

1 Introduction

In our previous paper [1] (below cited as paper II) we have considered constrained special-form theories with first- and second-class constraints (when the first-class primary constraints are the ideal of a quasi-algebra of all the first-class constraints) and have suggested the method of constructing the generator of local-symmetry transformations in both the phase and configuration space. It was proved that second-class constraints do not contribute to the transformation law of the local symmetry which entirely is stipulated by all the first-class constraints unlike the assertions appeared recently in the literature [2]-[4]. It was thereby shown that degeneracy of special-form theories with the first- and second-class constraints is due to their quasi-invariance under local-symmetry transformations. One must say the mentioned restriction on an algebra of constraints is fulfilled in most of the physically interesting theories, e.g., in electrodynamics, in the Yang – Mills theories, in the Chern – Simons theory, etc., and it has been used by us in previous works [5] in the case of dynamical systems only with first-class constraints and also by other authors at obtaining gauge transformations on the basis of different approaches [6]-[8], [9, 10]. However, in the existing literature there are examples of Lagrangians where this condition on constraints does not hold, e.g., Polyakov's string [11] and other model Lagrangians [10], [12]-[21]. Then it was natural to ask: Can the local-symmetry transformations be obtained in these theories? What is a role of second-class constraints under these transformations and, generally, what is the nature of the Lagrangian degeneracy in this case? For example, in ref.[19] it is stated that in the mentioned example the gauge transformation generators do not exist for the Hamiltonian formalism though for the Lagrangian one the gauge transformations may be constructed. In refs.[20, 21] in the case of theories only with first-class constraints we have shown that one can always pass to equivalent sets of constraints, for which the indicated condition holds valid, and, therefore, gauge transformations do exist both in the Hamiltonian and Lagrangian formalism. Therefore, the degeneracy of theories with the first-class constraints is due to their invariance under gauge transformations without restrictions on the algebra of constraints.

In the present paper it will be shown that, as in the presence only of first-class constraints, in the general case of systems with first- and second-class constraints, when the mentioned condition on constraints is not fulfilled, there always exist equivalent sets of constraints, for which the indicated condition holds valid. Therefore, the conclusions made in the former case about the existence of local-symmetry transformations in both the Hamiltonian and Lagrangian formalism and about the nature of degeneracy of theories hold valid also in the general case. Also the conclusion of paper II about the no influence of second-class constraints on local-symmetry transformations and the conclusion of ref.[22] about the mechanism of appearance of higher derivatives of coordinates and of group parameters in these transformations are valid in the general case.

One can see that in the case, when higher (than first order) derivatives of coordinates enter into the Noether transformation law in the configuration space, the generator of local-symmetry transformations in the phase space depends on derivatives of coordinates and momenta. Therefore, the Poisson brackets are not determined in this case, and there arises a question about the canonicity of the obtained transformations. Here we shall show that the difficulty with the Poisson brackets is surmounted through the extension by Ostrogradsky of phase space and the proof of canonicity of local-symmetry transformations in this phase space, which had been furnished by us earlier for theories only with first-class constraints [22], hold true also in the presence of second-class constraints in theory.

This paper is organized as follows. In section 2, for the general case of systems with first- and second-class constraints (without restriction on the algebra of first-class constraints) we derive the local-symmetry transformations from the requirement for them to map the solutions of the Hamiltonian equations of motion into the solutions of the same equations. The derivation of a generator from this requirement (unlike the one from quasi-invariance of the action functional in paper II) is made to establish a ratio of the groups of local-symmetry transformations under which the equations of motion and the action functional are invariant (as it is known, generally, the action functional is invariant under a more slender group of symmetry transformations than the corresponding equations

of motion do). As in paper II, these derivations are based substantially on results of our previous paper [23] (paper I) on the separation of constraints into the first- and second-class ones and on properties of the canonical set of constraints. In section 3, we consider the local-symmetry transformations in the extended (by Ostrogradsky) phase space. In the 4th section the method is illustrated by an example. In Appendix A, we describe the way of passing to an equivalent constraint set when all the primary constraints of the first class are momentum variables.

2 Local-Symmetry Transformations in General Case of Systems with First- and Second-Class Constraints

As in the special case (paper II; below we shall refer to formulas of papers I and II as $(\mathrm{I}.\cdots)$ and $(\mathrm{II}.\cdots)$), we shall consider a dynamical system with the canonical set $(\Phi_\alpha^{m_\alpha}, \Psi_{a_i}^{m_{a_i}})$ of first- and second-class constraints, respectively ($\alpha = 1,\cdots,F$, $m_\alpha = 1,\cdots,M_\alpha$; $a_i = 1,\cdots,A_i$, $m_{a_i} = 1,\cdots,M_{a_i}$, $i = 1,\cdots,n$), properties of which are expressed by the Poisson brackets among them and the Hamiltonian by the formulas (II.9)-(II.12):

$$\{\Phi_\alpha^{m_\alpha}, H\} = g_{\alpha}^{m_\alpha}{}_{\beta}^{m_\beta}\ \Phi_\beta^{m_\beta}, \quad m_\beta = 1,\cdots,m_\alpha+1, \tag{1}$$

$$\{\Psi_{a_i}^{m_{a_i}}, H\} = \bar{g}_{a_i}^{m_{a_i}}{}_{\alpha}^{m_\alpha}\ \Phi_\alpha^{m_\alpha} + \sum_{k=1}^{n} h_{a_i}^{m_{a_i}}{}_{b_k}^{m_{b_k}}\ \Psi_{b_k}^{m_{b_k}}, \quad m_{b_n} = m_{a_i}+1, \tag{2}$$

$$\{\Phi_\alpha^{m_\alpha}, \Phi_\beta^{m_\beta}\} = f_{\alpha}^{m_\alpha}{}_{\beta}^{m_\beta}{}_{\gamma}^{m_\gamma}\ \Phi_\gamma^{m_\gamma}, \tag{3}$$

$$\{\Psi_{a_i}^{m_{a_i}}, \Psi_{b_k}^{m_{b_k}}\} = \bar{f}_{a_i}^{m_{a_i}}{}_{b_k}^{m_{b_k}}{}_{\gamma}^{m_\gamma}\ \Phi_\gamma^{m_\gamma} + \sum_{l=1}^{n} k_{a_i}^{m_{a_i}}{}_{b_k}^{m_{b_k}}{}_{c_l}^{m_{c_l}}\ \Psi_{c_l}^{m_{c_l}} + D_{a_i}^{m_{a_i}}{}_{b_k}^{m_{b_k}} \tag{4}$$

with general properties of the structure functions given by the formulas (II.13)-(II.16)

$$g_{\alpha}^{m_\alpha}{}_{\beta}^{m_\beta} = 0, \quad \text{if } m_\alpha + 2 \le m_\beta\ , \tag{5}$$

$$\left\{\begin{array}{ll} \bar{g}_{a_i}^{m_{a_i}}{}_{\alpha}^{m_\alpha} = 0, & \text{if } \ m_\alpha \ge m_{a_i}\ , \\ & \\ h_{a_i}^{m_{a_i}}{}_{b_k}^{m_{b_k}} = 0, & \text{if } \ m_{a_i}+2 \le m_{b_k} \ \text{ or if } \ a_i = b_k, \quad m_{a_i} = M_{a_i}, \\ & m_{b_k} \ge M_{a_i}, \end{array}\right. \tag{6}$$

$$f_{\alpha}^{1}{}_{\beta}^{m_\beta}{}_{\gamma}^{m_\gamma} = 0 \quad \text{for} \quad m_\gamma \ge 2, \tag{7}$$

$$
\begin{cases}
F_{a_i}^{M_{a_i}-l\ l+1}{}_{b_i} = (-1)^l\, F_{a_i}^{1}{}_{b_i}^{M_{b_i}}, & l = 0, 1, \cdots, M_{a_i} - 1, \\
F_{a_i}^{j}{}_{b_i}^{k} = 0, & \text{if } j + k \neq M_{a_i} + 1, \\
F_{a_i}^{m_{a_i}}{}_{b_k}^{m_{b_k}} = 0, & \text{if } a_i, b_k \text{ refer to different chains (or doubled} \\
& \text{chains) of second-class constraints } \left(D_{a_i}^{m_{a_i}}{}_{b_k}^{m_{b_k}} \overset{\Sigma}{=} F_{a_i}^{m_{a_i}}{}_{b_k}^{m_{b_k}}\right)
\end{cases}
\tag{8}
$$

and with

$$
H = H_c + \sum_{k=1}^{n} (\mathbf{K}^{1\ k})^{-1}_{b_k\ a_k} \{\Psi^k_{a_k}, H_c\} \Psi^1_{b_k} \tag{9}
$$

being a first-class function [24]; H_c is the canonical Hamiltonian.
Passing to this set from the initial one is always possible in an arbitrary case by the method developed in paper I. Here we shall consider the general case (when first-class primary constraints are not the ideal of quasi-algebra of all the first-class constraints, i.e. the restriction (II.25) is not fulfilled) and derive local-symmetry transformations.

A group of phase-space coordinate transformations that maps each solution of the Hamiltonian equations of motion into the solution of the same equations will be called the symmetry transformation.

Consider the Hamiltonian equations of motion in the following form:

$$
\begin{cases}
\dot{q}_i \overset{\Sigma_1}{\approx} \{q_i, H_T\}, \quad \dot{p}_i \overset{\Sigma_1}{\approx} \{p_i, H_T\}, \quad i = 1, \cdots, N, \\
\Psi^1_{a_k} \overset{\Sigma_1}{\approx} 0, \quad a_k = 1, \cdots, A_k \ (k = 1, \cdots, n), \\
\Phi^1_\alpha \overset{\Sigma_1}{\approx} 0, \quad \alpha = 1, \cdots, F,
\end{cases}
\tag{10}
$$

where

$$
H_T = H + u_\alpha \Phi^1_\alpha, \tag{11}
$$

u_α are undetermined Lagrange multipliers; the symbol $\overset{\Sigma_1}{\approx}$ means weak equality on the primary-constraints surface Σ_1.

Consider also the infinitesimal transformations of the phase-space coordinates

$$
\begin{cases}
q'_i = q_i + \delta q_i, \qquad \delta q_i = \{q_i, G\}, \\
p'_i = p_i + \delta p_i, \qquad \delta p_i = \{p_i, G\}
\end{cases}
\tag{12}
$$

with the generator G sought in the form (II.4)

$$
G = \varepsilon^{m_\alpha}_\alpha \Phi^{m_\alpha}_\alpha + \eta^{m_{a_i}}_{a_i} \Psi^{m_{a_i}}_{a_i}. \tag{13}
$$

To recognize a role of the second-class constraints in the local-symmetry transformations in this general case, we consider them on the same basis as the first-class constraints.

Like in refs.[8, 9, 19, 21], we will require the transformed quantities $q'_i(t)$ and $p'_i(t)$ defined by (12) to be solutions of the Hamiltonian equations of motion (10) provided that the initial $q_i(t)$ and $p_i(t)$ do this, i.e.

$$
\begin{aligned}
&\dot{q}'_i \overset{\Sigma_1}{\approx} \frac{\partial H'_T}{\partial p_i}(q', p'), \quad \dot{p}'_i \overset{\Sigma_1}{\approx} -\frac{\partial H'_T}{\partial q_i}(q', p'), \quad i = 1, \cdots, N, \\
&\Psi^1_{a_k}(q', p') \overset{\Sigma_1}{\approx} 0, \quad a_k = 1, \cdots, A_k \ (k = 1, \cdots, n), \\
&\Phi^1_\alpha(q', p') \overset{\Sigma_1}{\approx} 0, \quad \alpha = 1, \cdots, F,
\end{aligned}
\tag{14}
$$

where

$$
H'_T = H_T + \delta u_\alpha(t) \Phi^1_\alpha(q, p) = H + u'_\alpha(t) \Phi^1_\alpha(q, p). \tag{15}
$$

Replacements in (15) of H_T by H and of $u_\alpha(t)$ by $u'_\alpha(t)$ are stipulated by that, generally speaking, different solutions that should be related with each other through the local-symmetry transformations correspond to different choices of the functions $u_\alpha(t)$ (the transformed quantities are denoted by the same letters with the prime). In equations (14) it is taken into consideration that the transformations (12) must conserve the primary-constraints surface Σ_1 (see the argument after formula (6) in paper II).

Equations (14) can be rewritten with taking account of (12) and (10) in the following form:

$$
\frac{d}{dt}\{q_i, G\} \overset{\Sigma_1}{\approx} \{\{q_i, H'_T\}, G\}, \tag{16}
$$

$$
\frac{d}{dt}\{p_i, G\} \overset{\Sigma_1}{\approx} \{\{p_i, H'_T\}, G\}, \quad i = 1, \cdots, N, \tag{17}
$$

$$
\{\Psi^1_{a_k}, G\} \overset{\Sigma_1}{\approx} 0, \quad a_k = 1, \cdots, A_k \ (k = 1, \cdots, n), \tag{18}
$$

$$
\{\Phi^1_\alpha, G\} \overset{\Sigma_1}{\approx} 0, \quad \alpha = 1, \cdots, F. \tag{19}
$$

We shall analyze consequences of the obtained equation system starting from the conditions of the primary-constraints surface conservation (18) and (19). As in the special case of paper II (the consideration is completely identical), from (18) we obtain that in expression (13) the coefficients of those i-ary constraints, which are the final stage of each chain of second-class constraints, and of those second-class primary constraints, which do not generate the secondary constraints, disappear:

$$\eta^{i}_{a_i} = 0 \quad \text{for } i = 1,\cdots,n, \tag{20}$$

As to the condition (19), we rewrite it in the form:

$$\{\Phi^1_\alpha, G\} = (f^{1}_{\alpha}{}^{m_\beta}_{\beta}{}^{1}_{\gamma}\,\Phi^1_\gamma + f^{1}_{\alpha}{}^{m_\beta}_{\beta}{}^{m_\gamma}_{\gamma}\,\Phi^{m_\gamma}_\gamma)\varepsilon^{m_\beta}_\beta + \{\Phi^1_\alpha, \Psi^{m_{a_i}}_{a_i}\}\eta^{m_{a_i}}_{a_i} \overset{\Sigma_1}{\approx} 0, \tag{21}$$

$$\alpha,\beta,\gamma = 1,\cdots,F; \quad m_\beta = 1,\cdots,M_\beta; \quad m_\gamma = 2,\cdots,M_\gamma.$$

The last term in (21) vanish for the canonical set of constraints (Φ,Ψ); therefore, the equality (21) were satisfied if $f^{1}_{\alpha}{}^{m_\beta}_{\beta}{}^{m_\gamma}_{\gamma} = 0$ for $m_\gamma \geq 2$ (i.e. the first-class primary constraints were the ideal of quasi-algebra of all the first-class constraints). This case is considered in paper II. Here we consider the general case of a constraint algebra when

$$f^{1}_{\alpha}{}^{m_\beta}_{\beta}{}^{m_\gamma}_{\gamma} \neq 0 \quad \text{for} \quad m_\gamma \geq 2. \tag{22}$$

For systems only with first-class constraints, the case (22) was investigated by us earlier [20, 21]. For systems with first- and second-class constraints, when (22) is the case, one can act in the same way as in the presence only of first-class constraints, i.e. using arbitrariness that is inherent in the generalized Hamiltonian formalism by Dirac, we shall pass to an equivalent set of constraints by the transformation that affects only first-class constraints:

$$\tilde{\Phi}^{m_\beta}_\beta = C_{\beta}^{m_\beta}{}_{\alpha}^{m_\alpha}\Phi^{m_\alpha}_\alpha, \qquad \det\left\|C_{\beta}^{m_\beta}{}_{\alpha}^{m_\alpha}\right\|_\Sigma \neq 0. \tag{23}$$

It is sufficient to consider a particular case of the transformation (23) when primary constraints remain unchanged, i.e.

$$C_{\beta}^{1}{}_{\alpha}^{m_\alpha} = \delta_{\beta\alpha} \qquad \text{for any} \quad m_\alpha.$$

It is not difficult to see that taking account of (3) we obtain

$$\begin{aligned}\{\Phi^1_\alpha, \tilde{\Phi}^{m_\beta}_\beta\} &= [\{\Phi^1_\alpha, C_{\beta}^{m_\beta}{}_{\gamma}^{m_\gamma}\} + f^{1}_{\alpha}{}^{m_\delta}_{\delta}{}^{m_\gamma}_{\gamma}C_{\beta}^{m_\beta}{}_{\delta}^{m_\delta}]\Phi^{m_\gamma}_\gamma \\ &+ f^{1}_{\alpha}{}^{m_\delta}_{\delta}{}^{1}_{\gamma}C_{\beta}^{m_\beta}{}_{\delta}^{m_\delta}\Phi^1_\beta, \qquad m_\beta, m_\delta, m_\gamma \geq 2.\end{aligned} \tag{24}$$

From the expression (24) it is clear that if we could choose $C_{\beta}^{m_\beta}{}_{\gamma}^{m_\gamma}$ so that the coefficients of secondary constraints vanish

$$\{\Phi^1_\alpha, C_{\beta}^{m_\beta}{}_{\gamma}^{m_\gamma}\} + f^{1}_{\alpha}{}^{m_\delta}_{\delta}{}^{m_\gamma}_{\gamma}C_{\beta}^{m_\beta}{}_{\delta}^{m_\delta} = 0, \tag{25}$$

for a new set of constraints $\tilde{\Phi}^{m_\beta}_\beta$ we were obtained $\tilde{f}^{1}_{\alpha}{}^{m_\beta}_{\beta}{}^{m_\gamma}_{\gamma} = 0$ (for $m_\gamma \geq 2$) and

$$\{\tilde{\Phi}^1_\alpha, \tilde{\Phi}^{m_\beta}_\beta\} = \tilde{f}^{1}_{\alpha}{}^{m_\delta}_{\beta}{}^{1}_{\gamma}\tilde{\Phi}^1_\gamma, \tag{26}$$

i.e. that is needed for the realization of (21). Thus, for $C_{\beta}^{m_\beta}{}_{\gamma}^{m_\gamma}$ we have derived the system of linear inhomogeneous equations in the first-order partial derivatives (25). This system can be shown to be fully integrable. The condition of integrability for systems of the type (25) looks as follows [25]

$$\{\Phi^1_\sigma, \{\Phi^1_\alpha, C_{\beta}^{m_\beta}{}_{\gamma}^{m_\gamma}\}\} - \{\Phi^1_\alpha, \{\Phi^1_\sigma, C_{\beta}^{m_\beta}{}_{\gamma}^{m_\gamma}\}\} = 0. \tag{27}$$

Using eq.(25), properties of the Poisson brackets and making some transformations we rewrite the relation (27) in the form

$$\begin{aligned}&[\{\Phi^1_\alpha, f^{1}_{\sigma}{}^{m_\delta}_{\delta}{}^{m_\gamma}_{\gamma}\} - f^{1}_{\alpha}{}^{m_\delta}_{\delta}{}^{m_\tau}_{\tau}f^{1}_{\sigma}{}^{m_\tau}_{\tau}{}^{m_\gamma}_{\gamma} - \{\Phi^1_\sigma, f^{1}_{\alpha}{}^{m_\delta}_{\delta}{}^{m_\gamma}_{\gamma}\} \\ &+ f^{1}_{\sigma}{}^{m_\delta}_{\delta}{}^{m_\tau}_{\tau}f^{1}_{\alpha}{}^{m_\tau}_{\tau}{}^{m_\gamma}_{\gamma}]C_{\beta}^{m_\beta}{}_{\delta}^{m_\delta} = 0, \qquad m_\beta, m_\delta, m_\tau \geq 2.\end{aligned} \tag{28}$$

Utilizing the Jacobi identity

$$\{\Phi^1_\alpha, \{\Phi^1_\sigma, \Phi^{m_\beta}_\beta\}\} + \{\Phi^{m_\beta}_\beta, \{\Phi^1_\alpha, \Phi^1_\sigma\}\} + \{\Phi^1_\sigma, \{\Phi^{m_\beta}_\beta, \Phi^1_\alpha\}\} = 0, \qquad m_\beta \geq 2$$

and the relation (3) we obtain

$$\begin{aligned}&[\{\Phi^1_\alpha, f^{1}_{\sigma}{}^{m_\delta}_{\delta}{}^{m_\gamma}_{\gamma}\} - f^{1}_{\alpha}{}^{m_\delta}_{\delta}{}^{m_\tau}_{\tau}f^{1}_{\sigma}{}^{m_\tau}_{\tau}{}^{m_\gamma}_{\gamma} - \{\Phi^1_\sigma, f^{1}_{\alpha}{}^{m_\delta}_{\delta}{}^{m_\gamma}_{\gamma}\} \\ &+ f^{1}_{\sigma}{}^{m_\delta}_{\delta}{}^{m_\tau}_{\tau}f^{1}_{\alpha}{}^{m_\tau}_{\tau}{}^{m_\gamma}_{\gamma}]\Phi^{m_\gamma}_\gamma = \{\{\Phi^1_\alpha, \Phi^1_\sigma\}, \Phi^{m_\delta}_\delta\},\end{aligned} \tag{29}$$

$$m_\beta \geq 2, \quad m_\gamma, m_\delta, m_\tau \geq 1.$$

Note that every primary constraint of first class contains at least one momentum variable, therefore, there always exist canonical transformations transforming the primary constraints into new momentum variables (see Appendix A). We shall regard such transformation to be carried out, therefore, the Poisson brackets between primary constraints may be considered to be strictly zero in the whole phase space. From here, the expressions in the square brackets in front of the constraints $\Phi^{m_\gamma}_\gamma$ on the left-hand side of the identity (29) being coefficients of the functionally independent quantities disappear each separately. As the condition (28) contains the same coefficients of $C_{\beta}^{m_\beta}{}_{\delta}^{m_\delta}$, it is satisfied identically, which proves the system of equations (25) to be fully integrable. Therefore, there always exists a set of constraints $\tilde{\Phi}^{m_\alpha}_\alpha$ equivalent to the initial set for which the condition (26) (and, therefore, (19)) holds valid. We shall below omit the mark "~".

Now, using the equality

$$\frac{d}{dt}\{A,B\} = \{\frac{\partial A}{\partial t}, B\} + \{A, \frac{\partial B}{\partial t}\} + \{\{A,B\}, H_T\} \tag{30}$$

(valid for arbitrary functions $A(q,p,t)$ and $B(q,p,t)$ given in the whole phase space) and the Jacobi identities for the quantities (q_i, G, H_T') and (p_i, G, H_T'), we represent equations (16) and (17) as

$$\{q_i, \frac{\partial G}{\partial t} + \{G, H_T'\}\} \overset{\Sigma_1}{\approx} 0, \tag{31}$$

$$\{p_i, \frac{\partial G}{\partial t} + \{G, H_T'\}\} \overset{\Sigma_1}{\approx} 0, \tag{32}$$

respectively. By virtue of an arbitrariness of the multipliers $u_\alpha(t)$, in what follows the prime will be omitted. If these equalities were the case in the whole phase space, it would follow from them that

$$\frac{\partial G(q,p,t)}{\partial t} + \{G(q,p,t), H_T(q,p,t)\} = f(t),$$

where $f(t)$ is an arbitrary function of time. However, since eqs.(31) and (32) are the case only on the surface Σ_1, we obtain that

$$\frac{\partial G(q,p,t)}{\partial t} + \{G(q,p,t), H_T(q,p,t)\} = f(t) + J(q,p,t), \tag{33}$$

where

$$J = c_\alpha(q,p,t)\Phi^1_\alpha(q,p) + d_{a_i}(q,p,t)\Psi^1_{a_i}(q,p),$$
$$\alpha = 1,\cdots,F, \quad a_i = 1,\cdots,A_i, \quad i = 1,\cdots,n.$$

However, both $f(t)$ and $J(q,p,t)$ are identity generators on the primary constraint surface, and can be ignored in subsequent discussions [9]. Note that equation (33) (with $f(t)$ ignored) is a necessary condition of that G is the generating function of infinitesimal transformations of local symmetry (12), and, furthermore, this is sufficient for a quasi-invariance (within a surface term) of the action functional

$$S = \int_{t_1}^{t_2} dt\ (p\dot{q} - H_T), \tag{34}$$

under these transformations. To see the latter, consider the variation of action, induced by the transformations (12),

$$\delta S = \int_{t_1}^{t_2} dt \Big[\frac{d}{dt}(p_i\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i}\dot{q}_i - \frac{\partial G}{\partial p_i}\dot{p}_i + \{G, H_T\}\Big]$$

which, with taking into account the relation

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial q_i}\dot{q}_i + \frac{\partial G}{\partial p_i}\dot{p}_i,$$

can be rewritten as

$$\delta S = \int_{t_1}^{t_2} dt \Big[\frac{d}{dt}(p_i\frac{\partial G}{\partial p_i} - G) + \frac{\partial G}{\partial t} + \{G, H_T\}\Big] \tag{35}$$

giving the desired result if eq.(33) is fulfilled.

Now, inserting the required form of the generator G (13) into (33), we obtain the equality (II.17) which must be satisfied by a proper inspection of the coefficients $\varepsilon^{m_\alpha}_\alpha$ and $\eta^{m_{a_i}}_{a_i}$. Further consideration repeats entirely the one of paper II resulting in that the second-class constraints do not contribute to the generator of local-symmetry transformations that is a linear combination of all the first-class constraints (and only of them)

$$G = B^{m_\alpha m_\beta}_{\alpha\ \beta}\phi^{m_\alpha}_\alpha \varepsilon^{(M_\alpha - m_\beta)}_\beta, \qquad m_\beta = m_\alpha, \cdots, M_\alpha. \tag{36}$$

with the coefficients

$$B^{m_\alpha m_\beta}_{\alpha\ \beta}\varepsilon^{(M_\alpha - m_\beta)}_\beta \quad \Big(\varepsilon^{(M_\alpha - m_\beta)}_\beta \equiv \frac{d^{M_\alpha - m_\beta}}{dt^{M_\alpha - m_\beta}}\varepsilon_\beta(t), \quad \varepsilon_\beta(t) \equiv \varepsilon^{M_\beta}_\beta\Big)$$

determined from the system of equations

$$\dot{\varepsilon}^{m_\alpha}_\alpha + \varepsilon^{m_\beta}_\beta g^{m_\beta m_\alpha}_{\beta\ \alpha} = 0, \qquad m_\beta = m_\alpha - 1, \cdots, M_\alpha, \tag{37}$$

with the help of the procedure of reparametrization described in paper II. The local-symmetry transformations of q and p determined by formulas (12) are also the quasi-invariance transformations of the action functional (34).

The corresponding transformations of local symmetry in the Lagrangian formalism are determined in the following way:

$$\delta q_i(t) = \{q_i(t), G\}\Big|_{p=\frac{\partial L}{\partial \dot{q}}}, \qquad \delta\dot{q}(t) = \frac{d}{dt}\delta q(t). \tag{38}$$

So, one can state that in the general case of theories with first- and second-class constraints (without restrictions on the constraint algebra) the representation of a certain quantity G as a linear combination of all the first-class constraints (and only of them) with the coefficients determined by the system of equations (37) is the necessary and sufficient condition for G to be the local-symmetry transformation generator. In addition, these are the necessary and sufficient conditions for (12) to be the quasi-invariance transformation of the functional of action in both the phase and $(q, \dot{q})$ space.

3 Local-Symmetry Transformations in the Extended Phase Space

One can see that in the case, when higher (than first order) derivatives of coordinates enter into the transformation law in the configuration spase and into the surface term in the action variation, the coefficients $B_{\alpha\ \beta}^{m_\alpha m_\beta}$ in expression (36) for G depend on the derivatives of q and p. It is clear, in this case there arises a question about "explicit" canonicity of the obtained transformations outside of the constraints surface. Therefore, it is clear that in the general case one should consider not only the violation of the condition (26) (the manner of the deed in this case is worked out in the previous section) but also that structure of constraints when there arise higher derivatives of coordinates in the law of local-symmetry transformations. Here we shall show how to construct these transformations in the latter case and prove the canonicity of gauge transformations in the extended (by Ostrogradsky) phase space, which has been shown by us earlier for theories with first-class constraints [22], to hold true also in the presence of second-class constraints in a theory.

Let us consider the singular Lagrangian $L(q, \dot{q})$, and let the higher (than first) derivatives of coordinates contribute to the corresponding law of local-symmetry transformations. Under these transformations we have

$$L' = L(q, \dot{q}) + \frac{d}{dt} F(q, \dot{q}, \ddot{q}, \cdots, \varepsilon, \dot{\varepsilon}, \cdots) \tag{39}$$

where $\varepsilon(t)$ are the group parameters. Adding to Lagrangian $L(q, \dot{q})$ the total time derivative of function which depends also on higher derivatives

does not change the Lagrangian equations of motion. As it is seen from (39), the theory with Lagrangian L' must be considered as the one with higher derivatives. Both Lagrangian and Hamiltonian formulations of the theories with L and L' are equivalent [27]. The Hamiltonian formulation of the theory with L' is built in the extended (by Ostrogradsky) phase space. An equivalence of Hamiltonian formulations of the theories with L and L' means that the Hamiltonian equations of motion of these both theories are related among themselves by canonical transformations. Therefore, the Hamiltonian formulation of the theory with the Lagrangian L must be built in the same extended phase space as it is the case for L'. Thus, the theory with L will be considered from the very beginning as the one with higher derivatives of the same order that they have in L'.

From the above reasoning it is clear that to require a canonicity of the local-simmetry transformations has the meaning only in the indicated extended phase space.

Let us construct the extended phase space using the formalism of theories with higher derivatives [26, 27, 28]. We shall determine the coordinates as follows

$$q_{1\ i} = q_i, \quad q_{s\ i} = \frac{d^{s-1}}{dt^{s-1}} q_i, \quad s = 2, \cdots, K, \quad i = 1, \cdots, N \tag{40}$$

where K equals the highest order of derivatives of q and p. The conjugate momenta defined by the formula [26, 27, 28]

$$p_{r\ i} = \sum_{l=r}^{K} (-1)^{l-r} \frac{d^{l-r}}{dt^{l-r}} \frac{\partial L}{\partial q_{r+1\ i}}$$

are

$$p_{1\ i} = p_i, \qquad p_{s\ i} = 0 \quad \text{for} \quad s = 2, \cdots, K. \tag{41}$$

The generalized momenta for $s \geq 2$ are extra primary constraints of the first class.

In the extended phase space the total Hamiltonian is written down as

$$\bar{H}_T = H_T(q_{1\ i}, p_{1\ i}) + \lambda_{s\ i}\, p_{s\ i}, \quad s \geq 2, \tag{42}$$

where H_T is of the same form as in the initial phase space (11) and $\lambda_{s\ i}$ are arbitrary functions of time.

Now the Poisson brackets are determined in the following way

$$\{A, B\} = \frac{\partial A}{\partial q_{r\ i}}\frac{\partial B}{\partial p_{r\ i}} - \frac{\partial A}{\partial p_{r\ i}}\frac{\partial B}{\partial q_{r\ i}}.$$

From (42) we may conclude that there do not appear additional secondary constraints corresponding to $p_{s\ i}$ for $s \geq 2$. The set of constraints in the extended phase space remains the same as in the initial phase space, obeys the same algebra (1)-(4), and does not depend on the new coordinates and momenta as also H_T does.

We shall seek a generator $\overline{G}$ in the extended phase space in the form, analogous to the one in the initial phase space (13). Then from the requirements of quasi-invariance of the action

$$\overline{S} = \int_{t_1}^{t_2} dt\big[p_{r\ i}\ q_{r+1\ i} + p_{K\ i}\ \dot{q}_{K\ i} - \bar{H}_T\big], \qquad r = 1, \cdots, K-1 \qquad (43)$$

and of conservation of the primary constraint surface $\overline{\Sigma}_1$ under the transformations generated by $\overline{G}$, we shall obtain the same relations (37) for determining $\varepsilon_\alpha^{m_\alpha}$ (with the help of the iterative procedure described in detail in paper II) and the same conclusion about no influence of second-class constraints on the local symmetries of a system.

Before to implement the above-mentioned iterative procedure that gives the result (36), we notice that the coefficients $B_{\alpha\ \beta}^{m_\alpha m_\beta}$ would depend only on $q_{1\ i}$ and $p_{1\ i}$ and on their derivatives. Now, carrying out the iterative procedure we shall exchange derivatives of $q_{1\ i}$ according to formula (40), and for derivatives of $p_{1\ i}$ we shall make the following replacements:

$$\begin{aligned} p_{1\ i} &= \frac{\partial L}{\partial q_{2\ i}} = h_0^i(q_{1\ k}, q_{2\ k}), \qquad i, k = 1, \cdots, N, \\ \dot{p}_{1\ i} &= \frac{\partial h_0^i}{\partial q_{1\ n}} q_{2\ n} + \frac{\partial h_0^i}{\partial q_{2\ n}} q_{3\ n} = h_1^i(q_{1\ k}, q_{2\ k}, q_{3\ k}), \qquad (44) \\ &\vdots \\ p_{1\ i}^{(M_\alpha - 2)} &= h_{M_\alpha - 2}^i(q_{1\ k}, q_{2\ k}, \cdots, q_{M_\alpha - 1\ k}). \end{aligned}$$

As a result, we shall obtain the expression for $\overline{G}$:

$$\overline{G} = B_{\alpha\ \beta}^{m_\alpha m_\beta} \Phi_\alpha^{m_\alpha} \varepsilon_\beta^{(M_\alpha - m_\beta)} + \varepsilon_{s\ i} p_{s\ i}, \qquad (45)$$

$$m_\beta = m_\alpha, \cdots, M_\alpha, \quad s = 2, \cdots, K,$$

where $B_{\alpha\ \beta}^{m_\alpha m_\beta}(q_{1\ i}, \cdots, q_{M_\alpha - 1\ i}; p_{1\ i})$, being just in the same forms as in the initial phase space, are written, however, with taking account of the above-indicated replacements; $\varepsilon_{s\ i}$ are the supplementary group parameters in the amount equal to the number of the supplementary primary constraints of first class $p_{s\ i}$. Note that the obtained generator (45) satisfies the group property

$$\{\overline{G}_1, \overline{G}_2\} = \overline{G}_3, \qquad (46)$$

where the transformation $\overline{G}_3$ (45) is realized by carrying out two successive transformations $\overline{G}_1$ and $\overline{G}_2$ (45). Now the local-symmetry transformations of the coordinates of the initial phase space in the extended one are of the form

$$\begin{cases} \delta q_{1\ k} = \varepsilon_\beta^{(M_\alpha - m_\beta)} \{q_{1\ k}, B_{\alpha\ \beta}^{m_\alpha m_\beta}(q_{1\ i}, \cdots, q_{M_\alpha - 1\ i}; p_{1\ i}) \phi_\alpha^{m_\alpha}(q_{1\ i}, p_{1\ i})\}, \\ \delta p_{1\ k} = \varepsilon_\beta^{(M_\alpha - m_\beta)} \{p_{1\ k}, B_{\alpha\ \beta}^{m_\alpha m_\beta}(q_{1\ i}, \cdots, q_{M_\alpha - 1\ i}; p_{1\ i}) \phi_\alpha^{m_\alpha}(q_{1\ i}, p_{1\ i})\}. \end{cases} \qquad (47)$$

One can verify that to within quadratic terms in $\delta q_{i\ k}$ and $\delta p_{j\ n}$

$$\{q_{i\ k} + \delta q_{i\ k}, p_{j\ n} + \delta p_{j\ n}\} = \delta_{ij}\delta_{kn},$$

i.e. the obtained infinitesimal transformations of local symmetry are canonical in the extended (by Ostrogradsky) phase space.

The local-symmetry transformations in the configuration space may be obtained if after calculating the Poisson brackets in the first formula (47) one takes account of the definitions (40) and of the generalized momenta p_i and make use of formula (38) for $\delta\dot{q}$. They are the Noether transformations. (Note that, as it is seen from (47), to reduce calculations in obtaining these transformations one may use formulas (12) in the initial phase space provided one applies the following "rule": derivatives of q and p are simply put outside the Poisson brackets.) In this case, if the coefficients $B_{\alpha\ \beta}^{m_\alpha m_\beta}$ depend explicitly on $q_{s\ i}$, where $s \geq 2$, then higher derivatives of coordinates $q_i^{(s)} (s \geq 2)$ are present in the transformation law in the configuration space. The functions $g_{\sigma\ \tau}^{m_\sigma m_\tau}$, arising in formula (1), signal to the appearance of that dependence. Moreover, the order of the highest derivative of coordinates may be established already at the beginning, when obtaining the explicit form of $g^{m_\sigma m_\tau}_{\sigma\ \ \tau}$. To this end, one ought to consider the systems of relations (1) and (??). One can see that if any of the coefficients $g^{M_\alpha-1\ M_\alpha}_{\alpha\ \ \ \ \beta}$ and $g^{M_\alpha\ M_\alpha}_{\alpha\ \ \beta}$ in front of the constraints of the last stage M_α depends on $q_{1\ i}$ and $p_{1\ i}$, the coefficients $B^{m_\alpha m_\beta}_{\alpha\ \ \beta}$ will depend on $q_{s\ i} (s = 2, \cdots, M_\alpha - 1)$, and the generator $\overline{G}$ will contain $q_{s\ i}$ $(s = 2, \cdots, K)$, as it is seen from (??). Then, taking account of (40), the order of the highest possible derivative of coordinates in the law of the Noether transformations in the configuration space is equal to $K \equiv \max_\alpha(M_\alpha - 1)$. If these coefficients are constants and any of coefficients $g^{M_\alpha-2\ M_\alpha-1}_{\alpha\ \ \ \ \beta}$, $g^{M_\alpha-1\ M_\alpha-1}_{\alpha\ \ \ \ \beta}$ and $g^{M_\alpha\ M_\alpha-1}_{\alpha\ \ \beta}$ in front of the constraints of the antecedent stage $\phi^{M_\alpha-1}_\beta$ depends on $q_{1\ i}$ and $p_{1\ i}$, then in the Noether transformations law the order of the highest possible derivative will be smaller by one: $\max_\alpha(M_\alpha - 2)$. And generally, in an arbitrary case, when any of coefficients in front of the constraints of k-th stage ϕ^k_β in the Dirac procedure of breeding the constraints depends on $q_{1\ i}$ and $p_{1\ i}$ and all the coefficients in front of the constraints $\phi^{k+i}_\beta (i = 1, \cdots, M_\alpha - k)$ are constants, the order of the highest possible derivative of coordinates in the Noether transformations law is $M_\alpha - k$.

The order of the highest derivative of $\varepsilon_\alpha(t)$ contained in the Noether transformations law is equal always to $M_\alpha - 1$. Note that the amount of group parameters ε_α and $\varepsilon_{s\ i}$ are equal to the number of primary constraints of first class.

4 Example

We consider the Lagrangian with constraints of first and second class when the first-class constraints make up a quasi-algebra of the general form (the restriction (26) is not fulfilled). Examples of that sort for systems only with first-class constraints are described in our previous works [20]-[21] including also the cases when the transformation law in the configuration space contains higher (than the first order) derivatives of coordinates and, therefore, for a canonicity of the local-symmetry transformations one must extend (by Ostrogradsky) the initial phase space.

So, consider the Lagrangian

$$L = \frac{1}{2}\dot{q}_1^2 + \frac{1}{2(q_4 + q_5)}\dot{q}_2^2 + \frac{1}{2}\dot{q}_3^2 + \frac{1}{2}q_2^2 + q_3(q_4 - q_5). \tag{48}$$

Then passing to the Hamiltonian formalism we obtain the generalized momenta

$$p_1 = \dot{q}_1, \quad p_2 = \frac{\dot{q}_2}{q_4 + q_5}, \quad p_3 = \dot{q}_3, \quad p_4 = 0, \quad p_5 = 0$$

and, thus, two primary constraints

$$\phi_1^1 = p_4, \qquad \phi_2^1 = p_5 \tag{49}$$

and the total Hamiltonian

$$H_T = \frac{1}{2}p_1^2 + \frac{1}{2}(q_4 + q_5)p_2^2 + \frac{1}{2}p_3^2 - \frac{1}{2}q_2^2 - q_3(q_4 - q_5) + u_1\phi_1^1 + u_2\phi_2^1. \tag{50}$$

From the self-consistency conditions of theory we obtain two secondary constraints

$$\phi_1^2 = -\frac{1}{2}p_2^2 + q_3, \qquad \phi_2^2 = -\frac{1}{2}p_2^2 - q_3,$$

two tertiary constraints

$$\phi_1^3 = -q_2p_2 + p_3, \qquad \phi_2^3 = -q_2p_2 - p_3$$

and two quaternary constraints

$$\phi_1^4 = -(q_4 + q_5)p_2^2 - q_2^2 + q_4 - q_5, \quad \phi_2^4 = -(q_4 + q_5)p_2^2 - q_2^2 - q_4 + q_5.$$

There do no longer arise constraints, because the conditions of the time conservation of constraints ϕ_1^4 and ϕ_2^4 determine one of the Lagrangian multipliers. Further one can see for oneself that $\mathrm{rank}\|\{\phi^{m_\alpha}_\alpha, \phi^{m_\beta}_\beta\}\| = 4$; therefore, four constraints are of second class. Now implementing our procedure of the constraint separation into first and second class, we obtain the following set of independent constraints: the first-class constraints

$$\Phi_1^1 = \frac{1}{2}(p_4 + p_5), \quad \Phi_1^2 = -\frac{1}{2}p_2^2, \quad \Phi_1^3 = -q_2p_2, \quad \Phi_1^4 = -(q_4 + q_5)p_2^2 - q_2^2$$

and the three-linked chain of second-class constraints

$$\Psi_1^1 = \frac{1}{2}(p_4 - p_5), \quad \Psi_1^2 = q_3, \quad \Psi_1^3 = p_3, \quad \Psi_1^4 = q_4 - q_5.$$

One can see that the first-class constraint Φ_1^4 violates the condition (26), namely,

$$\{\Phi_1^1, \Phi_1^4\} = -2\Phi_1^2. \tag{51}$$

Therefore we shall pass to an equivalent set of constraints by the transformation (23):

$$\tilde{\Phi}_1^{m_1} = C^{m_1 m_1'} \Phi_1^{m_1'}, \tag{52}$$

where the matrix $\mathbf{C}$ is the solution of the equation (25). Since from the quantities $f_{\alpha}^{1}{}_{\delta}^{m_\delta}{}_{\gamma}^{m_\gamma}$ in (25) the only non-vanishing one is $f_{1}^{1}{}_{1}^{4}{}_{1}^{2} = -2$, the matrix

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & 1 \end{pmatrix}, \tag{53}$$

where c is the solution of the equation $\{\Phi_1^1, c\} - 2 = 0$, e.g.

$$c = -2(q_5 + q_6), \tag{54}$$

can be taken as the particular solution of eq.(25). Thus we obtain the desired canonical set of constraints when the condition (26) holds valid $(\{\tilde{\Phi}_1^1, \tilde{\Phi}_1^{m_1}\} = 0, \quad m_1 = 2,3,4)$:

$$\tilde{\Phi}_1^{m_1'} = \Phi_1^{m_1'} \quad (m_1' = 1,2,3), \qquad \tilde{\Phi}_1^4 = -q_2^2.$$

This provide the fulfilment of the second condition (19) of the conservation of the primary-constraint surface Σ_1 under the transformations (12).

Further we seek the generator G in the form (13):

$$G = \eta_1^{k_1}\, \Psi_1^{k_1} + \varepsilon_1^{m_1}\, \tilde{\Phi}_1^{m_1}, \qquad k_1 = 1,\cdots,4, \quad m_1 = 1,\cdots,4. \tag{55}$$

Since in eq.(2) the only non-vanishing structure functions are $h_{1}^{3}{}_{1}^{4} = h_{1}^{2}{}_{1}^{3} = h_{1}^{2}{}_{1}^{2} = 1$, the system of equations (??) has the form

$$\left\{ \begin{array}{l} \dot{\eta}_1^4 + \eta_1^3 = 0, \\ \dot{\eta}_1^3 + \eta_1^2 = 0, \\ \dot{\eta}_1^2 + \eta_1^1 = 0. \end{array} \right. \tag{56}$$

Then, taking into account that the first condition (18) of the Σ_1 conservation under transformations (12) gives $\eta_1^4 = 0$, we verify on the basis of (56) that all $\eta_1^{k_1} = 0$, i.e. the second-class constraints of system do not contribute to the generator G.

As $g_{1}^{4}{}_{1}^{4} = g_{1}^{3}{}_{1}^{3} = g_{1}^{4}{}_{1}^{2} = g_{1}^{2}{}_{1}^{2} = 0$, $g_{1}^{3}{}_{1}^{4} = g_{1}^{2}{}_{1}^{3} = g_{1}^{1}{}_{1}^{2} = 1$ and $g_{1}^{4}{}_{1}^{3} = g_{1}^{3}{}_{1}^{2} = 2(q_4 + q_5)$ in eq.(1), the system of equations (37) for determining $\varepsilon_1^{m_1}$ becomes

$$\left\{ \begin{array}{l} \dot{\varepsilon}_1^4 + \varepsilon_1^3 = 0, \\ \dot{\varepsilon}_1^3 + 2(q_4+q_5)\varepsilon_1^4 + \varepsilon_1^2 = 0, \\ \dot{\varepsilon}_1^2 + 2(q_4+q_5)\varepsilon_1^3 + \varepsilon_1^1 = 0. \end{array} \right. \tag{57}$$

Denoting $\varepsilon_1^4 \equiv \varepsilon$, we obtain

$$\varepsilon_1^3 = -\dot{\varepsilon}, \quad \varepsilon_1^2 = \ddot{\varepsilon} - 2(q_4+q_5)\varepsilon, \quad \varepsilon_1^1 = \frac{d}{dt}[\ddot{\varepsilon} - 2(q_4+q_5)\varepsilon] + 2(\dot{q}_4+q_5)\dot{\varepsilon}. \tag{58}$$

We see that the quantity G in (55) depends on $\dot{q}_4$ and $\dot{q}_5$; therefore, for a canonicity of the desired local-symmetry transformations it is necessary to extend the phase space according to section 3. It is sufficient to carry out the following extension: Define the coordinares $\tilde{q}_i$ $(i = 1,\cdots,7)$:

$$\tilde{q}_i = q_i \quad (i = 1,\cdots,5), \qquad \tilde{q}_6 = \dot{q}_4, \qquad \tilde{q}_7 = \dot{q}_5, \tag{59}$$

and their conjugate momenta calculated in accordance with (41)

$$\tilde{p}_i = p_i \quad (i = 1,\cdots,5), \qquad \tilde{p}_6 = \tilde{p}_7 = 0. \tag{60}$$

The generalized momenta $\tilde{p}_6$ and $\tilde{p}_7$ are extra primary constraints of the first class.

In the extended phase space, one should carry out the procedure of reparametrization of the system of equations (57), although formally, to obtain the definite form, it is sufficient to express G in the coordinates of this extended space according to (59),(60) and (44):

$$\begin{aligned} G \;=\; & \left[-\frac{\dddot{\varepsilon}}{2} + (\tilde{q}_4 + \tilde{q}_5)\ddot{\varepsilon} + (\tilde{q}_4 + \tilde{q}_5 + \tilde{q}_6 + \tilde{q}_7)\,\dot{\varepsilon}\right](\tilde{p}_4 + \tilde{p}_5) \\ & + \left[-\frac{\ddot{\varepsilon}}{2} + (\tilde{q}_4 + \tilde{q}_5)\varepsilon\right] \tilde{p}_2^2 + \tilde{q}_2(\tilde{p}_2\dot{\varepsilon} + \tilde{q}_2\varepsilon). \end{aligned} \tag{61}$$

It can be seen that the local-symmetry transformations generated by this G (61) are already canonical.

In the $(q,\dot{q})$-space the local-symmetry transformations established with the help of formulas (38) have the form

$$\delta q_1 = \delta q_3 = 0, \quad \delta q_2 = -\frac{\dot{q}_2}{q_4 + q_5}\,\ddot{\varepsilon} + (2\dot{q}_2 + q_2)\dot{\varepsilon},$$

$$\delta q_5 = \delta q_6 = -\frac{\dddot{\varepsilon}}{2} + (q_4 + q_5)\,\ddot{\varepsilon} + (q_4 + q_5 + \dot{q}_4 + \dot{q}_5)\,\dot{\varepsilon}, \quad \delta\dot{q}_i = \frac{d}{dt}\,\delta q_i.$$

It is easy to verify that under these transformations

$$\delta L = \frac{1}{2}\frac{d}{dt}\left\{-\frac{\dot{q}_2^2}{(q_4+q_5)^2}[\ddot{\varepsilon} - 2(q_4+q_5)\dot{\varepsilon}] + q_2^2\varepsilon\right\},$$

i.e. the action is quasi-invariant.

5 Conclusion

In the framework of the generalized Hamiltonian formalism for dynamical systems with first- and second-class constraints, we have suggested the method of constructing the generator of local-symmetry transformations for arbitrary degenerate Lagrangians both in the phase and configuration space. The general case is considered including both the violation of the condition (26) (i.e. without restrictions on the algebra of first-class constraints) and the possibility of the presence of higher derivatives of coordinates in the local-symmetry transformation law; and the arising problem of canonicity of transformations in the latter case is solved.

The generator of local-symmetry transformations is derived from the requirement for them to map the solutions of the Hamiltonian equations of motion into the solutions of the same equations which must be supplemented by the demand on the primary-constraint surface Σ_1 to be conserved under these transformations. As it is discussed in paper II, the condition of the Σ_1 conservation actually is not an additional restriction on the properties of the local-symmetry transformation generator that naturally follows from the definition of the symmetry group of the action functional.

We have proved in the general case that the Dirac hypothesis [24] that all first-class constraints generate the local-symmetry transformations holds true also in the presence of second-class constraints and second-class constraints do not contribute to the law of these transformations and do not generate global transformations in lack of first-class constraints.

The generator of local-symmetry transformations is obtained for degenerate theories of general form, without restrictions on the algebra of constraints. We have shown that in this case (these are, e.g., Polyakov's string [11] and other model Lagrangians [10], [12]-[21]) one can always pass to an equivalent set of constraints, the algebra of which satisfies the

condition (26), and, therefore, now the method of constructing the generator developed for singular theories of special form in paper II can be applied. In Appendix A, the method of passing to one of the indicated equivalent sets when all the first-class primary constraints are momentum variables is given.

The corresponding transformations of local symmetry in the $(q, \dot{q})$-space are determined with the help of formulae (38).

When deriving the local-symmetry transformation generator the employment of obtained equation system (37) is important, the solution of which manifests a mechanism of appearance of higher derivatives of coordinates and group parameters in the Noether transformation law in the configuration space, the highest possible order of coordinate derivatives being determined by the structure of the first-class constraint algebra, and the order of the highest derivative of group parameters in the transformation law being by unity smaller than the number of stages in deriving secondary constraints of first class by the Dirac procedure.

We have shown the obtained local-symmetry transformations to be canonical in the extended (by Ostrogradsky) phase space where the time derivatives of coordinates (which have emerged in the transformation law) are taken as complementary coordinates and the conjugate momenta (defined by the formula of theories with higher derivatives [26, 27, 28]) are the initial momenta plus the extra first-class primary constraints (the number of the latter equals the number of complementary coordinates). In addition, the dynamics of a system remains to be fixed in the sector of the initial phase-space variables.

Obtained generator (45) ((36)) satisfies the group property (46). The amount of group parameters which determine the rank of quasigroup of these transformations equals the number of primary constraints of first class.

So, we can state in the general case of theories with first- and second-class constraints (without restrictions on the constraint algebra) that the necessary and sufficient condition for a certain quantity G to be the local-symmetry transformation generator is the representation of G as a linear combination of all the first-class constraints (and only of them) of the equivalent set of the special form (when the first-class primary constraints are the ideal of algebra of all the first-class constraints) with the coefficients determined by the system of equations (37). Passing to the indicated equivalent set of constraints is always possible, and the method is presented in this work. In addition, these are the necessary and sufficient conditions for (12) to be the quasi-invariance transformation of the functional of action in both the phase and $(q, \dot{q})$ space. It is thereby shown in the general case that the functional of action and the corresponding Hamiltonian equations of motion are invariant under the same quasigroup of local-symmetry transformations.

As it is known, gauge-invariant theories belong to the class of degenerate theories. In this paper, we have shown that the degeneracy of theories with the first- and second-class constraints in the general case is due to their quasi-invariance under local-symmetry transformations.

Acknowledgments

One of the authors (S.A.G.) thanks the Russian Foundation for Fundamental Research (Grant № 96-01-01223) for support.

Appendix A

Here we shall describe the way of passing to, at least, one separated set of equivalent constraints $\bar{\phi}_\alpha^{m_\alpha}$ when all the primary constraints of the first class are momentum variables. We shall consider that the initial set of first- and second-class constraints are canonical, i.e. the complete separation of constraints into first- and second-class ones is already carried out. Then, the formulated problem can be solved by the iteration procedure provided that we take into account the first-class primary constraints to make a subalgebra of quasi-algebra of all the first-class constraints (7):

$$\{\Phi_\alpha^1, \Phi_\beta^1\} = f_{\alpha\ \beta\ \gamma}^{1\ 1\ 1}\Phi_\gamma^1, \qquad \alpha, \beta = 1, \cdots, F.$$

This relation follows from the stationarity condition for Φ_α^1 and from from the properties of the canonical set of constraints. The iteration procedure can be first developed for first-class constraints, and second-class constraints can be taken into account at last stage. There always exist canonical transformations of the form [27, 29]

$$\bar{P}_1 = \Phi_1^1(q,p), \qquad \{\bar{Q}_1, \bar{P}_1\} = 1, \qquad \{\bar{Q}_\sigma, \bar{P}_\tau\} = \delta_{\sigma\tau},$$

$$\{\bar{P}_1, \bar{P}_\tau\} = \{\bar{Q}_1, \bar{P}_\tau\} = \{\bar{P}_1, \bar{Q}_\tau\} = \{\bar{Q}_1, \bar{Q}_\tau\} = 0, \tag{62}$$

$$\sigma, \tau = 2, \cdots, N.$$

(The bar over a letter means the first stage of the iteration procedure.) All the remaining primary constraints of first class assume the form

$$\varphi_\alpha^1(\bar{Q}, \bar{P}) = \Phi_\alpha^1\big(q(\bar{Q}, \bar{P}), p(\bar{Q}, \bar{P})\big)\Big|_{\bar{P}_1=0}, \qquad \alpha = 2, \cdots, F.$$

In view of the transformation being canonical, we can write

$$\{\bar{P}_1, \varphi_\alpha^1\} = -\frac{\partial \varphi_\alpha^1}{\partial \bar{Q}_1} = \bar{f}_{1\ \alpha\ \gamma}^{1\ 1\ 1}\varphi_\gamma^1, \qquad \alpha, \gamma \geq 2,.$$

with φ_α^1 having the structure [27]

$$\varphi_\alpha^1 = \bar{E}_{\alpha\ \gamma}^{1\ 1}\bar{\varphi}_\gamma^1, \qquad \det\bar{\mathbf{E}}\Big|_\Sigma \neq 0, \tag{63}$$

and obeying the conditions

$$\frac{\partial \bar{\varphi}_\gamma^1}{\partial \bar{Q}_1} = \frac{\partial \bar{\varphi}_\gamma^1}{\partial \bar{P}_1} = 0, \qquad \gamma \geq 2.$$

As all the constraints $\bar{\varphi}_\gamma^1$ do not depend upon $\bar{Q}_1$ and $\bar{P}_1$, we perform an analogous procedure for the constraint $\bar{\varphi}_2^1$ in the $2N-2$-dimensional subspace $(\bar{Q}_\sigma, \bar{P}_\sigma)(\sigma = 2, \cdots, N)$, i.e. without affecting $\bar{Q}_1$ and $\bar{P}_1$. Then the constraints $\bar{\bar{\varphi}}_\alpha^1(\alpha = 3, \cdots, F)$ arising in a formula analogous to formula (63) are independent of $\bar{Q}_1, \bar{P}_1$ and $\bar{\bar{Q}}_2, \bar{\bar{P}}_2$. Next, making this procedure step by step $(F-2)$ times, we finally obtain the first-class primary constraints to be momenta, and therefore they commute with each other (final momenta and coordinates will be denoted by Q_α and P_α, respectively, $\alpha = 1, \cdots, F$).

All secondary constraints of first class will then assume the form

$$\varphi_\alpha^{m_\alpha}(Q, P) = \varphi_\alpha^{m_\alpha}\Big(q(Q,P), p(Q,P)\Big)\Big|_{P_\alpha=0}, \qquad \alpha = 1, \cdots, F;\ m_\alpha = 2, \cdots, M_\alpha.$$

As the transformations are canonical, we can write

$$\{P_\alpha, \varphi_\beta^{m_\beta}\} = -\frac{\partial \varphi_\beta^{m_\beta}}{\partial \bar{Q}_\alpha} = f_{\alpha\ \beta\ \gamma}^{1\ m_\beta m_\gamma}\varphi_\gamma^{m_\gamma},$$

with $\varphi_\alpha^{m_\alpha}$ having the structure [27]

$$\varphi_\alpha^{m_\alpha} = A_{\alpha\ \beta}^{m_\alpha m_\beta} \widetilde{\varphi}_\beta^{m_\beta}, \qquad \det \mathbf{A}\Big|_\Sigma \neq 0, \tag{64}$$

and obeying the conditions

$$\frac{\partial \widetilde{\varphi}_\alpha^{m_\alpha}}{\partial Q_\beta} = \frac{\partial \widetilde{\varphi}_\alpha^{m_\alpha}}{\partial P_\beta} = 0, \qquad \alpha, \beta = 1, \cdots, F, \quad m_\alpha \geq 2.$$

And, finally, all second-class constraints will be expressed as

$$\psi_{a_i}^{m_{a_i}}(Q,P) = \Psi_{a_i}^{m_{a_i}}\Big(q(Q,P), p(Q,P)\Big)\Big|_{P_\alpha=0}, \quad \alpha = 1, \cdots, F;$$

$$i = 1, \cdots, n, \ a_i = 1, \cdots, A_i, \ m_{a_i} = 1, \cdots, M_{a_i}$$

with all (previously-established in paper I) features of the canonical set of constraints remaining valid.

The set of constraints thus constructed (primary constraints being momenta and secondary $\widetilde{\varphi}_\alpha^{m_\alpha}$) satisfies the condition (26) with vanishing right-hand side, i.e. we have derived the searched set of constraints. Note that $(A^{-1})_{\alpha\ \beta}^{m_\alpha m_\beta}$ in (64) is a solution to the system of equations (25).

References

[1] N.P. Chitaia, S.A. Gogilidze and Yu.S. Surovtsev, "Second-Class Constraints and Local Symmetries," Communication of Joint Institute for Nuclear Research E2-96-234, Dubna, 1996.

[2] R. Sugano and T. Kimura, *Phys. Rev. D* **41** (1990), 1247.

[3] A. Cabo and P. Louis-Martinez, *Phys. Rev. D* **42** (1990), 2726.

[4] L. Lusanna, *Riv. Nuovo Cimento* **14 (1991)**, 1.

[5] S.A. Gogilidze, V.V. Sanadze, Yu.S. Surovtsev and F.G. Tkebuchava, "The Theories with Higher Derivatives and Gauge-Transformation Construction," Preprint of Joint Institute for Nuclear Research E2-87-390, Dubna, 1987; *Int. J. Mod. Phys. A* **47** (1989), 4165.

[6] J.L. Anderson and P.G. Bergmann, *Phys. Rev.* **83** (1951), 1018. P.G. Bergmann and J. Goldberg, *Phys. Rev.* **98** (1955), 531.

[7] M. Henneaux, C. Teitelboim and J. Zanelli, *Nucl. Phys. B* **332** (1990), 169.

[8] A. Cabo and P. Louis-Martinez, *Phys. Rev. D* **42** (1990), 2726.

[9] L. Castellani, *Ann. Phys. (N.Y.)* **143** (1982) 357.

[10] R. Cawley, *Phys. Rev. Lett.* **42** (1979), 413; *Phys. Rev. D* **21** (1980), 2988.

[11] A.M. Polyakov, *Phys. Lett. B* **103** (1981), 207.

[12] A. Frenkel, *Phys. Rev. D* **21** (1980), 2986.

[13] R. Sugano and T. Kimura, *Prog. Theor. Phys.* **69** (1983), 1241.

[14] M.J. Gotay, *J. Phys. A.: Math. Gen.* **16** (1983), L-141.

[15] M.J. Gotay and J.M. Nester, *J. Phys. A.: Math. Gen.* **17** (1984), 3063.

[16] R. Di Stefano, *Phys. Rev. D* **27** (1983), 1758.

[17] V.V. Nesterenko and A.M. Chervyakov, "Singular Lagrangians. Classical Dynamics and Quantization," Preprint of Joint Institute for Nuclear Research P2-86-323, Dubna, 1986.

[18] C. Batlle, J. Gomis, X. Gràcia and J.M. Pons, *J. Math. Phys.* **30** (1989), 1345.

[19] X. Gràcia and J.M. Pons, *J. Phys. A: Math. Gen.* **25** (1992), 6357.

[20] S.A. Gogilidze, V.V. Sanadze, Yu.S. Surovtsev and F.G. Tkebuchava, *Theor. Math. Phys.* **102** (1995), 66.

[21] S.A. Gogilidze, V.V. Sanadze, Yu.S. Surovtsev and F.G. Tkebuchava, *J.Phys. A: Math. Gen.* **27** (1994), 6509.

[22] S.A. Gogilidze, V.V. Sanadze, Yu.S. Surovtsev and F.G. Tkebuchava, *Theor. Math. Phys.* **102** (1995), 56.

[23] N.P. Chitaia, S.A. Gogilidze and Yu.S. Surovtsev, "Constrained Dynamical Systems: Separation of Constraints into First and Second Classes," Preprint of Joint Institute for Nuclear Research E2-96-227, Dubna, 1996.

[24] P.A.M. Dirac, *Canad. J. Math.* **2** (1950), 129. "Lectures on Quantum Mechanics," Belfer Graduate School of Science, Monographs Series, Yeshiva University, New York, 1964.

[25] V.I. Smirnov, "Course of Higher Mathematics," Vol.4, Part 2, Nauka, Moscow, 1981 (in Russian).

[26] M.V. Ostrogradsky, *Mem. de l'Acad. Imper. des Sci. de St-Petersbourg* 4 (1850), 385.

[27] D.M. Gitman and I.V. Tyutin, "Canonical quantization of constrained fields," Nauka, Moscow, 1986 (in Russian).

[28] V.V. Nesterenko, *J.Phys. A: Math. Gen.* **22** (1989), 1673.

[29] L.P. Eisenhart, "Continuous Groups of Transformation," Princeton, N.J., Dover, New York, 1961.

Received by Publishing Department
on July 5, 1996.

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Квазигруппа преобразований локальной симметрии в теориях со связями

В обобщенном гамильтоновом формализме Дирака исследуются локальные симметрии систем со связями первого и второго рода в общем случае без ограничений на алгебру связей. Метод конструирования генератора преобразований локальной симметрии получен из требования, чтобы они отображали решения гамильтоновых уравнений движения в решения тех же уравнений. Доказано, что связи второго рода не дают вклада в закон преобразований локальной симметрии, который полностью определяется всеми связями первого рода (и только ими) из некоторого эквивалентного набора, переход к которому от первоначального набора связей всегда возможен и представлен здесь. Выяснен механизм появления высших производных от координат и групповых параметров в законе преобразования симметрии во второй теореме Нётер. В последнем случае показано, что полученные преобразования симметрии являются каноническими в расширенном (по Остроградскому) фазовом пространстве. В общем случае показано, что вырожденность теорий со связями первого и второго рода обусловлена их инвариантностью относительно преобразований локальной симметрии. Показано также, что функционал действия и соответствующие гамильтоновы уравнения движения инвариантны относительно одной и той же квазигруппы преобразований локальной симметрии.

Работа выполнена в Лаборатории теоретической физики им.Н.Н.Боголюбова ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна, 1996

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Quasigroup of Local-Symmetry Transformations in Constrained Theories

In the framework of the generalized Hamiltonian formalism by Dirac, the local symmetries of dynamical systems with first- and second-class constraints are investigated in the general case without restrictions on the algebra of constraints. The method of constructing the generator of local-symmetry transformations is obtained from the requirement for them to map the solutions of the Hamiltonian equations of motion into the solutions of the same equations. It is proved that second-class constraints do not contribute to the transformation law of the local symmetry entirely stipulated by all the first-class constraints (and only by them) of an equivalent set passing to which from the initial constraint set is always possible and is presented. A mechanism of occurrence of higher derivatives of coordinates and group parameters in the symmetry transformation law in the Noether second theorem is elucidated. In the latter case it is shown that the obtained transformations of symmetry are canonical in the extended (by Ostrogradsky) phase space. It is thereby shown in the general case that the degeneracy of theories with the first- and second-class constraints is due to their invariance under local-symmetry transformations. It is also shown in the general case that the action functional and the corresponding Hamiltonian equations of motion are invariant under the same quasigroup of local-symmetry transformations.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna, 1996