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M.Kibler[1], L.G.Mardoyan[2], G.S.Pogosyan[3]

ON A GENERALIZED OSCILLATOR SYSTEM: INTERBASIS EXPANSIONS

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[1]Institut de Physique Nucléaire de Lyon, IN2P3–CNRS et Université Claude Bernard, 43 Bd du 11 Novembre 1918, F-69622 Villeurbanne Cedex, France
[2]E-mail: mardoyan@thsun1.jinr.dubna.su
[3]E-mail: pogosyan@thsun1.jinr.dubna.su

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1 Introduction

The purpose of this paper is to study the quantum mechanical motion of a particle in the three-dimensional axially symmetric potential

$$V = \frac{\Omega^2}{2}(x^2+y^2+z^2) + \frac{P}{2}\frac{1}{z^2} + \frac{Q}{2}\frac{1}{x^2+y^2}, \tag{1}$$

where Ω, P, and Q are constants with $\Omega > 0$, $P > -\frac{1}{4}$, and $Q \geq 0$. In the last decade, this potential (including the case $P = 0$) has been the object of numerous studies [1-12]. The Schrödinger and Hamilton-Jacobi equations for this generalized oscillator potential are separable in spherical, cylindrical, and spheroidal (prolate and oblate) coordinates. In the case when $P = 0$ we get the well-known ring-shape oscillator potential which was investigated in many articles [1, 3, 4, 7, 8] in recent years as a companion of the Hartmann potential [4, 6, 8, 13-17]. If $P = Q = 0$ we have the ordinary isotropic harmonic oscillator in three dimensions.

The plan of this article is as follows. We solve the Schrödinger equation

$$H\Psi = E\Psi, \tag{2}$$

with

$$H = -\frac{1}{2}(\partial_{xx} + \partial_{yy} + \partial_{zz}) + V, \tag{3}$$

in spherical coordinates (in the second section) and in cylindrical coordinates (in the third section). (The constant $\hbar$ and the reduced mass are taken to be equal to 1. In the whole paper, we use Ψ to denote the total wavefunction whatsoever the coordinate system is; the wavefunctions Ψ in spherical, cylindrical, and spheroidal coordinates are then distinguished by the corresponding quantum numbers. Note also that we use s to denote the fraction $\frac{1}{2}$ in the following.) In the fourth section, we determine the interbasis expansion coefficients between the cylindrical and the spherical bases. The fifth and sixth sections deal with the generalized oscillator system in spheroidal coordinates. In particular, the prolate and oblate spheroidal bases are expanded in terms of both the spherical basis and the cylindrical basis. Two appendices close this article. The first one is devoted to the bi-orthogonality of the radial wavefunctions (in spherical coordinates) for the generalized oscillator system. The second appendix concerns a connection between the Smorodinsky-Winternitz system (that is a basic component for the generalized oscillator system) and the Morse system.

The generalized oscillator system constitutes a pending part to the generalized Kepler-Coulomb system studied in Ref. [11, 12, 18]. The latter two nonrelativistic systems generalize two important paradigms in quantum mechanics, namely, the oscillator system and the Kepler-Coulomb system. The main results of this work and the one in Ref. [18] concern the separability in spheroidal coodinates as well as the SU(2) approach and the three-term recursion relations for the interbasis expansion coefficients.

The authors are very pleased to contribute to this memorial volume in honour of Jean-Louis Calais. Professor Jean-Louis Calais achieved, among other important works, an original job [19] on the derivation of the SU(2) Clebsch-Gordan coefficients by the (Löwdin) projection operator method. We are glad to present here a work where an analytic continuation of SU(2) Clebsch-Gordan coefficients plays an important rôle in the analysis of interbasis expansions.

The use of spheroidal coordinates is now well established in quantum chemistry [20]. There exist now powerful techniques [21] for evaluating (angular and radial) prolate spheroidal wavefunctions from differential equations. It is hoped that this paper will shed some new light on expansions of spheroidal wavefunctions.

2. Spherical Basis

The Schrödinger equation (2) in spherical coordinates (r,θ,φ) for the potential (1), i.e.,

$$V = \frac{\Omega^2}{2}r^2 + \frac{P}{2}\frac{1}{r^2\cos^2\theta} + \frac{Q}{2}\frac{1}{r^2\sin^2\theta},$$

may be solved by seeking a wavefunction Ψ of the form

$$\Psi(r,\theta,\varphi) = R(r)\Theta(\theta)\frac{e^{im\varphi}}{\sqrt{2\pi}}, \tag{4}$$

with $m \in \mathbf{Z}$. This amounts to find the eigenfunctions of the set $\{H, L_z, M\}$ of commuting operators, where the constant of motion M reads

$$M = L^2 + \frac{P}{\cos^2\theta} + \frac{Q}{\sin^2\theta} \tag{5}$$

(L^2 is the square of the angular momentum and L_z its z-component). We are thus left with the system of coupled differential equations:

$$(M - A)\Theta \;=\; 0, \tag{6}$$

$$\left[\frac{1}{r^2}d_r(r^2 d_r) + 2E - \Omega^2 r^2 - \frac{A}{r^2}\right] R \;=\; 0, \tag{7}$$

where A is a (spherical) separation constant.

Let us consider the angular equation (6). By putting $\Theta(\theta) = f(\theta)/\sqrt{\sin\theta}$, we can rewrite Eq. (6) in the Pöschl-Teller form:

$$\left(d_{\theta\theta} + A + \frac{1}{4} - \frac{b^2-\frac{1}{4}}{\cos^2\theta} - \frac{c^2-\frac{1}{4}}{\sin^2\theta}\right) f = 0,$$

$$b = \sqrt{P + \frac{1}{4}}, \quad c = \sqrt{Q + m^2}. \tag{8}$$

In the case where $b > s$, the angular potential is repulsive for $\theta = \frac{\pi}{2}$. In this case, the θ domain is separated in two regions ($\theta \in]0, \frac{\pi}{2}[$ and $\theta \in]\frac{\pi}{2}, \pi[$) and the "motion" takes place in one or another region. Furthermore, in this case Eq. (8) corresponds to a genuine Pöschl-Teller potential. In the case where $0 < b < s$, we can call the angular potential an attractive Pöschl-Teller potential. When $b = s$, i.e., $P = 0$, we get the well-known ring-shape oscillator potential [1, 3, 4, 7, 8]. The solution $\Theta(\theta) \equiv \Theta_q(\theta; c, \pm b)$ of Eq. (6)

(for both $0 < b < s$ and $b > s$), with the conditions $\Theta(0) = \Theta(\frac{\pi}{2}) = 0$, is easily found to be (cf., [22, 23])

$$\Theta(\theta) = N_q(c, \pm b)(\sin\theta)^c(\cos\theta)^{s\pm b} P_q^{(c,\pm b)}(\cos 2\theta), \tag{9}$$

with $q \in \mathbf{N}$, where $P_n^{(\alpha,\beta)}$ denotes a Jacobi polynomial. Then, the constant A is quantized as

$$A_q(c, \pm b) = (2q + c \pm b + s)(2q + c \pm b + 3s). \tag{10}$$

The normalization constant $N_q(c, \pm b)$ in (9) is given (up to a phase factor) by

$$\int_0^{\frac{\pi}{2}} \Theta_{q'}\Theta_q \sin\theta d\theta = \frac{1}{2}\delta_{q'q}. \tag{11}$$

This leads to

$$N_q(c, \pm b) = \sqrt{\frac{(2q + c \pm b + 1)q!\Gamma(q + c \pm b + 1)}{\Gamma(q + c + 1)\Gamma(q \pm b + 1)}}. \tag{12}$$

Note that only the positive sign in front of b has to be taken when $b > s$ while both the positive and negative signs have to be considered for $0 < b < s$.

Let us go to the radial equation (7). The introduction of (10) into (7) yields an equation that is very reminiscent of the radial equation for the three-dimensional isotropic oscillator except that the orbital quantum number l is replaced by $2q + c \pm b + s$. The solution $R(r) \equiv R_{n_r q}(r; c, \pm b)$ of the obtained equation, in terms of Laguerre polynomials L_n^α, is

$$R(r) = N_{n_r q}(c, \pm b)(\sqrt{\Omega}r)^{2q+c\pm b+s} e^{-s\Omega r^2} L_{n_r}^{2q+c\pm b+1}(\Omega r^2), \tag{13}$$

with $n_r \in \mathbf{N}$. In Eq. (13), the radial wavefunctions $R_{n_r q}$ satisfy the orthogonality relation

$$\int_0^\infty R_{n_r' q} R_{n_r q} r^2 dr = \delta_{n_r' n_r} \tag{14}$$

[cf., Eq. (71)] so that the normalization factor $N_{n_r q}(c, \pm b)$ is

$$N_{n_r q}(c, \pm b) = \sqrt{\frac{2\Omega^{3s} n_r!}{\Gamma(n_r + 2q + c \pm b + 2)}}. \tag{15}$$

The normalized total wavefunction $\Psi(r,\theta,\varphi) \equiv \Psi_{n_r q m}(r,\theta,\varphi; c, \pm b)$ is then given by Eqs. (4), (9), (12), (13), and (15). The energies E corresponding to $n_r + q$ fixed are

$$E_n(c, \pm b) = \Omega\,(2n + c \pm b + 2), \tag{16}$$

with $n = n_r + q$. Equation (16) shows that, for each quantum number n, we have two levels (for $+b$ and $-b$) in the $0 < b < s$ region and one level (for $+b$) in the $b > s$ region. Note that this spectrum was obtained through a path integral approach in [2, 6] for the $b > s$ case and in [12] for the general case (see also Refs. [10, 11]).

In the $0 < b < s$ region, for the limiting situation where $b = s^-$, i.e., $P = 0^-$, we have for the separation constant A:

$$A_q(c, +s) = (2q + c + 1)(2q + c + 2), \qquad A_q(c, -s) = (2q + c)(2q + c + 1). \tag{17}$$

Then, by using the connecting formulas [24]

$$C_{2n+1}^{\lambda}(x) = \frac{(\lambda)_{n+1}}{(s)_{n+1}}\, x\, P_n^{(\lambda-s,+s)}(2x^2-1), \quad C_{2n}^{\lambda}(x) = \frac{(\lambda)_n}{(s)_n}\, P_n^{(\lambda-s,-s)}(2x^2-1),$$

between the Jacobi polynomial $P_n^{(\alpha,\beta)}$ and the Gegenbauer polynomial C_n^{λ}, we have the following odd and even angular solutions (with respect to $\cos\theta \mapsto -\cos\theta$)

$$\Theta_q(\theta;c,+s) = \sqrt{\frac{(4q+2c+3)(2q+1)!}{2\pi\Gamma(2q+2c+2)}}\, 2^c\Gamma(c+s)(\sin\theta)^c C_{2q+1}^{c+s}(\cos\theta), \tag{18}$$

$$\Theta_q(\theta;c,-s) = \sqrt{\frac{(4q+2c+1)(2q)!}{2\pi\Gamma(2q+2c+1)}}\, 2^c\Gamma(c+s)(\sin\theta)^c C_{2q}^{c+s}(\cos\theta). \tag{19}$$

Let us introduce (a new orbital quantum number) l and (a new principial quantum number) N through

$$l-|m| = \left\{\begin{array}{ll} 2q+1 & \text{for the } + \text{ sign} \\ 2q & \text{for the } - \text{ sign}\end{array}\right\}, \qquad N-|m| = \left\{\begin{array}{ll} 2n+1 & \text{for the } + \text{ sign} \\ 2n & \text{for the } - \text{ sign}\end{array}\right\}. \tag{20}$$

Note that $N = 2n_r + l$ both for the $+$ and $-$ signs. Then, the separation constant [Eq. (17)] and the energy [Eq. (16)] can be expressed as

$$A_q(c,\pm s) \equiv A_l(\delta) = (l+\delta)(l+\delta+1), \quad E_n(c,\pm s) \equiv E_N(\delta) = \Omega\,(N+\delta+3s), \tag{21}$$

respectively, where

$$\delta = \sqrt{Q+m^2} - |m|.$$

Thus, the two parts of the energy spectrum for the signs $\pm$ correspond now to odd (for $+$) and even (for $-$) values of $N-|m|$. In terms of N, l, and δ, the functions $R_{n_r q}(r;c,\pm s) \equiv R_{Nl}(r;\delta)$ [cf., Eq. (13)] and $\Theta_q(\theta;c,\pm s) \equiv \Theta_{lm}(\theta;\delta)$ [cf., Eqs. (18) and (19)] can be rewritten as

$$R_{Nl}(r;\delta) = \sqrt{\frac{2\Omega^{3s}(\frac{N-l}{2})!}{\Gamma(\frac{N+l}{2}+\delta+3s)}}\,(\sqrt{\Omega}r)^{l+\delta} e^{-s\Omega r^2} L_{\frac{N-l}{2}}^{l+\delta+s}(\Omega r^2), \tag{22}$$

$$\Theta_{lm}(\theta;\delta) = 2^{|m|+\delta}\Gamma(|m|+\delta+s)\sqrt{\frac{(2l+2\delta+1)(l-|m|)!}{2\pi\Gamma(l+|m|+2\delta+1)}}(\sin\theta)^{|m|+\delta} C_{l-|m|}^{|m|+\delta+s}(\cos\theta). \tag{23}$$

Equations (22) and (23 compare with the corresponding results for the ring-shape oscillator in [1, 3]. Note that (23) was given in terms of Legendre functions in Refs. [1] and [3] and was studied in details in Ref. [9].

In the $b > s$ region, for the limiting situation where $b = s^+$, i.e., $P = 0^+$, we have only odd solutions. In other words when $P \to 0^+$, the eigenvalues and eigenfunctions of the generalized oscillator do not restrict to the eigenvalues and eigenfunctions, respectively, of the ring-shape oscillator. This fact may be explained in the following manner. To make $P = 0$ in the wavefunction $\Psi_{n_r qm}(r,\theta,\varphi;c,+b)$ amounts to changing the Hamiltonian into a Hamiltonian corresponding to $P=0$ and to introducing an unpenetrable barrier. (Another way to describe this phenomenon is to say that for very small P, the potential V is infinite in the $\theta = \frac{\pi}{2}$ plan and equal to the ring-shape potential only for $P = 0$.) This phenomenon is known as the Klauder phenomenon [25].

A further limit can be obtained in the case when $\delta = 0$, i.e., $Q = 0$. It is enough to use the connecting formula [24]

$$P_l^{|m|}(x) = \frac{(-2)^{|m|}}{\sqrt{\pi}}\Gamma(|m|+s)(1-x^2)^{s|m|} C_{l-|m|}^{|m|+s}(x)$$

between the Gegenbauer polynomial C_n^{λ} and the Legendre polynomial $P_l^{|m|}$. In fact for $Q = 0$, Eq. (23) can be reduced to

$$\Theta_{lm}(\theta;0) = (-1)^{|m|}\sqrt{\frac{2l+1}{2}\frac{(l-|m|)!}{(l+|m|)!}}\,P_l^{|m|}(\cos\theta),$$

so that $\Theta_{lm}(\theta;0)e^{im\varphi}/\sqrt{2\pi}$ coincides with the usual spherical harmonic $Y_{lm}(\theta,\varphi)$ (up to a phase factor, e.g., see [26]). The wavefunctions $\Theta_q(\theta;c,\pm b)e^{im\varphi}/\sqrt{2\pi}$ may thus be considered as a generalisation of the spherical harmonics.

3 Cylindrical Basis

In the cylindrical coordinates (ρ,φ,z), the potential V reads

$$V = \frac{\Omega^2}{2}(\rho^2+z^2) + \frac{P}{2}\frac{1}{z^2} + \frac{Q}{2}\frac{1}{\rho^2}.$$

Equation (2), with this potential, admits a solution Ψ of the form

$$\Psi(\rho,\varphi,z) = R(\rho)Z(z)\frac{e^{im\varphi}}{\sqrt{2\pi}}, \tag{24}$$

where $m \in \mathbf{Z}$. In other words, we look for the eigenfunctions of the set $\{H, L_z, N\}$ of commuting operators, where the constant of motion N is

$$N = D_{zz} + \frac{P}{z^2}, \tag{25}$$

D_{zz} being the zz component of

$$D_{z_i z_j} = -\partial_{z_i z_j} + \Omega^2\, z_i z_j,$$

the so-called Demkov tensor [27] for the isotropic harmonic oscillator in $\mathbf{R}^3$. It is sufficient to solve the two coupled equations

$$
\begin{aligned}
(N - 2E_z)Z &= 0, \qquad (26)\\
\left[\frac{1}{\rho}d_\rho(\rho d_\rho) + 2E_\rho - \Omega^2\rho^2 - \frac{Q+m^2}{\rho^2}\right] R &= 0, \qquad (27)
\end{aligned}
$$

where the two cylindrical separation constants E_ρ and E_z obey $E_\rho + E_z = E$. The solutions $\Psi(\rho,\varphi,z) \equiv \Psi_{n_\rho pm}(\rho,\varphi,z;c,\pm b)$ of (26-27) lead to the normalized wavefunction

$$
\Psi(\rho,\varphi,z) = R_{n_\rho}(\rho;c) Z_p(z;\pm b)\frac{e^{im\varphi}}{\sqrt{2\pi}}, \qquad (28)
$$

where

$$
R_{n_\rho}(\rho;c) = \sqrt{\frac{2\Omega n_\rho!}{\Gamma(n_\rho+c+1)}} e^{-s\Omega\rho^2}(\sqrt{\Omega}\rho)^c L^c_{n_\rho}(\Omega\rho^2)
$$

and

$$
Z_p(z;\pm b) = (-1)^p \sqrt{\frac{\Omega^s p!}{\Gamma(p\pm b+1)}} e^{-s\Omega z^2}(\sqrt{\Omega}z)^{s\pm b} L^{\pm b}_p(\Omega z^2), \qquad (29)
$$

with $n_\rho \in \mathbf{N}$ and $p \in \mathbf{N}$. The normalization of the wavefunction (28) is ensured by

$$
\int_0^\infty R_{n'_\rho} R_{n_\rho} \rho d\rho = \delta_{n'_\rho n_\rho}, \quad \int_0^\infty Z_{p'} Z_p dz = \frac{1}{2}\delta_{p'p}.
$$

Furthermore, the constants E_ρ and E_z in (26-27) become

$$
E_\rho(n_\rho, c) = \Omega\,(2n_\rho + c + 1), \quad E_z(p,\pm b) = \Omega\,(2p \pm b + 1). \qquad (30)
$$

Therefore, the quantized values of the energy E are given by (16) where now the quantum number n is $n = n_\rho + p$. As in the second section, the sign in front of b in Eqs. (28)-(30) may be only positive when $b > s$. When $0 < b < s$, both the signs $+$ and $-$ are admissible.

In the $0 < b < s$ region, in the limiting case where $b = s^-$, due to the connecting formulas [24]

$$
\mathcal{H}_{2n+1}(x) = (-1)^n 2^{2n+1} n! x L^{+s}_n(x^2), \quad \mathcal{H}_{2n}(x) = (-1)^n 2^{2n} n! L^{-s}_n(x^2),
$$

between the odd $\mathcal{H}_{2n+1}$ and even $\mathcal{H}_{2n}$ Hermite polynomials and the Laguerre polynomials $L^{\pm s}_n$, we immediately have

$$
\begin{aligned}
Z_p(z;+s) &= \left(\frac{\Omega}{\pi}\right)^{\frac14} \frac{e^{-s\Omega z^2}}{\sqrt{2^{2p+1}(2p+1)!}} \mathcal{H}_{2p+1}(\sqrt{\Omega}z),\\
Z_p(z;-s) &= \left(\frac{\Omega}{\pi}\right)^{\frac14} \frac{e^{-s\Omega z^2}}{\sqrt{2^{2p}(2p)!}} \mathcal{H}_{2p}(\sqrt{\Omega}z).
\end{aligned}
$$

Introducing (a new quantum number) n_3 such that $n_3 = 2p+1$ for the $+$ sign and $n_3 = 2p$ for the $-$ sign, we obtain

$$
Z_p(z;\pm s) = \left(\frac{\Omega}{\pi}\right)^{\frac14} \frac{e^{-s\Omega z^2}}{\sqrt{2^{n_3} n_3!}} \mathcal{H}_{n_3}(\sqrt{\Omega}z).
$$

The energy is then given by (21) where $N = 2n_\rho + n_3 + |m|$. Note that the spectrum in the case $b = s^-$, which corresponds to the ring-shape oscillator system, was obtained in Refs. [1-3].

In the $b > s$ region, in the limiting situation where $b = s^+$, we get only the odd solution of the ring-shape oscillator system.

4 Connecting the Cylindrical and Spherical Bases

According to first principles, any cylindrical wavefunction (24) corresponding to a given value of E can be developed in terms of the spherical wavefunctions (4) associated to the eigenvalue E (see also Ref. [11]). Thus, we have

$$
\Psi_{n_\rho pm} = \sum_{q=0}^{n} W^q_{np}(c,\pm b)\Psi_{n_r qm}, \qquad (31)
$$

where $n_\rho + p = n_r + q = n$. In Eq. (31), it is understood that the wavefunctions in the left- and right-hand sides are written in spherical coordinates (r,θ,φ) owing to $\rho = r\sin\theta$ and $z = r\cos\theta$. The dependence on $e^{im\varphi}$ can be eliminated in both sides of Eq. (31). Furthermore, by using the formula $L^\alpha_n(x) \sim (-1)^n x^n/n!$, valid for x arbitrarily large, (31) yields an equation that depends only on the variable θ. Thus, by using the orthonormality relation (11), for the quantum numbers q, we can derive the following expression for the interbasis expansion coefficients

$$
W^q_{np}(c,\pm b) = (-1)^{q-p} B^q_{np}(c,\pm b)\, E^q_{np}(c,\pm b), \qquad (32)
$$

where

$$
\begin{aligned}
B^q_{np}(c,\pm b) &= \sqrt{\frac{(2q+c\pm b+1)(n-q)!q!\Gamma(q+c\pm b+1)\Gamma(n+q+c\pm b+2)}{(n-p)!p!\Gamma(q+c+1)\Gamma(q\pm b+1)\Gamma(n-p+c+1)\Gamma(p\pm b+1)}}\\
E^q_{np}(c,\pm b) &= 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2n-2p+2c}(\cos\theta)^{2p+1\pm 2b}\, P_q^{(c,\pm b)}(\cos 2\theta)\sin\theta d\theta.
\end{aligned} \qquad (33)
$$

By making the change of variable $x = \cos 2\theta$ and by using the Rodrigues formula for the Jacobi polynomial [24]

$$
P^{(\alpha,\beta)}_n(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}[(1-x)^{\alpha+n}(1+x)^{\beta+n}],
$$

Eqs. (32)-(33) lead to the integral expression

$$W^q_{np}(c,\pm b) = \frac{(-1)^p}{2^{n+q+c\pm b+1}} \int_{-1}^{1} (1-x)^{n-p}(1+x)^p \frac{d^q}{dx^q}[(1-x)^{q+c}(1+x)^{q\pm b}]dx$$

$$\times\sqrt{\frac{(2q+c\pm b+1)(n-q)!\Gamma(q+c\pm b+1)\Gamma(n+q+c\pm b+2)}{p!q!(n-p)!\Gamma(q+c+1)\Gamma(q\pm b+1)\Gamma(n-p+c+1)\Gamma(p\pm b+1)}} \tag{34}$$

for the coefficient $W^q_{np}(c,\pm b)$. Equation (34) can be compared with the integral representation [26]

$$(ab\alpha\beta|c\gamma) = \delta_{\alpha+\beta,\gamma}\sqrt{\frac{(2c+1)(J+1)!(J-2c)!(c+\gamma)!}{(J-2a)!(J-2b)!(a-\alpha)!(a+\alpha)!(b-\beta)!(b+\beta)!(c-\gamma)!}}$$

$$\times\frac{(-1)^{a-c+\beta}}{2^{J+1}}\int_{-1}^{1}(1-x)^{a-\alpha}(1+x)^{b-\beta}\frac{d^{c-\gamma}}{dx^{c-\gamma}}[(1-x)^{J-2a}(1+x)^{J-2b}]dx$$

(with $J=a+b+c$) for the Clebsch-Gordan coefficients $C^{c\gamma}_{a\alpha;b\beta}\equiv(ab\alpha\beta|c\gamma)$ of the compact Lie group SU(2). This yields

$$W^q_{np}(c,\pm b) = (-1)^{n-q}\,(a_0 b_0 \alpha\beta|c_0,\alpha+\beta),$$

$$a_0 = \frac{n\pm b}{2},\quad b_0 = \frac{n+c}{2},\quad c_0 = q+\frac{c\pm b}{2}, \tag{35}$$

$$\alpha = p - \frac{n\mp b}{2},\quad \beta = \frac{n+c}{2} - p.$$

Since the quantum numbers in Eq. (35) are not necessarily integers or half of odd integers, the coefficients for the expansion of the cylindrical basis in terms of the spherical basis may be considered as analytical continuation, for real values of their arguments, of the SU(2) Clebsch-Gordan coefficients. The inverse of Eq. (31), namely

$$\Psi_{n_r qm} = \sum_{p=0}^{n} \tilde{W}^p_{nq}(c,\pm b)\,\Psi_{n_\rho pm} \tag{36}$$

follows from the orthonormality property of the SU(2) Clebsch-Gordan coefficients. The expansion coefficients in (36) are thus

$$\tilde{W}^p_{nq}(c,\pm b) = W^q_{np}(c,\pm b).$$

Note that in order to compute the coefficients $W^q_{np}(c,\pm b)$ through (35), we can use the ${}_3F_2(a,b,c;d,e;1)$ representation [26] of the SU(2) Clebsch-Gordan coefficients.

We close this section with some considerations concerning the limiting cases ($P=0$, $Q\neq 0$) and ($P=0$, $Q=0$). It is to be observed that the passage from ($P\neq 0$, $Q\neq 0$) to ($P=0$, $Q\neq 0$) needs some caution. Indeed for $b=s^-$, Eq. (35) can be rewritten in terms of the quantum numbers N, l, and n_3 as

$$W^q_{np}(c,\pm s) = (-1)^{\frac{N-l}{2}}\,(a_0 b_0\alpha\beta|c_0,\alpha+\beta),$$

$$a_0 = \frac{N-|m|-s\pm s}{4},\quad b_0 = \frac{N+|m|-s\mp s}{4}+\frac{\delta}{2},\quad c_0 = \frac{2l-1}{4}+\frac{\delta}{2}, \tag{37}$$

$$\alpha = \frac{2n_3-N+|m|-s\pm s}{4},\quad \beta = \frac{-2n_3+N+|m|+s\pm s}{4}+\frac{\delta}{2}.$$

By using the ordinary symmetry property [26]

$$(ab\alpha\beta|c\gamma) = (-1)^{a+b-c}(ab,-\alpha,-\beta|c,-\gamma)$$

and the Regge symmetry [26]

$$(ab\alpha\beta|c\gamma) = \left(\frac{a+b+\gamma}{2},\frac{a+b-\gamma}{2},\frac{a-b+\alpha-\beta}{2},\frac{a-b-\alpha+\beta}{2}|c,a-b\right)$$

in Eq. (37) with the sign + and by using the ordinary symmetry property [26]

$$(ab\alpha\beta|c\gamma) = (-1)^{a+b-c}(ba\beta\alpha|c\gamma)$$

in Eq. (37) with the sign −, we get

$$W^q_{np}(c,\pm s) \equiv W^l_{Nmn_3}(\delta) = (a_0 b_0\alpha\beta|c_0,\alpha+\beta),$$

$$a_0 = \frac{N+|m|}{4}+\frac{\delta}{2},\quad b_0 = \frac{N-|m|-1}{4},\quad c_0 = \frac{2l-1}{4}+\frac{\delta}{2}, \tag{38}$$

$$\alpha = \frac{N+|m|-2n_3}{4}+\frac{\delta}{2},\quad \beta = \frac{2n_3-N+|m|-1}{4}.$$

As a conclusion, when $b=s^-$ we have an expansion of the type [3]

$$\Psi_{Nmn_3}(\rho,\varphi,z;\delta) = \sum_l W^l_{Nmn_3}(\delta)\,\Psi_{Nlm}(r,\theta,\varphi;\delta), \tag{39}$$

where the summation on l goes, by steps of 2, from $|m|$ or $|m|+1$ to N according to whether as $N-|m|$ is even or odd (because $N-l$ is always even). Equations (38)-(39) were obtained in Ref. [3] for the ring-shape oscillator system. Finally, the case $P=Q=0$ can be easily deduced from (38)-(39) by taking $\delta=0$: we thus recover the result obtained in Refs. [28, 29] for the isotropic harmonic oscillator in three dimensions. Note that in the case $P=Q=0$, the expansion coefficients in Eq. (39) become Clebsch-Gordan coefficients for the noncompact Lie group SU(1,1) (cf., Ref. [30]).

5 Prolate and Oblate Spheroidal Bases

5.1 Separation in Prolate Spheroidal Coordinates

The prolate spheroidal coordinates (ξ,η,φ) are defined via

$$\begin{aligned} x &= \frac{R}{2}\sqrt{(\xi^2-1)(1-\eta^2)}\cos\varphi, \\ y &= \frac{R}{2}\sqrt{(\xi^2-1)(1-\eta^2)}\sin\varphi, \\ z &= \frac{R}{2}\xi\eta, \end{aligned}$$

(with $1 \le \xi < \infty$, $-1 \le \eta \le 1$, and $0 \le \varphi < 2\pi$), where R is the interfocus distance. As is well-known [31], in the limits where $R \to 0$ and $R \to \infty$, the prolate spheroidal coordinates reduce to the spherical coordinates and the cylindrical coordinates, respectively. In prolate spheroidal coordinates, the potential V reads

$$V = \frac{\Omega^2 R^2}{8}(\xi^2 + \eta^2 - 1) + \frac{2}{R^2}\left[\frac{P}{\xi^2\eta^2} + \frac{Q}{(\xi^2-1)(1-\eta^2)}\right]. \quad (40)$$

By looking for a solution Ψ of Eq. (2), with the potential (40), in the form

$$\Psi(\xi,\eta,\varphi) = \psi_1(\xi)\psi_2(\eta)\frac{e^{im\varphi}}{\sqrt{2\pi}}, \quad (41)$$

with $m \in \mathbf{Z}$, we obtain the two ordinary differential equations

$$\left[d_\xi(\xi^2-1)d_\xi - \frac{Q+m^2}{\xi^2-1} + \frac{ER^2}{2}\xi^2 - \frac{\Omega^2R^4}{16}\xi^2(\xi^2-1) + \frac{P}{\xi^2}\right]\psi_1 = +\lambda(R)\psi_1, \quad (42)$$

$$\left[d_\eta(1-\eta^2)d_\eta - \frac{Q+m^2}{1-\eta^2} - \frac{ER^2}{2}\eta^2 - \frac{\Omega^2R^4}{16}\eta^2(1-\eta^2) - \frac{P}{\eta^2}\right]\psi_2 = -\lambda(R)\psi_2, \quad (43)$$

where $\lambda(R)$ is a separation constant in prolate spheroidal coordinates. The combination of Eqs. (42) and (43), leads to the operator

$$\Lambda = -\frac{1}{\xi^2-\eta^2}\left[\eta^2\partial_\xi(\xi^2-1)\partial_\xi + \xi^2\partial_\eta(1-\eta^2)\partial_\eta\right]$$

$$+\frac{\xi^2+\eta^2-1}{(\xi^2-1)(1-\eta^2)}(Q-\partial_{\varphi\varphi}) + \frac{\Omega^2R^4}{16}\xi^2\eta^2 + P\frac{\xi^2+\eta^2}{\xi^2\eta^2}$$

after eliminating the energy E. The eigenvalues of the operator Λ are $\lambda(R)$ while its eigenfunctions are given by (41). The significance of the (self-adjoint) operator Λ is to be found in the connecting formula

$$\Lambda = M + \frac{R^2}{4}N, \quad (44)$$

where M and N are the constants of motion (5) and (25). The operator Λ is of pivotal importance for the derivation of the interbasis expansion coefficients from the spherical basis or the cylindrical basis to the prolate spheroidal basis. In particular, it allows us to derive the latter coefficients without knowing the wavefunctions in prolate spheroidal basis. (In this respect, credit should be put on the work by Coulson and Joseph [32] who considered an operator similar to Λ for the hydrogen atom.) Therefore, we shall not derive the prolate spheroidal wavefunctions ψ_1 and ψ_2 which could be obtained by solving Eqs. (42) and (43). It is more economical to proceed in the following way that presents the advantage of giving, at the same time, the global wavefunction $\Psi(\xi,\eta,\varphi) \equiv \Psi(\xi,\eta,\varphi;R,c,\pm b)$ and the interbasis expansion coefficients.

5.2 Interbasis Expansions for the Prolate Spheroidal Wavefunctions

The three constants of motion M, N, and Λ, which occur in Eq. (44), can be seen to satisfy the following eigenequations

$$M\Psi_{n_r q m} = A_q(c,\pm b)\Psi_{n_r q m}, \quad (45)$$

$$N\Psi_{n_\rho p m} = 2E_z(p,\pm b)\Psi_{n_\rho p m}, \quad (46)$$

and

$$\Lambda\Psi_{nkm} = \lambda_k(R)\Psi_{nkm} \quad (47)$$

for the spherical, cylindrical, and prolate spheroidal bases, respectively. [In Eq. (47), the index k labels the eigenvalues of the operator Λ and varies in the range $0 \le k \le n$.] The spherical, cylindrical, and prolate spheroidal bases are indeed eigenbases for the three sets of commuting operators $\{H, L_z, M\}$, $\{H, L_z, N\}$, and $\{H, L_z, \Lambda\}$, respectively. We are now in a position to deal with the interbasis expansions

$$\Psi_{nkm} = \sum_{p=0}^{n} U_{nk}^p(R;c,\pm b)\Psi_{n_\rho p m} \quad (48)$$

$$\Psi_{nkm} = \sum_{q=0}^{n} T_{nk}^q(R;c,\pm b)\Psi_{n_r q m} \quad (49)$$

for the prolate spheroidal basis in terms of the cylindrical and spherical bases.

First, we consider Eq. (48). Let the operator Λ act on both sides of (48). Then, by using Eqs. (44), (46), and (47) along with the orthonormality property of the cylindrical basis, we find that

$$\frac{1}{2}\left[\lambda_k(R) - \frac{R^2}{2}E_z(p,\pm b)\right]U_{nk}^p(R;c,\pm b) = \sum_{p'=0}^{n} U_{nk}^{p'}(R;c,\pm b)M_{pp'}^{(\pm)}, \quad (50)$$

where

$$M_{pp'}^{(\pm)} = \int_0^\infty\int_0^{2\pi}\int_0^\infty \Psi_{n_\rho p m}^* M \Psi_{n_\rho p' m}\rho d\rho d\varphi dz. \quad (51)$$

The calculation of the matrix element $M_{pp'}^{(\pm)}$ can be done by expanding the cylindrical wavefunctions in (51) in terms of spherical wavefunctions [see Eq. (31)] and by making use of the eigenvalue equation for M [see Eq. (45)]. This leads to

$$M_{pp'}^{(\pm)} = \frac{1}{2}\sum_{q=0}^{n} A_q(c,\pm b)W_{np}^q(c,\pm b)W_{np'}^q(c,\pm b). \quad (52)$$

To calculate the sum in Eq. (52), we need some recursion relation for the coefficient $W_{np}^q(c,\pm b)$ involving $p-1$, p, and $p+1$. Owing to Eq. (35), this amounts to use the following recursion relations [33]:

$$[-a(a+1) - b(b+1) + c(c+1) - 2\alpha\beta](ab\alpha\beta|c\gamma)$$

$$= \sqrt{(a+\alpha)(a-\alpha+1)(b-\beta)(b+\beta+1)}(a,b,\alpha-1,\beta+1|c\gamma)$$

$$+\sqrt{(a-\alpha)(a+\alpha+1)(b+\beta)(b-\beta+1)}(a,b,\alpha+1,\beta-1|c\gamma). \quad (53)$$

Then, by introducing Eq. (53) into Eq. (52) and by using the orthonormality condition

$$\sum_{c,\gamma}(ab\alpha\beta|c\gamma)(ab\alpha'\beta'|c\gamma) = \delta_{\alpha\alpha'}\delta_{\beta\beta'}, \tag{54}$$

we find that $M^{(\pm)}_{pp'}$ is given by

$$\begin{aligned} M^{(\pm)}_{pp'} &= 2\left[p(p\pm b)(n-p+1)(n+c-p+1)\right]^s\delta_{p',p-1} \\ &+[s\,(c\mp b+s)\,(c\mp b+3s)+2(p+1)(n-p)+2(p\pm b)(n+c-p+1)]\delta_{p'p} \\ &+2\left[(p+1)(p+1\pm b)(n-p)(n+c-p)\right]^s\delta_{p',p+1}. \end{aligned} \tag{55}$$

Now by introducing (55) into (50), we get the following three-term recursion relation

$$\begin{aligned} &[(p+1)(n-p)+(p\pm b)(n+c-p+1) \\ &+\frac{1}{4}(c\mp b+s)(c\mp b+3s)+\frac{R^2}{8}E_z(p,\pm b)-\frac{1}{4}\lambda_k(R)]U^p_{nk} \\ &+\left[(p+1)(p+1\pm b)(n-p)(n+c-p)\right]^s U^{p+1}_{nk} \\ &+\left[p(p\pm b)(n-p+1)(n+c-p+1)\right]^s U^{p-1}_{nk}=0 \end{aligned} \tag{56}$$

for the expansion coefficients $U^q_{nk}\equiv U^q_{nk}(R;c,\pm b)$. The recursion relation (56) provides us with a system of $n+1$ linear homogeneous equations which can be solved by taking into account the normalization condition

$$\sum_{p=0}^{n}|U^p_{nk}(R;c,\pm b)|^2=1.$$

The eigenvalues $\lambda_k(R)$ of the operator Λ then follow from the vanishing of the determinant for the latter system.

Second, let us concentrate on the expansion (49) of the prolate spheroidal basis in terms of the spherical basis. By employing a technique similar to the one used for deriving Eq. (50), we get

$$[\lambda_k(R)-A_q(c,\pm b)]\,T^q_{nk}(R;c,\pm b)=\frac{R^2}{2}\sum_{q'=0}^{n}T^{q'}_{nk}(R;c,\pm b)\,N^{(\pm)}_{qq'}, \tag{57}$$

where

$$N^{(\pm)}_{qq'}=\int_0^\infty\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\Psi^*_{n_rqm}N\Psi_{n_rq'm}r^2\sin\theta\,dr\,d\theta\,d\varphi.$$

The matrix elements $N^{(\pm)}_{qq'}$ can be calculated in the same way as $M^{(\pm)}_{pp'}$ except that we must use the relation [26]

$$(ab\alpha\beta|c\gamma)=-\left[\frac{c^2(2c+1)(2c-1)}{(c^2-\gamma^2)(-a+b+c)(a-b+c)(a+b-c+1)(a+b+c+1)}\right]^s$$

$$\times\left\{\left[\frac{(c-\gamma-1)(c+\gamma-1)(-a+b+c-1)(a-b+c-1)(a+b-c+2)(a+b+c)}{(c-1)^2(2c-3)(2c-1)}\right]^s\right.$$
$$\left.\times(ab\alpha\beta|c-2,\gamma)-\frac{(\alpha-\beta)c(c-1)-\gamma a(a+1)+\gamma b(b+1)}{c(c-1)}(ab\alpha\beta|c-1,\gamma)\right\}$$

and the orthonormality condition

$$\sum_{\alpha,\beta}(ab\alpha\beta|c\gamma)(ab\alpha\beta|c'\gamma')=\delta_{c'c}\delta_{\gamma'\gamma},$$

instead of Eqs. (53) and (54). This produces the matrix element

$$\begin{aligned} N^{(\pm)}_{qq'} &= E_n(c,\pm b)\frac{2q(q+1)+(c\pm b)(2q\pm b+1)}{(2q+c\pm b)(2q+c\pm b+2)}\delta_{q'q} \\ &-2\Omega\left[A^{q+1}_n(c,\pm b)\delta_{q',q+1}+A^q_n(c,\pm b)\delta_{q',q-1}\right], \end{aligned} \tag{58}$$

where

$$A^q_n(c,\pm b)=\left[\frac{q(n-q+1)(q+c\pm b)(q\pm b)(q+c)(n+q+c\pm b+1)}{(2q+c\pm b)^2(2q+c\pm b-1)(2q+c\pm b+1)}\right]^s.$$

Finally, the introduction of (58) into (57) leads to the three-term recursion relation

$$\begin{aligned} &\left[\lambda_k(R)-A_q(c,\pm b)-\frac{R^2}{2}E_n(c,\pm b)\frac{2q(q+1)+(c\pm b)(2q\pm b+1)}{(2q+c\pm b)(2q+c\pm b+2)}\right]T^q_{nk} \\ &+\Omega R^2\left[A^{q+1}_n(c,\pm b)T^{q+1}_{nk}+A^q_n(c,\pm b)T^{q-1}_{nk}\right]=0 \end{aligned} \tag{59}$$

for the expansion coefficients $T^p_{nk}\equiv T^p_{nk}(R;c,\pm b)$. This relation can be iterated by taking into account the normalization condition

$$\sum_{q=0}^{n}|T^q_{nk}(R;c,\pm b)|^2=1.$$

Here again, the eigenvalues $\lambda_k(R)$ may be obtained from the vanishing of the determinant of a system of $n+1$ linear homogeneous equations.

5.3 Limiting Cases

Putting $b=s^-$, i.e., $P=0^-$, in the matrix element (58) with $Q\neq 0$ and by using (20), we have

$$N^{(\pm)}_{qq'}=E_N(\delta)\frac{2A_l(\delta)-2(|m|+\delta)^2-1}{(2l+2\delta-1)(2l+2\delta+3)}\delta_{l'l}-2\Omega\left[A^{l+2}_N(\delta)\delta_{l',l+2}+A^l_N(\delta)\delta_{l',l-2}\right],$$

where

$$A^l_N(\delta) = \left[\frac{l_-(l_- - 1)(l_+ + 2\delta)(l_+ + 2\delta - 1)(N - l + 2)(N + l + 2\delta + 1)}{4(2l + 2\delta - 1)^2(2l + 2\delta - 3)(2l + 2\delta + 1)}\right]^s$$

(with $l_\pm = l \pm |m|$) and finally we get the following three-term recursion relation

$$\left[\lambda_k(R) - A_l(\delta) - \frac{R^2}{2}E_N(\delta)\frac{2A_l(\delta) - 2(|m| + \delta)^2 - 1}{(2l + 2\delta - 1)(2l + 2\delta + 3)}\right]T^l_{Nk}(R;\delta)$$

$$+\Omega R^2\left[A^{l+2}_N(\delta)T^{l+2}_{Nk}(R;\delta) + A^l_N(\delta)T^{l-2}_{Nk}(R;\delta)\right] = 0$$

for $T^l_{Nk}(R;\delta) \equiv T^l_{Nk}(R;c,\pm s)$. By analogy it is easy to obtain a three-term recursion relation for the interbasis expansion coefficients $U^{n_3}_{Nk}(R;\delta) \equiv U^{n_3}_{Nk}(R;c,\pm s)$. We get

$$[(2n_3 + 1)(N - n_3 + \delta + 1) + (|m| + \delta)^2 - 1 + \frac{\Omega R^2}{4}(2n_3 + 1) - \lambda_k(R)]\,U^{n_3}_{Nk}(R;\delta)$$

$$+\left[(n_3 + 1)(n_3 + 2)(N - |m| - n_3)(N + |m| - n_3 + 2\delta)\right]^s U^{n_3+2}_{Nk}(R;\delta)$$

$$+\left[n_3(n_3 - 1)(N - |m| - n_3 + 2)(N + |m| - n_3 + 2\delta + 2)\right]^s U^{n_3-2}_{nk}(R;\delta) = 0.$$

Consequently, when $b = s^-$ we have the expansions [cf., Eqs. (48) and (49)]

$$\Psi_{Nkm} = \sum_{n_3}^{N} U^{n_3}_{Nk}(R;\delta)\Psi_{Nmn_3},$$

$$\Psi_{Nkm} = \sum_{l}^{N} T^l_{Nk}(R;\delta)\Psi_{Nlm},$$

for the ring-shape oscillator. The summations on l and n_3 go, by steps of 2, from $|m|$ or $|m| + 1$ to N and from 0 or 1 to $N - |m|$ according to whether as $N - |m|$ is even or odd (because $N - l$ and $N - |m| - n_3$ are always even).

The next limiting case $\delta = 0$, i.e., $Q = 0$, is trivial and the corresponding results for the isotropic harmonic oscillator agree with the ones obtained in Ref. [34].

Finally, it should be noted that the following two limits

$$\lim_{R\to 0} U^p_{nk}(R;c,\pm b) = \tilde{W}^p_{nk}(c,\pm b),$$

$$\lim_{R\to\infty} T^q_{nk}(R;c,\pm b) = W^q_{nk}(c,\pm b)$$

furnish a useful means for checking the calculations presented in the fourth and fifth sections.

5.4 Separation and Interbasis Expansions for the Oblate Spheroidal Wavefunctions

The oblate spheroidal coordinates $(\bar{\xi}, \bar{\eta}, \varphi)$ are defined by

$$x = \frac{\bar{R}}{2}\sqrt{(\bar{\xi}^2 + 1)(1 - \bar{\eta}^2)}\cos\varphi,$$

$$y = \frac{\bar{R}}{2}\sqrt{(\bar{\xi}^2 + 1)(1 - \bar{\eta}^2)}\sin\varphi,$$

$$z = \frac{\bar{R}}{2}\bar{\xi}\bar{\eta},$$

(with $0 \le \bar{\xi} < \infty$, $-1 \le \bar{\eta} \le 1$, and $0 \le \varphi < 2\pi$), where $\bar{R}$ is the interfocus distance in the oblate spheroidal coordinate system. As in the prolate system, in the limits $\bar{R} \to 0$ and $\bar{R} \to \infty$, the oblate spheroidal coordinates give the spherical and cylindrical coordinates, respectively [31, 34].

The potential V, the Schrödinger equation, the oblate spheroidal constant of motion $\bar{\Lambda}$, and the interbasis expansion coefficients for the oblate spheroidal coordinates can be obtained from the corresponding expressions for the prolate spheroidal coordinates by means of the trick: $\xi \to -i\bar{\xi}$ and $R \to i\bar{R}$.

6 Spheroidal Corrections for the Spherical and Cylindrical Bases

As we have already mentioned, the spheroidal system of coordinates is one of the most general one-parameter systems of coordinates which contains spherical and cylindrical coordinates as some limiting cases. Accordingly, the prolate spheroidal basis of the generalized oscillator as $R \to 0$ and $R \to \infty$ degenerates into the spherical and cylindrical bases that can be treated as zeroth order approximations in some perturbation series. The three-term recursion relations for the expansion coefficients of the prolate spheroidal basis in the cylindrical and spherical bases, which have been obtained in the fifth section, may serve as a basis for constructing an algebraic perturbation theory, respectively, at large ($R \gg 1$) and small ($R \ll 1$) values of the interfocus distance R. Thus it is possible to derive prolate spheroidal corrections for the spherical and cylindrical bases.

6.1 The Case $R \ll 1$

Let us rewrite the three-term recursion relation (59) in the following form

$$[\lambda_k(R) - A_q(c,\pm b) - \Omega R^2 B^q_n(c,\pm b)]T^q_{nk}$$

$$+\Omega R^2\left[A^{q+1}_n(c,\pm b)T^{q+1}_{nk} + A^q_n(c,\pm b)T^{q-1}_{nk}\right] = 0, \qquad (60)$$

where

$$B^q_n(c,\pm b) = \frac{1}{2}(2n + c \pm b + 2)\frac{2q(q+1) + (c \pm b)(2q \pm b + 1)}{(2q + c \pm b)(2q + c \pm b + 2)}.$$

The zeroth order approximation for the separation constant $\lambda_k(R)$ and the coefficients $T^p_{nk}(R;c,\pm b)$ can immediately be derived from the recursion relation (60). Indeed, from Eq. (60), we obtain

$$\lim_{R\to 0}\lambda_k(R) = A_k(c,\pm b), \quad \lim_{R\to 0} T^q_{nk}(R;c,\pm b) = \delta_{kq},$$

so that, for the wavefunction, we have

$$\lim_{R\to 0}\Psi_{nkm}(\xi,\eta,\varphi;R,c,\pm b) = \Psi_{n_r km}(r,\theta,\varphi,z;c,\pm b).$$

As is seen from these limiting relations, the quantum number k, labeling the spheroidal separation constant and being (according to the oscillation theorem [31]) the number of zeros of the prolate angular spheroidal function $\psi_2(\eta)$ in the interval $-1\le\eta\le 1$, turns into a spherical quantum number determining the number of zeros of the angular function (9). It is clear that this fact is a consequence of the independence of the number of zeros of the wavefunction on R.

In order to calculate higher order corrections, we represent the interbasis coefficients $T^q_{nk}(R;c,\pm b)$ and the spheroidal separation constant $\lambda_k(R)$ as expansions in powers of ΩR^2:

$$T^q_{nk}(R;c,\pm b) = \delta_{kq} + \sum_{j=1}^{\infty} T^{(j)}_{kq}\ (\Omega R^2)^j, \tag{61}$$

$$\lambda_k(R) = A_k(c,\pm b) + \sum_{j=1}^{\infty}\lambda^{(j)}_k\ (\Omega R^2)^j. \tag{62}$$

Substituting Eqs. (62) and (62) into the three-term recursion relation (60) and equating the coefficients with the same power of R, we arrive at the equation for the coefficients $T^{(j)}_{kq}$ and $\lambda^{(j)}_k$

$$\begin{aligned}4(k-q)(k+q+c\pm b+1)T^{(j)}_{kq} &= -A^{q+1}_n(c,\pm b)T^{(j-1)}_{k,q+1} + B^q_n(c,\pm b)T^{(j-1)}_{kq}\\ &-A^q_n(c,\pm b)T^{(j-1)}_{k,q-1} - \sum_{t=0}^{j-1}\lambda^{(j-t)}_k T^{(t)}_{kq}.\end{aligned} \tag{63}$$

Equation (63) with the initial condition $T^{(0)}_{kq} = \delta_{kq}$ and the condition $T^{(j)}_{qq} = \delta_{j0}$ arising in the standard perturbation theory [35] allow us to derive a formula expressing $\lambda^{(j)}_k$ for $j\ge 1$ through the coefficients $T^{(j-1)}_{kk}$ and $T^{(j-1)}_{k,k\pm1}$:

$$\lambda^{(j)}_k = -A^{k+1}_n(c,\pm b)T^{(j-1)}_{k,k+1} + B^k_n(c,\pm b)T^{(j-1)}_{k,k} - A^k_n(c,\pm b)T^{(j-1)}_{k,k-1}. \tag{64}$$

This gives a possibility to determine, step by step, the coefficients $\lambda^{(j)}_k$ and $T^{(j)}_{kq}$ in Eqs. (62) and (62). As an example, let us write down the first and second order corrections in (62) for $\lambda_k(R)$ and the first order correction in (62) for $T^q_{nk}(R;c,\pm b)$. It follows from Eq. (64) that

$$\begin{aligned}\lambda^{(1)}_k &= B^k_n(c,\pm b),\\ \lambda^{(2)}_k &= -A^{k+1}_n(c,\pm b)T^{(1)}_{k,k+1} - A^k_n(c,\pm b)T^{(1)}_{k,k-1}\end{aligned}$$

and Eq. (63) for $j=1$ results in

$$T^{(1)}_{kq} = -\frac{A^k_n(c,\pm b)}{4(2k+c\pm b)}\delta_{q,k-1} + \frac{A^{k+1}_n(c,\pm b)}{4(2k+c\pm b+2)}\delta_{q,k+1}. \tag{65}$$

Thus, for the spheroidal separation constant, with an accuracy up to the term $(\Omega R^2)^2$, we get

$$\lambda_k(R) = A_k(c,\pm b) + \Omega R^2 B^k_n(c,\pm b) + \frac{\Omega^2R^4}{4}\left[\frac{A^k_n(c,\pm b)^2}{2k+c\pm b} - \frac{A^{k+1}_n(c,\pm b)^2}{2k+c\pm b+2}\right].$$

Introducing (65) into (62) and then using (49) for the expansion of the prolate spheroidal basis over the spherical one, we get the following approximate formula

$$\begin{aligned}\Psi_{nkm}(\xi,\eta,\varphi;R,c,\pm b) &= \Psi_{nkm}(r,\theta,\varphi;c,\pm b)\\ -\frac{\Omega^2R^4}{4}&\left[\frac{A^k_n(c,\pm b)}{2k+c\pm b}\Psi_{n,k-1,m}(r,\theta,\varphi;c,\pm b) - \frac{A^{k+1}_n(c,\pm b)}{2k+c\pm b+2}\Psi_{n,k+1,m}(r,\theta,\varphi;c,\pm b)\right].\end{aligned}$$

6.2 The Case $R \gg 1$

Now let us consider the case $R\gg 1$. The three-term recursion relation (56) can be written as

$$\left[D^p_n(c,\pm b) + \frac{R^2}{8}E_z(p,\pm b) - \frac{\lambda_k(R)}{4}\right]U^p_{nk} + \left[C^{p+1}_n(c,\pm b)U^{p+1}_{nk} + C^p_n(c,\pm b)U^{p-1}_{nk}\right] = 0, \tag{66}$$

where

$$C^p_n(c,\pm b) = \left[p(p\pm b)(n-p+1)(n+c-p+1)\right]^s,$$

$$D^p_n(c,\pm b) = (p+1)(n-p) + (p\pm b)(n+c-p+1) + \frac{1}{4}(c\mp b+s)(c\mp b+3s).$$

It follows from Eq. (66) that

$$\lim_{R\to\infty}\frac{\lambda_k(R)}{R^2} = \frac{1}{2}E_z(k,\pm b), \qquad \lim_{R\to\infty}U^p_{nk}(R;c,\pm b) = \delta_{kp}.$$

For $R\gg 1$, the interbasis expansion coefficients and the spheroidal separation constant are developed in negative powers of ΩR^2:

$$U^p_{nk}(R;c,\pm b) = \delta_{kp} + \sum_{j=1}^{\infty}U^{(j)}_{kp}(\Omega R^2)^{-j}, \tag{67}$$

$$\frac{\lambda_k(R)}{\Omega R^2} = \frac{1}{2\Omega}E_z(k,\pm b) + \sum_{j=1}^{\infty}\lambda^{(j)}_k(\Omega R^2)^{-j}. \tag{68}$$

Substituting Eqs. (68) and (68) into Eq. (66), we get

$$\frac{1}{4}(p-k)U_{kp}^{(j)}+C_n^{p+1}(c,\pm b)U_{k,p+1}^{(j-1)}+D_n^p(c,\pm b)U_{kp}^{(j-1)}$$

$$+C_n^p(c,\pm b)U_{k,p-1}^{(j-1)}-\frac{1}{4}\sum_{t=1}^{j-1}\lambda_k^{(j-t)}U_{kp}^{(t)}=0. \qquad (69)$$

Using the conditions $U_{kp}^{(0)}=\delta_{kp}$ and $U_{pp}^{(j)}=\delta_{j0}$, one easily obtain

$$\frac{1}{4}\lambda_k^{(j)}=C_n^{p+1}(c,\pm b)U_{k,p+1}^{(j-1)}+D_n^p(c,\pm b)U_{kp}^{(j-1)}+C_n^p(c,\pm b)U_{k,p-1}^{(j-1)}. \qquad (70)$$

Equations (69) and (70) completely solve the problem of determining the expansion coefficients $\lambda_k^{(j)}$ and $U_{kp}^{(j)}$. For instance, we have the approximate formulae

$$\frac{\lambda_k(R)}{\Omega R^2}=\frac{1}{2\Omega}E_z(k,\pm b)+\frac{4}{\Omega R^2}D_n^k(c,\pm b)+\frac{16}{(\Omega R^2)^2}\left[C_n^k(c,\pm b)^2-C_n^{k+1}(c,\pm b)^2\right],$$

$$\Psi_{nkm}(\xi,\eta,\varphi;R,c,\pm b)=\Psi_{nkm}(\rho,\varphi,z;c,\pm b)$$

$$+\frac{4}{\Omega R^2}\left[C_n^k(c,\pm b)\Psi_{n,k-1,m}(\rho,\varphi,z;c,\pm b)-C_n^{k+1}(c,\pm b)\Psi_{n,k+1,m}(\rho,\varphi,z;c,\pm b)\right].$$

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Appendix A: Bi-Orthogonality of the Radial Wavefunctions

Besides the orthonormality relation (14) in the quantum numbers n_r for the function R_{n_rq}, we also have an orthogonality relation in the quantum numbers q, viz.,

$$J_{qq'}^{(\pm)}=\int_0^\infty R_{n_r'q'}R_{n_rq}dr=\frac{\Omega}{2q+c\pm b+1}\delta_{q'q}, \qquad (71)$$

for a given value $n_r'+q'=n_r+q$ of the principal quantum number n. The proof of (71) is as follows. In the integral in Eq. (71), we replace the two radial wavefunctions by their expressions (13). Then, with the help of the formula [36]

$$\int_0^\infty e^{-cx}x^{\alpha-1}L_m^\gamma(cx)L_n^\lambda(cx)dx=\frac{(\gamma+1)_m(\lambda-\alpha+1)_n\Gamma(\alpha)}{m!n!c^\alpha}$$

$$\times{}_3F_2(-m,\alpha,\alpha-\lambda;\gamma+1,\alpha-\lambda-n;1),$$

we arrive at

$$J_{qq'}^{(\pm)}=\Omega\frac{\Gamma(q'+q+c\pm b+1)}{\Gamma(2q'+c\pm b+2)\Gamma(q-q'+1)}\sqrt{\frac{(n-q')!\Gamma(n+q'+c\pm b+2)}{(n-q)!\Gamma(n+q+c\pm b+2)}}$$

$$\times{}_2F_1(-q+q',q+q'+c\pm b+1;2q'+c\pm b+2;1). \qquad (72)$$

By using the Gauss summation formula [24]

$${}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

we can rewrite (72) as

$$J_{qq'}^{(\pm)}=\frac{\Omega}{q+q'+c\pm b+1}\sqrt{\frac{(n-q')!\Gamma(n+q'+c\pm b+2)}{(n-q)!\Gamma(n+q+c\pm b+2)}}$$

$$\cdot\quad[\Gamma(q-q'+1)\Gamma(q'-q+1)]^{-1}$$

This completes the proof of Eq. (71) since $[\Gamma(q-q'+1)\Gamma(q'-q+1)]^{-1}=\delta_{qq'}$.

Appendix B: The Smorodinsky-Winternitz and Morse Systems

The Morse system with the potential

$$V_{\rm M}=V_0(e^{-2ax}-2e^{-ax})$$

can be connected to the dynamical system with the potential

$$V_{\rm SW}=\frac{\Omega^2}{2}z^2+\frac{P}{2}\frac{1}{z^2}.$$

(The latter potential may be considered as a one-dimensional component of the so-called Smorodinsky-Winternitz [37, 38, 39] potential. The potential $V_{\rm SW}$ was investigated by Calogero [40].)

The Schrödinger equation for the Morse potential $V_{\rm M}$, i.e.,

$$\left[d_{xx}+2E-2V_0(e^{-2ax}-2e^{-ax})\right]\psi=0 \qquad (73)$$

admits a discrete spectrum (with $E<0$) and a continuous spectrum. For the discrete spectrum, by making the change of variable

$$y=ax,\quad y\in\mathbf{R},\quad z=e^{-sy},\quad z\in\mathbf{R}^+$$

and the change of function

$$\psi(x)=\frac{1}{\sqrt{z}}f(z)$$

in Eq. (73), we get

$$\left[d_{zz}+4\lambda^2(2-z^2)+\left(\frac{8E}{a^2}+\frac{1}{4}\right)\frac{1}{z^2}\right]f=0, \qquad (74)$$

where

$$\lambda = \frac{\sqrt{2V_0}}{a}.$$

Equation (74) has the same form as Eq. (26) for $z > 0$ with

$$E_z = 4\lambda^2, \quad \Omega = 2\lambda, \quad P = -\frac{8E}{a^2} - \frac{1}{4}.$$

Therefore, we must consider two admissible regions for the energy E: (i) $-32E > a^2$ and (ii) $0 < -32E < a^2$.

In the case (i), by employing the energy formula (30) for E_z, we obtain that E is determined by the relation

$$\frac{\sqrt{-2E}}{a} = \lambda - (p+s), \quad p = 0, 1, \cdots, [\lambda - s]. \tag{75}$$

In Eq. (75), $[x]$ stands for the integral value of x. As a result, we have

$$E = -V_0 \left[1 - \frac{1}{\lambda}(p+s)\right]^2, \quad p = 0, 1, \cdots, [\lambda - s]. \tag{76}$$

Equation (76) is in agreement with the well-known result according to which the discrete spectrum of the Morse system has a finite number (here $[\lambda - s] + 1$) of energy levels with the condition $\lambda > s$.

In the case (ii), we have

$$\frac{\sqrt{-2E}}{a} = \pm[\lambda - (p+s)],$$

which has no solution for $p \in \mathbf{N}$.

The connection just described between the Morse and Smorodinsky-Winternitz systems can be used also to deduce the wavefunctions of one system from the wavefunctions of the other. For instance, from Eq. (29), we immediately get the normalized solution $\psi(x) \equiv \psi_p(z;\lambda)$ of (73):

$$\psi_p(z;\lambda) = (-1)^p (2\lambda)^{\lambda-p} \sqrt{\frac{ap!}{\Gamma(2\lambda - p)}} e^{-\lambda z^2} z^{2\lambda-2p-1} L_p^{2\lambda-2p-1}(2\lambda z^2), \tag{77}$$

with

$$z = e^{-sax}, \quad p = 0, 1, \cdots, [\lambda - s].$$

Our result (77) differs from the one of Nieto and Simmons [41] (by the fact that the factor $p!$ in (77) is $2p - \lambda$ in Ref. [41]).

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Киблер М., Мардоян Л.Г., Погосян Г.С. E2-96-242
Обобщенная осцилляторная система: межбазисные разложения

Проведено исследование нерелятивистской квантовомеханической динамической системы, которая обобщает трехмерный изотропный гармонический осциллятор. Переменные в уравнении Шредингера для этой обобщенной осцилляторной системы разделяются в сферических, цилиндрических и сфероидальных (вытянутых и сплюснутых) координатах. Полностью решена проблема межбазисных разложений волновых функций. Найдено, что коэффициенты прямого и обратного разложений цилиндрического базиса по сферическому выражаются через коэффициенты Клебша—Гордана для группы SU(2), аналитически продолженные по своим индексам в область произвольных вещественных значений момента. Показано, что коэффициенты межбазисного разложения вытянутого и сплюснутого сфероидальных базисов по сферическому и цилиндрическому базисам удовлетворяют трехчленным рекуррентным соотношениям. Обсуждается связь между обобщенной осцилляторной системой (проецированной на ось z) и одномерной системой Морса.

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Kibler M., Mardoyan L.G., Pogosyan G.S. E2-96-242
On a Generalized Oscillator System: Interbasis Expansions

This article deals with a nonrelativistic quantum mechanical study of a dynamical system which generalizes the isotropic harmonic oscillator system in three dimensions. The Schrödinger equation for this generalized oscillator system is separable in spherical, cylindrical, and spheroidal (prolate and oblate) coordinates. The quantum mechanical spectrum of this system is worked out in some details. The problem of interbasis expansions of the wavefunctions is completely solved. The coefficients for the expansion of the cylindrical basis in terms of the spherical basis, and vice-versa, are found to be analytic continuations (to real values of their arguments) of Clebsch—Gordan coefficients for the group SU(2). The interbasis expansion coefficients for the prolate and oblate spheroidal bases in terms of the spherical or the cylindrical bases are shown to satisfy three-term recursion relations. Finally, a connection between the generalized oscillator system (projected on the z-line) and the Morse system (in one dimension) is discussed.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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