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L.G.Mardoyan¹, A.N.Sissakian, V.M.Ter-Antonyan²

OSCILLATOR AS A HIDDEN
NON-ABELIAN MONOPOLE

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¹E-mail: mardoyan@thsun1.jinr.dubna.su

²E-mail: terant@thsun1.jinr.dubna.su

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Systems with nontrivial topology are the most interesting objects of quantum mechanics [1,2], quantum field theory [3,4], and condensed matter physics [5,6].

In pioneering papers [7,8], these systems were merely postulated; later on, they were deduced in quantum field theory from the first principles [9,10].

The problem of generation of systems with nontrivial topology in quantum mechanics was formulated in refs. [11, 12]; while in refs. [13,14], a four-dimensional isotropic oscillator was related to an Abelian system "charge-dyon".

This note is aimed at constructing a non-Abelian $SU(2)$ model that describes the system "charge-dyon".

Our consideration is essentially based on the Hürwitz transformation [15-18]:

$$\begin{aligned} x_0 &= u_0^2 + u_1^2 + u_2^2 + u_3^2 - u_4^2 - u_5^2 - u_6^2 - u_7^2 \\ x_1 &= 2(u_0u_4 + u_1u_5 - u_2u_6 - u_3u_7) \\ x_2 &= 2(u_0u_5 - u_6u_4 + u_2u_7 - u_3u_6) \\ x_3 &= 2(u_0u_6 + u_1u_7 + u_2u_4 + u_3u_5) \\ x_4 &= 2(u_0u_7 - u_1u_6 - u_2u_5 + u_3u_4) \end{aligned} \quad (1)$$

Transformation (1) obeys the identity

$$r \equiv (x_0^2 + \dots + x_4^2)^{1/2} = u_0^2 + \dots + u_7^2 \equiv u^2$$

We complement (1) with the angular coordinates [19]

$$\begin{aligned} \alpha_1 &= \frac{i}{2} \ln \frac{(u_0 + iu_1)(u_2 - iu_3)}{(u_0 - iu_1)(u_2 + iu_3)} \\ \beta_1 &= 2 \arctan \left[\frac{(u_0 + iu_1)(u_0 - iu_1)}{(u_2 + iu_3)(u_2 - iu_3)} \right]^{1/2} \\ \gamma_1 &= \frac{i}{2} \ln \frac{(u_0 + iu_1)(u_2 + iu_3)}{(u_0 - iu_1)(u_2 - iu_3)} \end{aligned} \quad (2)$$

Transformations (1)-(2) transform the space $R^8(\vec{u})$ into the direct product of the space $R^5(\vec{x})$ and three-dimensional sphere $S^3(\alpha_1, \beta_1, \gamma_1)$. From (2) it follows that $\alpha_1 \in [0, 2\pi)$, $\beta_1 \in [0, \pi]$, $\gamma_1 \in [0, 4\pi)$.

Consider, in $R^8(\vec{u})$, the quantum oscillator

$$\frac{\partial^2 \psi}{\partial u_\mu^2} + \frac{2M}{\hbar^2} \left(E - \frac{M\omega^2 u^2}{2} \right) \psi = 0, \quad u_\mu \in R^8 \quad (3)$$

Formulae (1)-(3) provide the basic result of our paper

$$\frac{1}{2M} \left(-i\hbar \frac{\partial}{\partial x_j} - \hbar A_j^a \hat{T}_a \right)^2 \psi + \frac{\hbar^2}{2Mr^2} \hat{T}^2 \psi - \frac{\alpha}{r} \psi = \epsilon \psi \quad (4)$$

Here $j = 1, 2, \dots, 5$; $a = 1, 2, 3$; parameters ϵ and α are connected with the oscillator parameters

$$\epsilon = -M\omega^2/8, \quad \alpha = E/4$$

Equations (3) and (4) are dual to each other. In equation (3) ω is fixed and E is quantized, whereas in eq. (4) E is fixed and ω is quantized. These situations are alternative to each other: that in eq. (3) having the meaning of energy acquires the meaning of coupling constant in eq. (4), and vice versa. For a fixed E equation (3) selects an infinite set of monoenergetic states with frequencies $\omega_N = E/\hbar(N+4)$, $N \in \mathbb{Z}$. These states form a complete system of functions describing the object connected with eq.(4).

Further, $\hat{T}^2 = \hat{T}_1^2 + \hat{T}_2^2 + \hat{T}_3^2$, and the operators \hat{T}_a satisfy the commutation relations

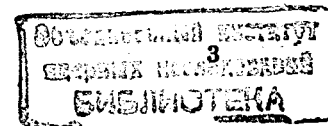
$$[\hat{T}_a, \hat{T}_b] = i\epsilon_{abc} \hat{T}_c$$

and are explicitly written as follows:

$$\begin{aligned} \hat{T}_1 &= i \left(\cos \alpha_1 \cot \beta_1 \frac{\partial}{\partial \alpha_1} + \sin \alpha_1 \frac{\partial}{\partial \beta_1} - \frac{\cos \alpha_1}{\sin \beta_1} \frac{\partial}{\partial \gamma_1} \right) \\ \hat{T}_2 &= i \left(\sin \alpha_1 \cot \beta_1 \frac{\partial}{\partial \alpha_1} - \cos \alpha_1 \frac{\partial}{\partial \beta_1} - \frac{\sin \alpha_1}{\sin \beta_1} \frac{\partial}{\partial \gamma_1} \right) \\ \hat{T}_3 &= -i \frac{\partial}{\partial \alpha_1} \end{aligned} \quad (5)$$

The triplet of five-dimensional vectors A_j^a is given by the expressions

$$\begin{aligned} A_j^1 &= \frac{1}{r(r+x_0)} (0, -x_4, -x_3, x_2, x_1) \\ A_j^2 &= \frac{1}{r(r+x_0)} (0, x_3, -x_4, -x_1, x_2) \\ A_j^3 &= \frac{1}{r(r+x_0)} (0, x_2, -x_1, x_4, -x_3) \end{aligned}$$



The vectors A_j^a are orthogonal to each other,

$$A_j^a A_j^b = \frac{1}{r^2} \frac{(r - x_0)}{(r + x_0)} \delta_{ab}$$

and also to the vector $\vec{x} = (x_0, \dots, x_4)$.

Every term of the triplet coincides with the vector potential of a 5-dimensional Dirac monopole with a unit magnetic charge and the line of singularity extending along the negative axis x_0 , including point $x_0 = 0$.

Equation (4) describes a 5-dimensional system "charge-dyon". The operators \hat{T}_a represent components of the total momentum operator of that system and are generators of the $SU(2)$ group of invariance of equation (4). The term $\hbar^2 \hat{T}^2 / 2Mr^2$ represents the correction introduced by Goldhaber [20].

In the space $R^5(\vec{x})$ we introduce the hyperspherical coordinates

$$\begin{aligned} x_0 &= r \cos \theta \\ x_1 + ix_2 &= r \sin \theta \cos \frac{\beta}{2} e^{i\frac{\alpha+\gamma}{2}} \\ x_3 + ix_4 &= r \sin \theta \sin \frac{\beta}{2} e^{i\frac{\alpha-\gamma}{2}} \end{aligned}$$

It is to be noted that $\theta \in [0, \pi]$, $\alpha \in [0, 2\pi]$, $\beta \in [0, \pi]$, $\gamma \in [0, 4\pi]$.

Equation (4) in these coordinates assumes the form

$$\left(\Delta_{r\theta} - \frac{\hat{j}^2}{r^2 \sin^2 \theta / 2} - \frac{\hat{L}^2}{r^2 \cos^2 \theta / 2} \right) \psi + \frac{2M}{\hbar^2} \left(\epsilon + \frac{\alpha}{r} \right) \psi = 0 \quad (6)$$

where

$$\hat{j}^2 = \hat{j}_1^2 + \hat{j}_2^2 + \hat{j}_3^2, \quad \hat{L}^2 = \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2, \quad \hat{L}_a = \hat{J}_a + \hat{T}_a$$

and the operators \hat{J}_a follow from formulae (5) under the change $\alpha_1 \rightarrow \alpha$, $\beta_1 \rightarrow \beta$, $\gamma_1 \rightarrow \gamma$. Then,

$$\Delta_{r\theta} = \frac{1}{r^4} \frac{\partial}{\partial r} \left(r^4 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin^3 \theta} \frac{\partial}{\partial \theta} \left(\sin^3 \theta \frac{\partial}{\partial \theta} \right)$$

By G we shall denote common eigenfunctions of the commuting operators \hat{L}^2 and \hat{J}^2 corresponding to the eigenvalues $L(L+1)$ и $J(J+1)$.

If we look for the solution to eq. (4) in the form

$$\Psi = R(r)Z(\theta)G(\alpha, \beta, \gamma; \alpha_1, \beta_1, \gamma_1)$$

then upon separating the variables r and θ and introducing the constant of separation $\lambda(\lambda+3)$ (for the time being λ is arbitrary), we arrive at two ordinary differential equations whose solutions are given by the following expressions

$$Z(\theta) = (1 - \cos \theta)^J (1 + \cos \theta)^L$$

$${}_2F_1 \left(-\lambda + J + L, \lambda + J + L + 3, 2J + 2; \frac{1 - \cos \theta}{2} \right)$$

$$R_{n_r, \lambda}(r) = \text{const} (2\kappa r)^\lambda e^{-\kappa r} F \left(\lambda + 2 - \frac{1}{r_0 \kappa}, 2\lambda + 4; 2\kappa r \right)$$

Here we introduced the notation:

$$r_0 = \hbar^2 / M\alpha^2, \quad \kappa^2 = -2M\epsilon / \hbar^2$$

From standard conditions it follows that λ should be either integer (including $\lambda = 0$) or half-integer and the energy spectrum of the 5-dimensional bound system "charge-dyon" is of the form

$$\epsilon = -\frac{M\alpha^2}{2\hbar^2(n_r + \lambda + 2)^2}, \quad n_r \in \mathbb{Z}, \quad \lambda \in \mathbb{Z} \quad (7)$$

The energy levels (7) are degenerate with the multiplicity

$$g_N = \frac{(N+7)!}{7!N!}, \quad N = 2(n_r + \lambda)$$

Instead of an oscillator, one could consider a system with the potential energy

$$M\omega^2 u^2 / 2 + V(u^2)$$

where $V(u^2)$ is an arbitrary function. Then, an extra term $V(r)/4r$ would arise in equation (4), though equations (3) and (4) would remain dual to each other.

There are no a priori reasons to consider that equation (3) is in a way related to the problem of magnetic charge. To obtain a non-Abelian monopole from an oscillator, we should apply to the nontrivial system of transformations (1)-(2); and therefore, we called the 8D-oscillator the hidden non-Abelian monopole.

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Мардоян Л.Г., Сисакян А.Н., Тер-Антонян В.М.
Осциллятор как скрытый неабелевый монополяр

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Из восьмимерного изотропного осциллятора преобразованием Гурвица построена неабелева $SU(2)$ -модель, описывающая пятимерную связанную систему заряд-дион. Сформулирован принцип дион-осцилляторной дуальности. Вычислен энергетический спектр и волновые функции системы заряд-дион.

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Mardoyan L.G., Sissakian A.N., Ter-Antonyan V.M.
Oscillator as a Hidden Non-Abelian Monopole

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A non-Abelian $SU(2)$ model is constructed for a five-dimensional bound system «charge-dyon» on the basis of the Hurwitz-transformed eight-dimensional isotropic quantum oscillator. The principle of dyon-oscillator duality is formulated; the energy spectrum and wave functions of the system «charge-dyon» are calculated.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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