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ОБЪЕДИНЕННОГО
ИНСТИТУТА
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

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THE BOGOLIUBOV RENORMALIZATION GROUP

Second English printing

1996

Comment to the second English printing

This article originally appeared in August 1994 in Russian as a preprint of Joint Institute for Nuclear Research No. P2-94-310 being simultaneously submitted to 'Uspekhi Math. Nauk'. Unfortunately, in the second part of published version ['Успехи Мат. Наук', т.49, No.5, (1994) 147-164] by some pure technical reasons there appeared many (more than 25) misprints related to the references of papers. In spite of the author's signal for the UMN Editorial Board, these errors have been reproduced in the American translation.

Moreover, this latter English-language publication ['Russian Math. Surveys', 49:5 (1994) 155-176], due to the translator's poor qualification both in subject terminology and Russian, contains a lot (around 50) of errors distorting the author's text.

By these reasons the author decided to present corrected English text. It results from the editing of the RMS publication and is adequate to the Russian preprint P2-94-310.

Contents

1 INTRODUCTION

1.1 Quantum fields

Nikolai Nikolaevich Bogoliubov (N.N. in what follows) took up problems in quantum field theory (QFT) in real earnest in the late 40s, possibly under the influence of the well–known series of articles by Schwinger which were presented at N.N.'s seminar in the Steklov (Mathematical) Institute. In any case, the first publications on quantum field theory by N.N., which appeared in the early 50s[1], were devoted to variational–derivative equations of Tomonaga–Schwinger type and were based on the axiomatic definition of the scattering matrix as a functional of the interaction domain $g(x)$, generalizing Schwinger's surface function $\sigma(t)$.

In the first half of the 50s N.N. made an active entry from the mathematical direction into a developing science, namely, renormalizable QFT, progressing more rapidly and reaching more deeply than other scientists who moved from mathematics into theoretical physics. As is known, he developed his own renormalization method based on the theory of Sobolev–Schwartz distributions. His approach makes it possible to dispense with "bare" fields and particles and the physically unsatisfactory picture of infinite renormalizations.

It was N.N.'s custom from time to time to present lectures containing surveys of large portions of QFT such as "renormalizations", "functional integral", or "surface divergences". Those who listened to the whole series of surveys were under the impression that N.N. "saw" these outwardly so different fragments from a single viewpoint, perceiving them as parts of the same picture.

This was at a time when the pre-war edition of Heitler's "Quantum Theory of Radiation" served as a textbook on the theory of elementary particles. Akhiezer and Berestetskii's "Quantum Electrodynamics" (1953) and the first volume of "Mesons and Fields" by Bethe, Hoffman and Schweber (1955) were yet to appear.

One day in the autumn of 1953, being under the influence of one of these lectures, I asked: "N.N., why don't you write a textbook on the new QFT?". His answer was: "That's not a bad idea, perhaps we should do it together?". At first I did not take it seriously. It should be explained that it was only in May of that memorable year that one of the co–authors of the future book defended his candidate dissertation in diffusion and neutron thermalization theory and he did not have a single publication in quantum field theory, while already in October of the same year the other became an academician.

Nevertheless, the conversation was resumed the following week and we begun to discuss the details of the project. The time frame of these events is quite well defined, firstly, by the fact that the above conversation took place in a car when going to N.N.'s flat in Shchukinskii passage (near the Kurchatov Institute), that is, before N.N. moved to the Moscow State University high–rise building in Lenin Hills at the end of 1953. Secondly, Akhiezer and Berestetskii's book had appeared just before our proposal was submitted at Gostekhizdat at the beginning of 1954. At the same time, the first version of the subsequent presentation of the axiomatic scattering S-matrix was put forward for publication in Uspekhi Fiz. Nauk at the end of 1954.

The initial draft of the book, apart from an introductory part presenting the Lagrangian formalism for relativistic fields and Schwinger's quantization scheme, included the original axiomatic construction of the scattering matrix based essentially on Bogoliubov's causality principle, the renormalization method resting on the distribution theory, as well as the functional integral method and the generalized Tomonaga–Schwinger equation.

Technically, the book was written as follows. I would visit N.N. in Lenin Hills and we would talk for an hour or two sketching the next chapter. Then, in my place I would write the first version of the text. At the next meeting this piece would be discussed and frequently altered substantially. When approved, the rewritten fair copy of the manuscript would be put in the left corner on the top of a large wardrobe. It would be taken from there to be typed by Evgeniya Aleksandrovna. Lightly embossed multicoloured paper was used for typing. Such paper, made in a factory in Riga, was purchased specially for our work. N.N. liked it very much. Different sections of the manuscript had different colours: blue, yellow, light green, violet... . Three copies were typed at once. I would collect the typed sections from the opposite, right corner of the wardrobe to enter the formulae.

The third copy of the coloured sections collected into chapters was read critically by colleagues working at N.N.'s department in the Steklov Institute. This reading provided the first "grinding–in". Two large articles in Uspekhi Fiz. Nauk[1] were intended to provide the second one. The text of the book[4] which appeared in September 1957, was therefore, in principle,

[1]Published in 1955[2],[3].

quite well "ironed out" and, except for the last two chapters containing new material on the renormalization group and dispersion relations, constituted, so to say, the "third approximation".

Looking back, equipped with my later experience as an author, I would say that the monograph consisting of 30–odd printer's sheets was created rather quickly. This was because from the very beginning N.N. had a clear plan and later the entire written text in his mind.

1.2 The birth of Bogoliubov's renormalization group

In the spring of 1955 a small conference on "Quantum Electrodynamics and Elementary Particle Theory" was organized in Moscow. It took place at the Lebedev (Physical) Institute in the first half of April. Among the participants there were several foreigners, including Hu Ning and Gunnar Källén. My brief presentation touched upon the consequences of finite Dyson transformations for renormalized Green functions and matrix elements in quantum electrodynamics (QED).

Landau's survey lecture "Fundamental Problems in QFT", in which the ultraviolet (UV) behaviour in quantum field theory was discussed, constituted the central event of the conference. Not long before, the problem of short-distance behaviour in QED was advanced substantially in a series of articles by Landau, Abrikosov, and Khalatnikov. They succeeded in constructing a closed approximation of the Schwinger–Dyson equations, which turned out to be compatible both with renormalizability and gauge covariance. This so-called "three–gamma" approximation admitted an explicit solution in the massless limit and, in modern language, it resulted in the summation of the leading UV logarithms.

The most remarkable fact was that this solution turned out to be self–contradictory from the physical point of view because it contained a "ghost pole" in the renormalized amplitude of the photon propagator, the difficulty of "zero physical charge".

At that time our meetings with N.N. were regular and intensive because we were busy preparing the final text of the book. N.N. was very interested in the results of Landau's group and presented me with the general problem of evaluating their reliability by constructing, for example, the second approximation (including ***next-to-leading UV logarithms***) of the Schwinger–Dyson equations to verify the stability of the UV asymptotics and the existence of a ghost pole.

At that time I would sometimes meet Alesha Abrikosov, whom I had known well since our student years. Shortly after the conference at the Lebedev Institute, Alesha told me about Gell–Mann and Low's article, which had just appeared. The same physical problem was considered in this paper, but, as he put it, it was complex to understand and they had not succeeded in combining it with the results obtained by their group.

I looked through the article and presented my teacher with a brief report on the methods and results, which included some general assertions on the scaling properties of charge distribution at short distances and rather complex functional equations.

The scene that followed my report was quite impressive. On the spot, N.N. announced that Gell–Mann and Low's approach was correct and very important: it was a realization of the *normalization group (la groupe de normalisation)* discovered a couple of years earlier by Stueckelberg and Petermann in the course of discussing the structure of the finite arbitrariness in the matrix elements arising upon removal of the divergences. This group is an example of the continuous transformation groups studied by Sophus Lie. This implied that functional group equations similar to those obtained in the article by Gell–Mann and Low must take place not only in the UV limit, but also in the general case.

Then N.N. added that differential equations corresponding to infinitesimal group transformations constitute the most powerful tool in the theory of Lie groups.

Fortunately, I was also familiar with the foundations of group theory. Within the next few days I succeeded in recasting Dyson's finite transformations and obtaining the desired functional equations for the scalar propagator amplitudes in QED, which have group properties, as well as the corresponding differential equations, that is, the Lie equations of the renormalization group. Each of these resulting equations contained a specific object, namely, the product of the squared electron charge $\alpha = e^2$ and the transverse photon propagator amplitude $d(Q^2)$. We called this product, $e^2(Q^2) = e^2 d(Q^2)$, the *invariant charge*. From the physical point of view it is an analogue of the so–called *effective charge* of an electron, first considered by Dirac in 1933, which describes the effect of charge screening due to quantum vacuum polarization. Also, the term "renormalization group" was first introduced by us in the original publication [5] in Doklady Akademii Nauk SSSR in 1955 (and in Nuovo Cimento[7] in 1956).

1.3 Episode with a "ghost pole"

At the above–mentioned conference at the Lebedev Institute Gunnar Källén presented a paper written in collaboration with Pauli on the so–called "Lee model", the exact solution of which contained a *ghost pole* (which, in contrast to the physical one corresponding to a bound state, had negative residue) in the nucleon propagator. Källén and Pauli's analysis led to the conclusion that the Lee model is physically void.

In view of the result on the presence of a similar pole in the photon propagator in QED (which follows from the solution of Landau's group as well as an independent analysis by Fradkin) obtained a little earlier in Moscow, Källén's report resulted in a heated discussion on the possible inconsistency of QED. I remember particularly well a scene by a blackboard on which Källén was presenting an example of a series converging non-uniformly with respect to a parameter (the terms of the series being dependent on the parameter) to support the claim that no rigorous conclusion about the properties of an infinite sum can be drawn from the analysis of a finite number of terms.

The parties left without convincing one another and before long a publication by Landau and Pomeranchuk appeared with a statement that not only quantum electrodynamics, but also local quantum field theory were self–contradictory.

Without going into details, let me remark that the analysis of this problem carried out by N.N. with the aid of the renormalization group formalism just developed by himself led to the conclusion that such a claim cannot have the status of a *rigorous result*, *independent of perturbation theory*.

Nevertheless, like Källén's arguments, our work also failed to convince the opponents. It is well known that Isaak Yakovlevich Pomeranchuk even closed his quantum field theory seminar shortly after these events.

2 HISTORY OF THE RENORMALIZATION GROUP IN QUANTUM FIELD THEORY

2.1 Renormalizations and renormalization invariance

As is known, the regular formalism for eliminating ultraviolet divergences in quantum field theory (QFT) was developed on the basis of covariant perturbation theory in the late 40s. This breakthrough is connected with the names of Tomonaga, Feynman, Schwinger and some others. In particular, Dyson and Abdus Salam carried out the general analysis of the structure of divergences in arbitrarily high orders of perturbation theory. Nevertheless, a number of subtle questions concerning so-called overlapping divergences in the scattering matrix, as well as surface divergences, discovered by Stueckelberg[8] in the Tomonaga–Schwinger equation, remained unclear.

An important contribution in this direction based on a thorough analysis of the mathematical nature of UV divergences was made by Bogoliubov. This was achieved on the basis of a branch of mathematics which was new at that time, namely, the Sobolev–Schwartz *theory of distributions*. The point is that propagators in local QFT are distributions (similar to the Dirac delta–function) and their products appearing in the coefficients of the expansion of the scattering matrix require supplementary definitions. In view of this the UV divergence existence reflects the ambiguity in the definition of the products in the case when their arguments coincide or lie on the light cone.

In the mid 50s on the basis of this approach Bogoliubov and his disciples developed a technique of supplementing the definition of the products of singular Stueckelberg–Feynman propagators [2] and proved a theorem [9, 10] on the finiteness and uniqueness (for renormalizable theories) of the scattering matrix in any order of perturbation theory. The prescription part of this theorem, namely, *Bogoliubov's R-operation*, still remains a practical means of obtaining finite and unique results in perturbative calculations in QFT.

The R–operation works, essentially, as follows. To remove the UV divergences, instead of introducing some regularization, for example, the momentum cutoff, and handling quasi-infinite counterterms, it suffices to complete the definition of divergent Feynman integrals by subtracting from them certain polynomials in the external momenta which in the simplest case are reduced to the first few terms of the Taylor series of the integral. The uniqueness of computational results is ensured by special conditions imposed on them. These conditions contain specific degrees of freedom[2] that can be used to establish the relationships between the Lagrangian parameters (masses, coupling constants) and the corresponding physical quantities. The fact that physical predictions are independent of the arbitrariness in the renormalization conditions, that is, they are *renormalization–invariant*, constitutes the conceptual foundation of the renormalization group.

An attractive feature of this approach is that it is free from any auxiliary nonphysical attributes such as bare masses, coupling constants, and regularization parameters which turn out to be unnecessary in computations employing Bogoliubov's approach. As a whole, this method can be regarded as *renormalization without regularization and counterterms.*

2.2 The discovery of the renormalization group

The renormalization group approach has been known in theoretical physics since the mid 50s. The renormalization group was discovered by Stueckelberg and Petermann [11] in 1953 as a group of infinitesimal transformations related to the finite arbitrariness arising in the elements of the scattering S-matrix upon elimination of the ultraviolet divergences. This arbitrariness can be fixed by means of certain parameters c_i:

> "... *we must expect that a group of infinitesimal operators* $\mathbf{P}_i = (\partial/\partial c_i)_{c=0}$, *exists, satisfying*
>
> $$\mathbf{P}_i S = h_i(m,e)\partial S(m,e,...)/\partial e ,$$
>
> *admitting thus a renormalization of e.*"

These authors introduced the *normalization group* generated (as a Lie group) by the infinitesimal operators $\mathbf{P}_i$ connected with the renormalization of the coupling constant e.

[2]These degrees of freedom correspond to different renormalization schemes and momentum scales.

In the following year, on the basis of Dyson's transformations written in the regularized form, Gell-Mann and Low [12] derived functional equations for QED propagators in the UV limit. For example, for the renormalized transverse part d of the photon propagator they obtained an equation of the form

$$d\left(\frac{k^2}{\lambda^2}, e_2^2\right) = \frac{d_C(k^2/m^2, e_1^2)}{d_C(\lambda^2/m^2, e_1^2)}\ , \quad e_2^2 = e_1^2 d_C(\lambda^2/m^2, e_1^2)\ , \qquad (1)$$

where λ is the cutoff momentum and e_2 is the physical electron charge. The appendix to this article contains the general solution (obtained by T.D.Lee) of this functional equation for the photon amplitude $d(x, e^2)$ written in two equivalent forms:

$$e^2 d\left(x, e^2\right) = F\left(xF^{-1}\left(e^2\right)\right)$$

and

$$\ln x = \int\limits_{e^2}^{e^2 d} \frac{dy}{\psi(y)}\ , \qquad (2)$$

where

$$\psi(e^2) = \frac{\partial(e^2 d)}{\partial \ln x} \quad \text{at} \quad x = 1\ .$$

A qualitative analysis of the behaviour of the electromagnetic interaction at small distances was carried out with the aid of (2). Two possibilities, namely, infinite and finite charge renormalizations were pointed out:

Our conclusion is that the **shape** *of the charge distribution surrounding a test charge in the vacuum does not, at small distances, depend on the coupling constant except through the scale factor. The behavior of the propagator functions for large momenta is related to the magnitude of the renormalization constants in the theory. Thus it is shown that the unrenormalized coupling constant $e_0^2/4\pi\hbar c$, which appears in perturbation theory as a power series in the renormalized coupling constant $e_1^2/4\pi\hbar c$ with divergent coefficients, many behave either in two ways:*

It may really be infinite as perturbation theory indicates;
It may be a finite number independent of $e_1^2/4\pi\hbar c$.

The latter possibility corresponds to the case when ψ vanishes at a finite point:[3] $\psi(\alpha_\infty) = 0$.

We remark that paper [12] neither paid attention to the group character of the analysis and the results obtained, nor paper [11] quoted. Moreover, the authors did not recognize that the Dyson transformations used by them are valid only for the transverse scaling of the electromagnetic field. Maybe this is why they failed to establish a connection between their results and the standard perturbation theory and they did not discuss the possibility that a ghost pole might exist.

The final step was taken by Bogoliubov and Shirkov [5], [6] [4]. Using the group properties of finite Dyson transformations for the coupling constant and the fields, the authors obtained functional group equations for the propagators and vertices in QED in the general case (that is, with mass taken into account). For example, the equation for the transverse amplitude of the photon propagator was obtained in the form

$$d(x, y; e^2) = d(t, y; e^2) d\left(x/t, y/t; e^2 d(t, y; e^2)\right)\ ,$$

in which the dependence of d not only on $x = k^2/\mu^2$ (where μ is a certain normalizing scale factor), but also on the mass variable $y = m^2/\mu^2$ is taken into account.

In the modern notation, the above relation [5] is an equation for the square of the effective electromagnetic coupling constant $\bar{\alpha} = \alpha d(x, y; \alpha = e^2)$:

$$\bar{\alpha}(x, y; \alpha) = \bar{\alpha}\left(x/t, y/t; \bar{\alpha}(t, y; \alpha)\right)\ . \qquad (3)$$

The term *"renormalization group"* was introduced and the notion of *invariant charge* [6] was defined in [5].

Let us emphasize that, in contrast to the Gell–Mann and Low approach, in our case there are no simplifications due to the massless nature of the ultraviolet asymptotics. Here the homogeneity of the mass scale is violated explicitly by the scale term m. Nevertheless, the symmetry (even though a bit more complex one) underlying the renormalization group can, as before, be stated as an *exact symmetry* of the solutions of the quantum

[3] Here α_∞ is the so-called fixed point of the renormalization group transformations.
[4] See also the survey [7] published in English in 1956.
[5] In the massless case $y = 0$ it is equivalent to (4).
[6] This notion is now known as the effective or running coupling constant.

field problem.[7]. This is what we mean when using the term *Bogoliubov's renormalization group* or **Renorm-group** for short.

The following differential group equations for $\bar{\alpha}$:

$$\frac{\partial \bar{\alpha}(x,y;\alpha)}{\partial \ln x} = \beta\left(\frac{y}{x}, \bar{\alpha}(x,y;\alpha)\right) \quad (4)$$

in the nonlinear form, which is standard in Lie theory, and for the electron propagator $s(x,y;\alpha)$:

$$\frac{\partial s(x,y;\alpha)}{\partial \ln x} = \gamma\left(\frac{y}{x}, \bar{\alpha}(x,y;\alpha)\right) s(x,y;\alpha)\ , \quad (5)$$

where

$$\beta(y,\alpha) = \frac{\partial \bar{\alpha}(\xi,y;\alpha)}{\partial \xi}\ , \quad \gamma(y,\alpha) = \frac{\partial s(\xi,y;\alpha)}{\partial \xi} \quad \text{at } \xi = 1\ . \quad (6)$$

were first obtained by differentiating the functional equations. In this way an explicit realization of the differential equations mentioned in the citation from [11] was obtained. These results established a conceptual link between the Stueckelberg–Petermann and Gell-Mann–Low approaches.

2.3 Creation of the RG method

Another important achievement of [5] consisted in formulating a simple algorithm for improving an approximate perturbative solution by combining it with the Lie equations[8]:

> *Formulae (4) and (5) show that to obtain expressions for $\bar{\alpha}$ and s valid for all values of their arguments one has only to define $\bar{\alpha}(\xi,y,\alpha)$ and $s(\xi,y,\alpha)$ in the vicinity of $\xi = 1$. This can be done by means of the usual perturbation theory.*

In the next publication [6] this algorithm was used effectively to analyse the ultraviolet and infrared (IR) asymptotic behaviour in QED in transverse gauge. The one-loop and two-loop UV asymptotics

$$\bar{\alpha}^{(1)}_{RG}(x,0,\alpha) = \frac{\alpha}{1 - \frac{\alpha}{3\pi}\cdot \ln x} \quad (7)$$

[7]See equation (11) below.

[8]Modern notation is used in this quotation from [5]

and

$$\bar{\alpha}^{(2)}_{RG}(x,0,\alpha) = \frac{\alpha}{1 - \frac{\alpha}{3\pi}\ln x + \frac{3\alpha}{4\pi}\ln(1 - \frac{\alpha}{3\pi}\ln x)} \quad (8)$$

of the photon propagator as well as the IR asymptotics

$$s(x,y;\alpha) \approx (p^2/m^2 - 1)^{-3\alpha/2\pi}$$

of the electron propagator were obtained. At that time these expressions had already been known only at the one–loop level. It should be noted that in the mid 50s the problem of the UV behaviour in local QFT was quite urgent. Substantial progress in the analysis of QED at small distances was made by Landau and his collaborators [13] by solving an approximate version of the Schwinger–Dyson equations including only the two-point functions ("dressed" propagators) $\Delta_i(...,\alpha)$ and the three-point function $\Gamma(...,\alpha)$, that is, the so-called "three-gamma equations". The authors obtained asymptotic expressions for QED propagators and 3-vertex, in which (using modern language) the leading UV logarithms were summed [9]. However, Landau's approach did not provide a prescription for constructing subsequent approximations.

An answer to this question was given only within the new renormalization group method. The simplest UV asymptotics of QED propagators obtained in our paper [6], for example, expression (7), agreed precisely with the results of Landau's group.

Within the renormalization group approach these results can be obtained in just a few lines of argument. To this end, the one-loop approximation

$$\bar{\alpha}^{(1)}_{PTh}(x,0;\alpha) = \alpha + \frac{\alpha^2}{3\pi}\ell + \ldots\ , \quad \ell = \ln x$$

of perturbation theory should be substituted into the right-hand side of the first equation in (6) to compute the generator $\beta(0,\alpha) = \psi(\alpha) = \alpha^2/3\pi$, followed by an elementary integration.

Moreover, starting from the two-loop expression

$$\bar{\alpha}^{(2)}_{PTh}(x,0;\alpha) = \alpha + \frac{\alpha^2}{3\pi}\ell + \frac{\alpha^2}{\pi^2}\left(\frac{\ell^2}{9} + \frac{\ell}{4}\right) + \ldots\ ,$$

we arrive at the second renormalization group approximation (8) corresponding to the summation of the next-to-leading UV logarithms. This

[9]These results were obtained under arbitrary covariant gauge.

demonstrates that the RG method is a regular procedure, within which it is quite easy to estimate the range of applicability of the results.

The second-order renormalization group solution (8) for the effective coupling constant first obtained in [6] contains the nontrivial log-of-log dependence which is now widely known as the two-loop approximation for the running coupling constant in quantum chromodynamics (QCD).

Before long [14] this approach was formulated for the case of QFT with two coupling constants g and h, namely, for a model of pion-nucleon interactions with self-interaction of pions [10]. The following system of two coupled equation:

$$\bar{g}^2\left(x,y;g^2,h\right) = \bar{g}^2\left(\frac{x}{t},\frac{y}{t},\bar{g}^2(t,y;g^2,h),\bar{h}\left(t,y;g^2,h\right)\right) ,$$
$$\bar{h}\left(x,y;g^2,h\right) = \bar{h}\left(\frac{x}{t},\frac{y}{t},\bar{g}^2\left(t,y;g^2,h\right),\bar{h}\left(t,y;g^2,h\right)\right) .$$

was first obtained. The corresponding system of nonlinear differential equations from [14] was used in [15] to carry out the UV analysis of the renormalizable model of pion-nucleon interactions based on one-loop perturbative computations.

In [5, 6] and [14] the renormalization group approach was thus directly connected with practical computations of the UV and IR asymptotics. Since then this technique, known as the *renormalization group method* (RGM), has become the sole means of asymptotic analysis in local QFT.

2.4 Other early RG applications

The first general theoretical application of the RG method was made in the summer of 1955 in connection with the (then topical) so-called ghost pole problem (also known as the "zero-charge trouble"). This effect, first discovered in QED [16, 17], was at first thought [17] to indicate a possible difficulty in quantum electrodynamics, and then [18, 19] as a proof of the inconsistency of the whole local QFT.

However, the renormalization group analysis of the problem carried out in [20] on the basis of (2) demonstrated that no conclusion obtained with the aid of finite-order computations within perturbation theory can be regarded as a complete proof. This corresponds precisely to the impression, one can get when comparing (7) and (8). In the mid 50s this result was very significant, for it restored the reputation of local QFT. Nevertheless, in the course of the following decade the applicability of QFT in elementary particle physics remained doubtful in the eyes of many theoreticians.

In the general case of arbitrary covariant gauge the renormalization group analysis in QED was carried out in [21]. Here the problem is that the charge renormalization is connected only with the transverse part of the photon propagator. Therefore, under nontransverse (for example, Feynman) gauge the Dyson transformation has a more complex form. Logunov proposed to solve this problem by considering the gauge parameter is another coupling constant.

Ovsyannikov [22] found the general solution of the functional RG equations taking mass into account:

$$\Phi(y,\alpha) = \Phi\left(y/x,\bar{\alpha}(x,y;\alpha)\right)$$

in terms of an arbitrary function Φ of two arguments, reversible in its second argument. To solve the equations, he used the differential group equations represented as linear partial differential equations of the form[11]:

$$\left\{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \beta(y,\alpha)\frac{\partial}{\partial\alpha}\right\}\bar{\alpha}(x,y,\alpha) = 0 .$$

The results of this "period of pioneers" were collected in the chapter "Renormalization group" in the monograph [23], the first edition of which appeared in 1957 [12], and very quickly acquired the status of "quantum-field folklore".

3 FURTHER RG DEVELOPMENT

3.1 Quantum field theory

The next decade brought a calm period, during which there was practically no substantial progress in the renormalization group method. An important exception, which ought to be mentioned here, was Weinberg's

[10]It is essential that for the Yukawa PS πN-interaction $\sim g$ to be renormalizable, it is necessary to add to the Lagrangian a quartic pion self-interaction term with an independent, that is, a second, coupling constant h. At that time this was not fully recognized: compare the given system with equations (4.19)′-(4.21)′ in [12], and the discussion in [18].

[11]Which are now known as the Callan—Symanzik equations.

[12]Shortly after that is was translated into English and French [24].

article [25], in which the idea of a *running mass* of a fermion was proposed. If considered from the viewpoint of [21], this idea can be formulated as follows:

> *any parameter of the Lagrangian can be treated as a (generalized) coupling constant, and so can be included into the renormalization group formalism.*

However, the results obtained in the framework of this approach turned out the same as before. For example, the most familiar expression for the fermion running mass

$$\bar{m}(x,\alpha) = m_\mu \left(\frac{\alpha}{\bar{\alpha}(x,\alpha)}\right)^\nu ,$$

in which the leading UV logarithms are summed, was known for the electron mass in QED (with $\nu = 9/4$) since the mid 50s (see [13] и [6]).

New possibilities for applying the RG method were discovered when the technique of operator expansion at small distances (on the light cone) appeared. The idea of this approach stems from the fact that the RG transform, regarded as a Dyson transformation of the renormalized vertex function, involves the simultaneous scaling of all its invariant arguments (normally, the squares of the momenta) of this function. The expansion on the light cone, so to say, "separates the arguments", as a result of which it becomes possible to study the physical UV asymptotic behaviour by means of the expansion coefficients (when some momenta are fixed and lie on mass shell). The argument–separation method for functions of several variables proposed by Wilson makes it possible to study a number of cases important from the physical point of view.

The end of the calm period can be marked well enough by the year 1971, when the renormalization group method was applied in the quantum theory of non-Abelian gauge fields, in which the famous effect of asymptotic freedom was discovered [27].

The renormalization group expression

$$\bar{\alpha}_s^{(1)} = \frac{\alpha_s}{1+\beta_1 \ln x} ,$$

for the effective coupling constants $\bar{\alpha}_s$ in QCD, computed in the one-loop approximation, exhibits a remarkable UV asymptotic behaviour thanks to β_1 being positive. This expression implies, in contrast to Eq. (7), that the effective QCD constant decreases as x increases and tends to zero in the UV

limit. This discovery, which has become technically possible only because of the RG method use, is the most important physical result obtained with the aid of the RG approach in particle physics.

3.2 Spin lattice

At the same time Wilson [28] succeeded in transplanting the RG philosophy from relativistic QFT to another branch of modern theoretical physics, namely, the theory of phase transitions in spin systems on a lattice. This new version of the RG was based on Kadanoff's idea[29] of joininig in "blocks" of few neighbouring spins with appropriate change (renormalization) of the coupling constant.

To realize this idea, it is necessary to average the spins in each block. This operation reducing the number of degrees of freedom and simplifying the system under consideration, preserves all its long-distance properties under a suitable renormalization of the coupling constant. Along with this, the above procedure gives rise to a new theoretical model of the original physical system.

In order that the system obtained by averaging be similar to the original one, one must also discard some terms of the new effective Hamiltonian which turns out to be unimportant in the description of infrared properties. As a result of this *Kadanoff – Wilson decimation*, we arrive at a new model system characterized by new values of the elementary scale and coupling constant. By iterating this operation, one can construct a discrete ordered set of models. From the physical point of view the passage from one model to some other one is an irreversible approximate operation. Two passages of that sort applied in sequence are equivalent to one, which gives rise to a group structure in the set of models. However, in this case the renormalization group is an approximative and is realized as a semigroup.

This construction, obviously in no way connected with UV properties, was much clearer from the general physical point of view and could therefore be readily understood by many theoreticians. Because of this, in the seventies the concept of the renormalization group and its algorithmic structure were rather quickly and successfully carried over to new branches of theoretical physics such as polymer physics [30], the theory of noncoherent transfer [31], and so on.

Apart from constructions analogous to those of Kadanoff and Wilson, in a number of cases the connection with the original quantum field RG was established.

3.3 Turbulence

This has been done with help of the functional integral representation. For example, the classic Kolmogorov–type turbulence problem was connected with the RG approach by the following steps [32]:

1. Define the generating functional for correlation functions.
2. Write for this functional the path integral representation.
3. By a change of functional integration variable establish an equivalence of the given classical statistical system with some QFT model.
4. Construct the Schwinger–Dyson equations for this equivalent QFT.
5. Apply the Feynman diagram technique and perform the finite renormalization procedure.
6. Write down the standard RG equations and use them to find fixed point and scaling behavior.

The physics of renormalization transformation in the turbulence problem is related to a change of UV cutoff in the wave-number variable.

3.4 Ways of the RG expanding

As we can see, in different branches of physics the renormalization group developed in two directions:

- The construction of a set of models for the physical problem at hand by direct analogy with the Kadanoff – Wilson construction (averaging over certain degrees of freedom) — in polymer physics, noncoherent transfer theory, percolation theory, and others;
- The search for an exact RG symmetry in the theory directly or by proving its equivalence to some QFT: for example, in turbulence theory [32, 33], turbulence in plasma [35], phase transition physics (based on a model of a continuous spin field).

What is the nature of the symmetry underlying the renormalization group?

a) In QFT the RG symmetry is an exact symmetry of a solution described in terms of the fundamental notions of the theory and some boundary value(s).

b) In turbulence and some other branches of physics it is a symmetry of a solution of an equivalent quantum field model.

c) In spin lattice theory, polymer theory, noncoherent transfer theory, percolation theory, and so on (in which the blocking concept of Kadanoff and Wilson is applied) the RG transformation involves transitions between various auxiliary models (constructed especially for this purpose). To formulate RG, it is necessary to construct an ordered set $\mathcal{M}$ of models M_i. The RG transformation connecting various models has the form

$$R(n)M_i = M_{ni} \ .$$

In this case the RG symmetry can thus be realized only on the whole set $\mathcal{M}$.

There is also a purely mathematical difference between the aforesaid realizations of the renormalization group. In field theory the RG is a continuous symmetry group. On the contrary, in the theory of critical phenomena, polymers, and other similar cases (when an averaging operation is necessary) we have an approximate discrete semigroup. It must be pointed out that in dynamical chaos theory, in which renormalization group ideas and terminology can sometimes be applied too, functional iterations do not constitute a group at all, in general. An entirely different terminology is sometimes adopted in the above–mentioned domains of theoretical physics. Expressions such as "the real–space renormalization group", "the Wilson RG", "the Monte–Carlo RG", or "the chaos RG" are used.

Nevertheless, the affirmative answer to the question

"Are there distinct renormalization groups?"

implies no more than what has just been said about the differences between cases a) and b) on the one hand and c) on the other.

3.5 Two faces of the renormalization group in QFT

As has been mentioned above, invariance under RG transformations, that is, renorm-group invariance, is a very important notion in renormalized quantum field theory. It means that all physical results are independent of the choice of the renormalization scheme and the subtraction point. The latter corresponds to a symmetry whose presence is embodied in the renormalization group. In QFT the RG transformations can be considered in two different ways.

The existence of virtual states and virtual processes is a characteristic feature of quantum field theory. For example, virtual transformations of a photon into an electron–positron pair and vice versa can take place in QED. This vacuum polarization process gives rise to the notion of effective charge. In the classical theory of electromagnetism a test electric charge placed in a polarizable medium attracts nearby charges of opposite sign, which leads to partial screening of the test charge. In QED the vacuum, that is, the very space between the particles, serves as a polarizable medium. The electron charge is screened by the vacuum fluctuations of the electromagnetic field. Dirac was the first to demonstrate in 1933 [37] that the electron charge in momentum representation depends on Q^2 according to the formula

$$e(Q^2,\Lambda^2) = e_0\left\{1+\frac{\alpha_0}{6\pi}\ln\frac{Q^2}{\Lambda^2}+...\right\}, \qquad (9)$$

where $e_0 = \sqrt{\alpha_0}$ is the bare charge and Λ is the cutoff momentum.

The first attempt to formulate renormalization group ideas in this context was undertaken by Stueckelberg and Petermann [11]. In their pioneering work the RG transformations were introduced somewhat formally being related to the procedure for divergences eliminating, the result of which contains a finite arbitrariness. It is this "degree of freedom" in the finite renormalized expressions that was used in our papers [5], [6]. Roughly speaking, our result corresponds to a parameter change ($\Lambda \to \mu$) describing the degree of freedom, so that the "finite representation"

$$e(Q^2,\mu^2) = e_\mu\left\{1+\frac{\alpha_\mu}{6\pi}\ln\frac{Q^2}{\mu^2}+...\right\}, \qquad (10)$$

is obtained in place of Eq. (9), $e_\mu = \sqrt{\alpha_\mu}$ being the physical charge of an electron measured at $Q^2 = \mu^2$. Here the renormalization group symmetry can therefore be expressed in terms of the momentum transfer scale, that is renormalization point μ.

Gell-Mann and Low used another representation. In their article the small distance behaviour in QED is analysed in terms of Λ, the momentum transfer cutoff. In this approach the electron charge could be represented by the expression (9) that is singular in the limit $\Lambda \to \infty$.

We shall present a simple physical picture (which can be derived from Wilson's Nobel lecture) to illustrate this approach. Imagine an electron of finite dimensions distributed in a small volume of radius $R_\Lambda = \hbar/c\Lambda$ with $\ln(\Lambda^2/m_e^2) \gg 1$. We assume that the electric charge of such a nonlocal electron depends on the cutoff momentum so that this dependence accumulates the effects of vacuum polarization taking place at distances

not exceeding R_Λ from the centre of the electron. We thus obtain a set of models with a nonlocal electron of charges $e_i = \sqrt{\alpha_i}$ corresponding to different values of the cutoff parameter Λ_i.

Here α_i depends on R_i as the effects of vacuum polarization in the excluded volume $r < R_i$ must be subtracted. In this picture the RG transformation can be thought of as the passage from one radius $R_i = \hbar/c\Lambda_i$ to another R_j accompanied by a simultaneous change of the effective electron charge

$$e_i \equiv e(\Lambda_i) \to e_j = e_i\left(1+\frac{\alpha_i}{6\pi}\ln\frac{\Lambda_j^2}{\Lambda_i^2}+...\right),$$

which can be determined with the aid of (9). In other words, here the RG plays the role of a symmetry of operations in the space of nonlocal QED models constructed so that each model is equivalent to the true local theory at long distances. It is right to say that in these two approaches the renormalization groups differ from one another.

3.6 Functional self–similarity

An attempt to analyse the relationship between these formulations on a simple common basis was undertaken about ten years ago [38]. In this paper (see also our surveys [39, 40, 41]) it was demonstrated that all the above–mentioned realizations of the renormalization group could be considered in a unified manner by using only some common notions of mathematical physics.

In the general case it proves convenient to discuss the symmetry underlying the renormalization group with the aid of a continuous one–parameter transformation of two variables x and g written as

$$R_t : \{x \to x' = x/t,\ g \to g' = \bar{g}(t,g)\}\ . \qquad (11)$$

Here x is the basic variable subject to a scaling transformation, while g is a physical quantity undergoing a more complicated functional transformation. To form a group, the transform R_t must satisfy the multiplication law

$$R_t R_\tau = R_{t\tau}\ ,$$

which leads to the following functional equation for $\bar{g}$:

$$\bar{g}(x,g) = \bar{g}\left(x/t, \bar{g}(t,g)\right)\ . \qquad (12)$$

This equation has the same form as the functional equation (3) for the effective coupling in QFT in the massless case, that is, when $y = 0$. It is also fully equivalent to the Gell-Mann–Low functional equation (1). It is therefore clear that the contents of RG equation can easily be reduced to the group multiplication law.

In physical problems the second argument g of the transformation is usually the boundary value of a dynamical function, that is, a solution of the problem under investigation. This means that the symmetry underlying the RG approach is a symmetry of the solution (not of the equation) describing the physical system at hand, involving a transformation of the parameters entering the boundary conditions.

As an illustration, we consider a solution $f(x)$ defined by the boundary condition $f(x_0) = f_0$. Among the arguments of f we also include the boundary parameters: $f(x) = f(x, x_0, f_0)$. In this case the RG transformation corresponds to altering the parametrization of the solution, say, from $\{x_0, f_0\}$ to $\{x_1, f_1\}$. In other words, the value of x for which the boundary condition is given should not be equal to x_0 (that is, another point x_i can also be used). We now assume that f can be represented as $F(x/x_0, f_0)$ with $F(1,\gamma) = \gamma$. The equality

$$F(x/x_0, f_0) = F(x/x_i, f_i)$$

reflects the fact that the function itself is not modified under that change of the boundary condition[13]. Setting $f_1 = F(x_1/x_0, f_0)$, $\xi = x/x_0$ and $t = x_1/x_0$, we get $F(\xi, f_0) = F(\xi/t, F(t, f_0))$, which is equivalent to (12). The group operation can now be defined by analogy with Eq. (11):

$$R_t \ : \ \{ \ \xi \to \xi/t \ , \quad f_0 \to f_1 = F(t, f_0) \ \} \ .$$

Therefore, in the simplest case the RG can be defined as a continuous one–parameter group of transformations of a solution of the physical problem fixed by a boundary condition. The RG transformation affects the parameters of the boundary condition and corresponds to changing the way in which this condition is introduced for *one and the same solution.*

Special cases of such transformations have been known for a long time. If we assume that $F = \bar{g}$ is a factored function of its arguments, then from Eq.(12) it follows that $F(z, f) = fz^k$, with k being a number. In this particular case the group transform takes the form

$$P_t \ : \ \{ \ z \to z' = z/t \ , \quad f \to f' = ft^k \ \} \ ,$$

[13]As, for example, in the case $F(x,\gamma) = \Phi(\ln x + \gamma)$.

that is known in mathematical physics as a power *self-similarity transformation.* More general case R_t with functional transformation law can be characterized as a *functional self-similarity* (FSS) transformation [38].

4 CONCLUSION

We can now answer the question concerning the physical meaning of the symmetry that underlies functional self-similarity and the renormalization group. Consider the case when the RG is equivalent to FSS. As we have already mentioned, it is not a symmetry of the physical system or the equations of the problem at hand, but a symmetry of a solution considered as a function of the essential physical variables and suitable boundary conditions. A symmetry like that can be defined, in particular, as the invariance of a physical quantity described by this solution with respect to the way in which the boundary conditions are inposed. Changing this way constitutes a group operation in the sense that the group property can be considered as the transitivity property of such changes.

Homogeneity is an important feature of the physical systems under consideration. However, homogeneity can be violated in a discrete manner. Let us imagine that such a discrete inhomogeneity is connected with a certain value of x, say, $x = y$. In this case the RG transformation with canonical parameter t will have the form:

$$R_t \ : \ \{ \ x' = x/t \ , \quad y' = y/t \ , \quad g' = \bar{g}(t, y; g) \ \} \ .$$

The group multiplication law yields precisely the functional equation (3).

The symmetry connected with functional self-similarity is a very simple and frequently encountered property of physical phenomena. It can easily be "discovered" in many very different problems of theoretical physics: in classical mechanics, transfer theory, classical hydrodynamics, and so on [42, 40, 41, 43].

Recently, interesting attempts have been made [45, 46] to use the RG concept in classical mathematical physics, in particular, to solve nonlinear differential equations. These articles discuss the possibility of establishing a regular method for finding a special class of symmetries of the equations in modern mathematical physics, namely, RG-type symmetries. The latter are defined as solution symmetries with respect to transformations involving parameters that enter into the solutions through the equations as well as through the boundary conditions in addition to (or even rather than) the natural variables of the problem present in the equations (see [47, 48]).

As is well known, the aim of modern group analysis [49, 50], which goes back to works of S. Lie[51], is to find symmetries of differential equations (DE). This approach does not include a similar problem of studying the symmetries of solutions of these equations. Beside the main direction of both the classical and modern analysis, there also remains the study of solution symmetries with respect to transformations involving not only the variables present in the equations, but also parameters appearing in the solutions, including the boundary conditions.

From the aforesaid it is clear that the symmetries which attracted attention in the 50s in connection with the discovery of the RG in QFT were those involving the parameters of the system in the group transformations. It is natural to refer to these symmetries related to functional self-similarity as the *RG-type symmetries.* As we have already mentioned, they are inherent in many problems of mathematical and theoretical physics. It is therefore important to establish, on the basis of modern group analysis, a regular method for finding RG symmetries for various classes of mathematical problems, including those whose formulation goes beyond the scope of systems of a finite number of partial differential equations.

The timeliness of the search for RG symmetries is connected with the effectiveness of the RG method, which makes it possible to improve the properties of approximate solutions of problems possessing the symmetry and, in particular, to reconstruct the correct structure of the behaviour of the solution in a neighbourhood of a singularity, which is, as a rule, disturbed by the approximation.

In problems admitting description in terms of DE's a regular algorithm for finding RG-type symmetries can be constructed by combining the group analysis [48, 52] with Ambartsumyan invariant embedding method [53]. In those cases when the embedding of the boundary–value problem for a DE leads to an integral formulation, it is required that the algorithms of group analysis should be extended to integro–differential systems of equations. Taking into account that recently some progress has also been made [54, 55, 56] in extending the range of applicability of the established methods of group analysis, one can say that the above combination turns out to be constructive enough also for integro–differential equations. We recall that the first embedding with a physical end in view was realized for the integral equation of radiative transfer [53].

At the same time, the embedding of the Cauchy problem for systems of

ordinary differential equations brings us back to the origins of the theory of such equations. This is because it can be realized within the framework of the well–known theorem on the existence of derivatives with respect to the initial values of the solutions of the system. Here it proves fruitful to treat the parameters (such as a coupling constant) as new variables introduced into the group transformations and/or the embedding procedure.

The differential formulation of RG symmetries employs an infinitesimal operator (tangent vector field), which, in general, combines the symmetry of the original problem with a symmetry (approximate or exact) of its solution taking the boundary conditions into account. Algorithmically, the invariant embedding procedure contains the operation of including these data among the variables taking part in group transformations. Here the object of group analysis is the system of equations consisting of the initial system and the embedding equations corresponding to it and to the boundary–value problem. The latter can be constructed on the basis of both the original equations and the boundary conditions. From the viewpoint of group analysis, combining the original system with the embedding equations changes the differential manifold (as a rule, quite substantially).

The symmetry group of the combined system can be found by the usual methods of Lie analysis and its modern modifications with the aid of the solution of the determining equation for the coordinates of the infinitesimal operator corresponding to the condition that ensures the invariance of this new manifold. As a matter of fact, the RG can be obtained (see Refs.[57,58]) by a suitable restriction of the resulting group to a solution, the representation of which can be quite diverse: as an exact integral or an algebraic expression, as a final portion of a perturbation series or another approximation formula, as a functional integral, and so on.

The author would like to express his gratitude to Drs. B.V. Medvedev and V.V. Pustovalov for helpful remarks.

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Received by Publishing Department
on February 6, 1996.