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COMMUTING CONSERVED QUANTITIES IN NONLINEAR REALIZATIONS OF W_3

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The KdV and Boussinesq equations, their modified versions, their higher order hierarchies, etc., are the most well studied $1+1$ dimensional integrable equations in the context of (W)-string and 2D (W)-gravity theories [1-5]. It is fairly well established that the second Hamiltonian structure for these equations are very intimately connected to the spin-2 Virasoro algebra W_2 (i.e. conformal symmetry) and its higher spin generalization W_3 algebra [4,5]. The complete integrability of these equations stems from the existence of infinite number of commuting conserved quantities. To gain an insight into these quantities has been an outstanding problem and many authors have explored their existence and properties from various points of view at the classical as well as at the quantum level (see, e.g.,[6,7] and references therein).

It is very essential to understand the geometry underlying $1+1$ dimensional integrable equations associated with W-type symmetries as it might shed light on 2D gravity theories. We studied the W_3 algebra of Zamolodchikov as well as its superextension and explained the geometrical origin for the associated Boussinesq equations [8] as well as super Boussinesq equations [9] in the framework of the universal geometric approach of nonlinear realization (NLR) method [10]. Under NLR scheme, the geometrical origin for the zero curvature representation, Hamiltonian, Miura maps, Drinfeld-Sokolov type of Lax-pair formulation, etc., emerge in a transparent manner. As a step in the direction of integrability, it is an interesting venture to understand the involuting conserved quantities in the framework of NLR method as it will provide a geometrical interpretation for the integrability criteria itself.

The main objective of the present paper is to provide a geometrical origin for the involuting conserved quantities for the Boussinesq equations in the framework of the coset space construction (group realizations) on homogeneous spaces [10]. The conserved quantities are derived by exploiting the inverse Higgs-covariant reduction constraints [11,12] and the Maurer-Cartan equation for the one-differential Cartan form in terms of which the curvature tensor, torsion, complex structure, etc., of the coset manifold are expressed (see, e.g., Ref.[12]). Besides establishing the complete integrability of the above equation, the derivation of the commuting conserved quantities under NLR scheme also provides a way to obtain higher order Boussinesq hierarchies. For instance, we derive the Boussinesq and its first hierarchy equation under the universal geometric approach of NLR by using the first nontrivial conserved quantity. The latter equation has recently been shown to be the Ward-identity for the W_3- gravity [2]. It turns out that the commuting quantities correspond to the translation generators on the coset manifold and their involution properties are equivalent to the linear independence of the coördinate directions on the coset manifold. The application of inverse Higgs-covariant reduction constraints on the infinite dimensional coset manifold singles out a 2D geodesic surface. The embedding conditions on this 2D surface corresponds to the Boussinesq equation [8]. In a similar manner, the next equation in the Boussinesq hierarchies can be geometrically interpreted as the embedding conditions on a 2D geodesic surface that intersects the previous one in a linearly independent way along the space axis.

The W_3 algebra is a nonlinear algebra. One of the key ingredients in the application of the NLR method to the classical (centrally extended) W_3 algebra

$$\begin{aligned}
[L_n, L_m] &= (n-m)\, L_{n+m} + \frac{c}{12}(n^3-n)\, \delta_{n+m,0}, \\
[L_n, W_m] &= (2n-m)\, W_{n+m}, \\
[W_n, W_m] &= 16(n-m)\, \Lambda_{n+m} - \frac{8}{3}(n-m)(n^2+m^2-\frac{nm}{2}-4)\, L_{n+m} \\
&\quad -\frac{c}{9}(n^3-n)(n^2-4)\, \delta_{n+m,0},
\end{aligned} \tag{1}$$

is to obtain a linear W_3^∞ algebra by treating all the higher spin composite generators as independent generators. For instance, the spin-4 generator $\Lambda_n = -\frac{8}{c}\sum_m L_{n-m}L_m$ and all other higher spin J_n^h $(h \geq 5)$ generators emerging due to commutation relations of $W's$ and/or $L's$ with the higher spin composite generators are to be treated as independent. Unlike the nonlinear algebra (1), the ensuing algebra

$$W_3^\infty = \Big\{L_n, W_n, \Lambda_n, J_n^h(h\geq 5)....\Big\}, \tag{2}$$

is an infinite dimensional linear algebra [13,8] where we can apply the whole arsenal of the standard techniques of coset space construction (or group realizations) on homogeneous spaces [10].

The most important subalgebra of W_3^∞ for our purposes is the one in which the Laurent indices of the generators with spin h vary from $-(h-1)$ to ∞. This subalgebra is special in some sense [13] because the explicit central charge term drops out from all the commutators. However, it is implicitly present in the commutators between the basic generators $(W's, L's)$ and the higher spin composite generators. We shall concentrate on this truncated version of W_3^∞ ("contact" W_3^∞) for the nonlinear realization method. Another subalgebra of our interest is the wedge algebra $W_{3\wedge}^\infty$ consisting of the generators with spin h where the Laurent indices vary from $-(h-1)$ to $+(h-1)$. The higher spin composite generators form a closed algebra in their own right. This can be readily checked by taking a single contraction in the OPEs. Thus, the higher spin composite generators form an ideal in this subalgebra and, therefore, the factor algebra generated by these ideals is the $sl(3,R) = W_{3\wedge}^\infty/\{\Lambda_n^4, J_n^h(h\geq 5)...\} \equiv \{W_{\pm 2}, W_{\pm 1}, W_0, L_{\pm 1}, L_0\}$.

It is a rather difficult task to obtain all the possible subalgebras in this infinite dimensional linear algebra W_3^∞. However, the stability subalgebra $\mathcal{H}$ of our interests that contains the maximum number of generators is [8]

$$\mathcal{H} = \{W_{-1}+2L_{-1}, W_0, W_1, W_2, L_1, L_0, \Lambda_n(n\geq -3), J_n^h(h\geq 5, n\geq -h+1)....\}. \tag{3}$$

With this stability subalgebra, the element g of the coset space can be parametrized in terms of 2D coordinates x , t and infinite set of 2D coset fields as

$$g = e^{tW_{-2}}\, e^{xL_{-1}}\, e^{\psi_3 L_3} \left(\Pi_{n\geq 4}\, e^{\psi_n L_n}\, e^{\xi_n W_n}\right) e^{uL_2}\, e^{vW_3}, \tag{4}$$

where an infinite tower of parameter fields $(u, v, \psi_3, \psi_4, \psi_5........)$ are the embedding fields in the infinite dimensional coset manifold. The commutativity of W_{-2} and L_{-1} implies that the x and t directions are linearly independent and, therefore, a point on the coset manifold can be parameterized by these coordinates. Furthermore, all the 2D fields $(u, v, \psi_3, \psi_4, \psi_5........)$ are any arbitrary functions of these coordinates. At this stage, there is no dynamics involved at the coset manifold and the coordinates and the coset fields are treated on an equal footing.

The basic geometrical object in NLR method is the one-differential Cartan form $\Omega = g^{-1}dg$ which can be decomposed as a sum over all the spin-h generators of the "contact" W_3^∞ with indices varying from $-(h-1)$ to ∞ as given below

$$\Omega = g^{-1}dg \equiv \sum_{n=-1}^{\infty} \omega_n L_n + \sum_{n=-2}^{\infty} \theta_n W_n + \text{higher spin contributions}. \tag{5}$$

By definition, this one-differential form satisfies the Maurer-Cartan equation

$$d^{ext}\Omega + \Omega\wedge\Omega = 0. \tag{6}$$

As higher spin composite fields form an ideal, it is essential to know only some of the lower order forms to obtain the dynamical equations on the coset manifold. However, in order to derive the conserved quantities for the above dynamical equations, we require higher order forms as well. These forms are

$$\begin{aligned}
\omega_{-1} &= dx, \quad \omega_0 = 0, \quad \omega_1 = -3udx + 160vdt, \quad \omega_2 = du - 4\psi_3 dx + 320\xi_4 dt, \\
\omega_3 &= d\psi_3 + (\frac{3}{2}u^2 - 5\psi_4)dx + (560\xi_5 - 240uv)dt, \\
\omega_4 &= d\psi_4 - 6\psi_5 dx + (896\xi_6 - 192v\psi_3 - 768u\xi_4)dt, \\
\omega_5 &= d\psi_5 + ud\psi_3 + (\frac{1}{2}u^3 - 5u\psi_4 + 2\psi_3^2 - 40v^2 - 7\psi_6)dx \\
&+ (192u^2v - 336u\xi_5 - 704\psi_3\xi_4 - 160v\psi_4 + 1344\xi_7)dt, \\
\omega_6 &= d\psi_6 + 2ud\psi_4 + (8\psi_3\psi_4 - 12u\psi_5 - 8\psi_7)dx \\
&+ (1920\xi_8 + 768u\xi_6 + 768u^2\xi_4 - 640\psi_4\xi_4 - 1664\psi_3\xi_5)dt, \\
\omega_7 &= d\psi_7 + 3ud\psi_5 - 40vd\xi_4 - \psi_4 d\psi_3 + \frac{3}{2}u^2 d\psi_3 + \Big[280v\xi_5 + 6u\psi_3^2 \\
&- 120\xi_4^2 - 60uv^2 + 12\psi_3\psi_5 - 21u\psi_6 + \frac{3}{8}u^4 + \frac{5}{2}\psi_4^2 - \frac{15}{2}u^2\psi_4 - 9\psi_8\Big]dx \\
&+ 320\Big[\frac{33}{4}\xi_9 + 9u\xi_7 - 51\psi_5\xi_4 - 5\psi_4\xi_5 - 10\psi_3\xi_6 + u^3v + 3u\psi_3\xi_4 \\
&- \frac{5}{2}v\psi_3^2 - \frac{5}{2}uv\psi_4 + \frac{10}{3}v^3 + \frac{21}{8}u^2\xi_5 + \frac{7}{4}v\psi_6\Big]dt, \\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{7}$$

$$\theta_{-2} = dt, \quad \theta_{-1} = 0, \quad \theta_0 = -6udt, \quad \theta_1 = -8\psi_3 dt,$$

$$
\begin{aligned}
\theta_2 &= -5\, v dx + \Big[12u^2 - 10\psi_4\Big] dt, \quad \theta_3 = dv - 6\, \xi_4 dx + \Big[24u\psi_3 - 12\psi_5\Big] dt, \\
\theta_4 &= d\xi_4 + (3uv - 7\xi_5)dx + \Big[20\psi_3^2 + 20u\psi_4 - 14\psi_6 - 8u^3 - 80v^2\Big] dt, \\
\theta_5 &= d\xi_5 + vdu + (6u\xi_4 - 4v\psi_3 - 8\xi_6)dx \\
&+ \Big[56\psi_3\psi_4 + 320v\xi_4 + 12u\psi_5 - 12u^2\psi_3 - 16\psi_7\Big] dt, \\
\theta_6 &= d\xi_6 + 3vd\psi_3 + (\tfrac{9}{2}u^2 v - 15v\psi_4 - 9\xi_7)dx \\
&+ \Big[72\psi_3\psi_5 + 1680v\xi_5 + 30\psi_4^2 - 360v^2 u - 18\psi_8\Big] dt, \\
\theta_7 &= d\xi_7 + 5vd\psi_4 + u\, d\xi_5 + 2\xi_4\, d\psi_3 + \Big[3u^2\xi_4 + 4\psi_3\xi_5 - 10\xi_4\psi_4 - 8u\xi_6 \\
&- 30v\psi_5 - 10\xi_8\Big] dx + \Big[56u\psi_3\psi_4 + 88\psi_3\psi_6 + 80\psi_4\psi_5 + 4480v\xi_6 + 6u^2\psi_5 \\
&- \frac{40}{3}\psi_3^2 - 480v^2\psi_3 - 4u^3\psi_3 - 3840uv\xi_4 - 320\xi_4\xi_5 - 16u\psi_7 - 20\psi_9\Big] dt, \\
\theta_8 &= d\xi_8 + 7uvd\psi_3 + 7v\, d\xi_5 + 2u\, d\xi_6 + \xi_5\, d\psi_3 + 4\xi_4 d\psi_4 + \Big[\frac{7}{2}u^3 v - 5\psi_4\xi_5 \\
&+ 42\xi_4\psi_5 + 8\xi_6\psi_3 + 14v\psi_3^2 - 18u\xi_7 - 49v\psi_6 - 35uv\psi_4 - 11\xi_9 - \frac{280}{3}v^3\Big] dx \\
&+ \Big[144u\psi_3\psi_5 + 104\psi_3\psi_7 + 80\psi_4\psi_5 + 4480v\xi_6 + 6u^2\psi_5 + 60u\psi_4^2 + 1344v\xi_7 \\
&+ 280\xi_5^2 + 42\psi_5^2 - 560v^2\psi_4 - 2352uv\xi_5 - 4928v\psi_3\xi_4 - 36u\psi_8 - 22\psi_{10}\Big] dt, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \qquad (8)
\end{aligned}
$$

As a result of the inverse Higgs-covariant reduction constraints [11,12], one obtains a relationship between higher spin goldstone fields and essential coset (goldstone) fields together with the dynamical equations of motion. Basically, in this procedure, all the components of the forms associated with the coset generators are set equal to zero as they transform homogeneously under the left action of W_3^∞ symmetry. To make this statement more transparent, it can be seen that the following inverse Higgs-covariant reduction constraints

$$
\omega_n = 0 \ \forall \ n \geq 2, \quad \theta_n = 0 \ \forall \ n \geq 3, \qquad (9)
$$

lead to the kinematical inverse Higgs relations

$$
\begin{aligned}
\xi_4 &= \frac{v'}{6}, \quad \xi_5 = \frac{\xi_4'}{7} + \frac{3}{7}u\, v, \quad \xi_6 = \frac{\xi_5'}{8} + \frac{3}{4}u\, \xi_4, \quad \xi_7 = \frac{\xi_6'}{9}, \\
\xi_8 &= \frac{1}{10}\Big[\xi_7' - 6\, u^2\, \xi_4 + 4\, \psi_3\xi_5\Big], \\
\xi_9 &= \frac{1}{11}\Big[\xi_8' - \frac{3}{2}\, u^2\, \xi_5 + 8\, \psi_3\xi_6 + 66\psi_4\xi_5 + \frac{560}{3}\, v^3\Big], \\
\xi_{10} &= \frac{1}{12}\Big[\xi_9' + 3\, u^3\, \xi_4 + 54\, \psi_6\xi_4 + 12\psi_3\xi_7 + 12\, \psi_3^2\, \xi_4 + 18u\, v\, \psi_5 - 42\, \xi_5\psi_5\Big], \\
&\ldots\ldots\ldots\ldots\ldots\ldots \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
\psi_3 &= \frac{u'}{4}, \quad \psi_4 = \frac{\psi_3'}{5} + \frac{3}{10}u^2, \quad \psi_5 = \frac{\psi_4'}{6}, \\
\psi_6 &= \frac{\psi_5'}{7} + \frac{2\, \psi_3^2}{7} - \frac{u^3}{7} - \frac{40v^2}{7}, \\
\psi_7 &= \frac{\psi_6'}{8} + \psi_3\psi_4 = \frac{d}{dx}\Big[\frac{\psi_6}{8} + \frac{\psi_3^2}{10} + \frac{u^3}{40}\Big], \\
\psi_8 &= \frac{1}{9}\Big[\psi_7' + \frac{9}{8}\, u^4 - \frac{5}{4}\, \psi_4^2 + \frac{3}{2}\, u^2\, \psi_4 - 120\, \xi_4^2 + 12\psi_3\psi_5 + 180u\, v^2\Big], \\
\psi_9 &= \frac{1}{10}\Big[\psi_8' + \frac{4}{3}\, \psi_3^3 + 3\, u^2\, \psi_5 - 720\, \xi_4\, \xi_5 + 16\psi_3\, \psi_6\Big], \\
\psi_{10} &= \frac{1}{11}\Big[\psi_9' + 20\psi_3\psi_7 + 2\psi_3^2\psi_4 + 384u\, \xi_4^2 + 252\xi_5^2 - 3\psi_5^2 \\
&-1536\xi_4\xi_6 - 60u\psi_3\psi_5 - 600u^2 v^2 - 216uv\xi_5 + \frac{9}{2}u^2\psi_6 - \frac{3}{2}u^5\Big], \\
&\ldots\ldots\ldots\ldots\ldots\ldots \qquad (11)
\end{aligned}
$$

when dx projections of the Cartan forms are set equal to zero. Here and in what follows, primes stand for the derivatives w.r.t. the space coordinate x. On the other hand, using the inverse Higgs relations, we obtain the following dynamical equations when dt projections of ω_2 and θ_3 are set equal to zero

$$
\frac{\partial u}{\partial t} = -\frac{160}{3}\frac{\partial v}{\partial x}, \qquad \frac{\partial v}{\partial t} = \frac{1}{10}\frac{\partial^3 u}{\partial x^3} - \frac{24}{5}u\frac{\partial u}{\partial x}. \qquad (12)
$$

With proper rescaling, the above equations can be readily recognized as the Boussinesq equation since the coset fields u and v in (4) have the correct conformal properties of a spin-2 stress energy tensor and a spin-3 primary field [8,13]. It is well known that these equations are completely integrable i.e. there exist infinite number of involuting conserved quantities for these equations. In Ref [8], the zero curvature representation, the existence of Drinfeld-Sokolov type of Lax-pairs on the algebra $sl(3, R)$ and connection of these equations with the second Hamiltonian structure of W_3, etc., have been established under NLR scheme. Here we go a step further to derive commuting conserved quantities in the framework of NLR method.

To obtain commuting conserved quantities, we exploit inverse Higgs- covariant reduction constraints (9) and the Maurer-Cartan equations (6). The latter can be expressed for the $SL(3, R)$ generators $W_{\pm 2}, W_{\pm 1}, W_0, L_0, L_{\pm 1}$ as

$$
\begin{aligned}
& d\theta_n + \sum_{m \geq -2}^{n+1} (m - 2k)\, \theta_m \wedge \omega_k\, \delta_{m+k,n} = 0, \\
& d\omega_n + \sum_{m \geq -1, m < k} (m - k)\, \omega_m \wedge \omega_k\, \delta_{m+k,n} \\
& -\frac{8}{3} \sum_{m \geq -2, m < k} (m - k)(m^2 + k^2 - \frac{mk}{2} - 4)\, \theta_m \wedge \theta_k\, \delta_{m+k,n} = 0. \qquad (13)
\end{aligned}
$$

By exploiting the covariant reduction constraints (9), it can be readily seen that the Maurer-Cartan equations for the coset space generators reduce to $d\omega_n = 0 \; \forall n \geq 2, d\theta_n = 0 \; \forall n \geq 3$. Actually these equations lead to conservation laws beyond Boussinesq equations (12) (which can be thought to be the conservation laws realized on the essential coset fileds u and v). For instance, the following equations

$$d\omega_k = 0, \quad d\theta_n = 0, \tag{14}$$

with the help of inverse Higgs-covariant reduction constraints (9), lead to the conservation laws when $k = 5, 7..$ and $n = 4, 8...$ These are succinctly expressed as

$$\begin{aligned}
H_1 &= \frac{c}{2}\int dx u(x,t), \quad H_2 = -\frac{40c}{3}\int dx v(x,t), \quad H_4 = c\int dx (uv)(x,t),\\
H_5 &= -c\int dx\Big[\frac{(u')^2}{20} + \frac{4\,u^3}{5} + \frac{80\,v^2}{3}\Big],\\
H_7 &= c\int dx\Big[\frac{u''u''}{3200} + \frac{9\,u\,(u')^2}{400} + \frac{(v')^2}{6} + \frac{3\,u^4}{50} + 4\,u\,v^2\Big],\\
H_8 &= -c\int dx\Big[\frac{u''v''}{30} + \frac{800\,v^3}{9} - \frac{3\,v\,(u')^2}{2} - 2\,u\,u''\,v + 8\,u^3\,v\Big],\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned} \tag{15}$$

Here the subscripts of these conserved quantities stand for the naive conformal dimensions. For instance, H_1 has the dimension cm^{-1}. The explicit form of the some of the nontrivial conserved quantities, whose "time" derivative equals a total space derivative for the Boussinesq equations (12), are

$$\begin{aligned}
&\frac{\partial}{\partial t}(u\cdot v) = \frac{\partial}{\partial x}\Big[\frac{uu''}{10} - \frac{8\,u^3}{5} - \frac{80\,v^2}{3} - \frac{(u')^2}{20}\Big],\\
&\frac{\partial}{\partial t}\Big[\frac{(u')^2}{20} + \frac{4\,u^3}{5} + \frac{80\,v^2}{3}\Big] = \frac{\partial}{\partial x}\Big[\frac{16\,v\,u''}{3} - \frac{16\,u'v'}{3} - 128u^2\,v\Big],\\
&\frac{\partial}{\partial t}\Big[\frac{u''u''}{3200} + \frac{9\,u\,(u')^2}{400} + \frac{(v')^2}{6} + \frac{3\,u^4}{50} + 4\,u\,v^2\Big]\\
&= \frac{\partial}{\partial x}\Big[\frac{u'''v'}{30} - \frac{u''v''}{30} + \frac{4uu''v}{5} - \frac{12uu'v'}{5} - \frac{2v(u')^2}{5} - \frac{64u^3v}{5} - \frac{640v^3}{9}\Big],\\
&\frac{\partial}{\partial t}\Big[\frac{u''v''}{30} - \frac{3\,v\,(u')^2}{2} + \frac{800\,v^3}{9} - 2uu''v + 8\,u^3\,v\Big]\\
= &\frac{\partial}{\partial x}\Big[\frac{u''''u''}{300} - \frac{u'''u'''}{600} - \frac{8v''v''}{9} - \frac{3u''(u')^2}{20} - \frac{9u(u'')^2}{50} - \frac{192u^5}{25} - 640u^2v^2\\
&+ \frac{80u''v^2}{3} + \frac{320v''vu}{3} + \frac{160v'vu'}{3} - \frac{160v'v'u}{3} + \frac{4u''u^3}{5} + \frac{18(u')^2u^2}{5}\Big].
\end{aligned} \tag{16}$$

A few remarks are in order. First of all, at a given conformal weight, it can be checked that only one of the forms ($\theta's$ or $\omega's$) lead to the conservation law. Secondly, there are no conservation laws for the conformal dimensions $3, 6, 9, 12..(n = 0, \mod 3)$.

Thirdly, the series of values $k = 5, 7, 11, 13, 17, 19...$ and $n = 4, 8, 10, 14, 16...$ in equation (14), lead to the conserved quantities when the inverse Higgs-covariant reduction constraints are appropriately chosen. Finally, the conservation laws for H_1 and H_2 are the Boussinesq equations (12) modulo some constant scale factors.

To demonstrate the validity of the above statements, we elaborate on a few derivations. The Maurer-Cartan equations $d\omega_1 = 0, d\theta_2 = \theta_1 \wedge \omega_1$, emerging from the general equation (13), lead to the conserved quantities H_1 and H_2. We have used in the above equations the inverse Higgs-covariant constraints $\omega_2 = 0, \theta_3 = 0$, and $\omega_2 = 0, \omega_3 = 0, \omega_4 = 0, \theta_3 = 0$, respectively. Furthermore, the explicit expressions from (7) and (8) for the appropriate forms have also been used. It will be noticed that the above conserved quantities also result in when the dt components of the forms ω_2, θ_3 are set equal to zero due to covariant reduction procedure. Here too, the appropriate expressions for the inverse Higgs relations are to be used. The counterpart of $d\omega_1 = 0$ is the equation $d\theta_1 = 2\theta_0 \wedge \omega_1 - \theta_1 \wedge \omega_0 - 4\theta_2 \wedge \omega_{-1}$ where covariant reduction constraints $\omega_2 = 0, \omega_3 = 0$ have been used. It does not lead to any conservation law but implies the inverse Higgs relation $\psi_4 = \frac{\psi_3'}{5} + \frac{3}{10}u^2$. The next equation $d\omega_2 = 0$ leads to $\dot{\psi}_3 = -80\,\xi_4'$. However, it is not a new conserved quantity because the inverse Higgs relations for ψ_3 and ξ_4 are such that it reexpresses the first conserved quantity H_1. The forthcoming equations $d\theta_3 = 0$ and $d\omega_3 = 0$ do not lead to any conservation law. It is straightforward to see that $d\,\theta_3 = 0$ implies $-6\dot{\xi}_4 = \frac{\partial}{\partial x}[24u\psi_3 - 12\psi_5]$ which is a modified version of the second conservation law H_2 when we use the appropriate inverse Higgs constraints from (10) and (11). At the same conformal weight, equation $d\omega_3 = 0$ is such that either it leads to $\dot{\psi}_3 = -80\xi_4'$, which implies the first conservation law, or it can not be expressed as a total space derivative of any quantity. Thus, for conformal dimension 3, there is no conservation law and H_3 does not exist.

The next nontrivial conserved quantity emerges from the equation $d\theta_4 = 0$, i.e.,

$$\frac{\partial}{\partial t}\Big[3uv - 7\xi_5\Big] = \frac{\partial}{\partial x}\Big[20\psi_3^2 + 20u\psi_4 - 14\psi_6 - 8u^3 - 80v^2\Big]. \tag{17}$$

Due to the covariant reduction constraint, when dt projection of the form θ_5 is set equal to zero, we obtain on Boussinesq equations

$$\frac{\partial \xi_5}{\partial t} = 16\psi_7 + 12u^2\psi_3 - 12u\psi_5 - 56\psi_3\psi_4. \tag{18}$$

Using required inverse Higgs relations from (10) and (11), it can be checked that the above equation is a total derivative, i.e.,

$$\frac{\partial \xi_5}{\partial t} = \frac{\partial}{\partial x}\Big[2\psi_6 + \frac{1}{5}u^3 - 2u\psi_4 - \frac{16}{3}\psi_3^2\Big]. \tag{19}$$

Ultimately equations (17) and (19) lead to

$$\frac{\partial(u\,v)}{\partial t} = \frac{\partial}{\partial x}\Big[2u\psi_4 - \frac{11}{5}u^3 - \frac{4}{5}\psi_3^2 - \frac{80}{3}v^2\Big], \tag{20}$$

as the next conservation law. The above equation has been expressed in equation (16) in a different form because we have exploited there the inverse Higgs relations. It will be noticed that this conservation law can also be derived directly from (19) if we use the relation $\xi_5 = \frac{1}{7}(\xi_4 + 3uv)$. The equation $d\omega_4 = 0$ does not lead to any conservation law. Thus, H_4 only emerges from $d\theta_4 = 0$. The next equation in this sequence, i.e., $d\omega_5 = 0$, leads to

$$\frac{\partial}{\partial t}\Big[\frac{4}{5}\psi_3^2 + \frac{4}{5}u^3 + \frac{80}{3}v^2\Big] = \frac{\partial}{\partial x}\Big[\frac{320}{3}v\psi_4 - 160u^2v - 128\psi_3\xi_4\Big], \qquad (21)$$

as conservation law which determines the form of H_5 in (15) if we use the inverse Higgs relations. The argument for the nonexistence of H_6 is the same as that for the absence of H_3. The derivation of the higher order conserved quantities follows the same logic as argued in the earlier cases.

The commutativity of the conserved quantities in (15) (i.e., $\{H_i, H_j\} = 0$ for $i,j = 1,2,4,....$) can be established if we exploit the following second Hamiltonian structure associated with u and v fields for the classical W_3 Poisson brackets [1]

$$\begin{aligned}
\{u(x,t), u(y,t)\} &= \frac{2}{c}\Big[\frac{1}{6}\frac{\partial^3}{\partial y^3} - 2\,u(y)\,\frac{\partial}{\partial y} - \frac{\partial u}{\partial y}\Big]\,\delta(x-y),\\
\{u(x,t), v(y,t)\} &= -\frac{2}{c}\Big[3\,v(y)\,\frac{\partial}{\partial y} + \frac{\partial v}{\partial y}\Big]\,\delta(x-y),\\
\{v(x,t), v(y,t)\} &= \frac{3}{100c}\Big[-\frac{1}{48}\,\frac{\partial^5}{\partial y^5} + \frac{5}{4}\,u(y)\,\frac{\partial^3}{\partial y^3} + \frac{15}{8}\,\frac{\partial u}{\partial y}\,\frac{\partial^2}{\partial y^2}\\
&+ (\frac{9}{8}\,\frac{\partial^2 u}{\partial y^2} - 12\,u^2)\,\frac{\partial}{\partial y} + (\frac{1}{4}\frac{\partial^3 u}{\partial y^3} - 12\,u\,\frac{\partial u}{\partial y})\Big]\delta(x-y). \qquad (22)
\end{aligned}$$

This commutativity can also be understood in terms of the generators of W_3^∞. For instance, if we take the Laurent mode decompositions for the spin-2 fields u and spin-3 fields v and consider the holomorphic and antiholomorphic parts together, the contour integrations in (15) will lead to the following set of generators modulo some constant scale factors

$$\{\,L_{-1},\;\; W_{-2},\;\; \Phi_{-4},\;\; S_{-5}.....................\} \qquad (23)$$

where $\Phi = \frac{48}{c}(TW), S = \frac{1}{c}(W^2 - \frac{128}{3c}\,T^3 + \frac{4}{3}\,(\partial T)^2)...$, etc. These generators form the Cartan subalgebra in W_3^∞ as they commute among themselves. For instance, besides commutativity of L_{-1} and W_{-2}, it can be seen that $L_{-1}, W_{-2}, \Phi_{-4}$ commute with each other in the following commutation relations

$$\begin{aligned}
[L_n, \Phi_m] &= (4n - m)\,\Phi_{n+m} \;+\; 4(n^3 - n)\,W_{n+m},\\
[W_n, \Phi_m] &= 4\Big[(n+m+4)(n+m+5)(n+m+6)
\end{aligned}$$

[1]Note that there are some printing errors in Ref.[8] in these Poisson brackets.

$$\begin{aligned}
&-15(n+1)(n+2)(m - \frac{n}{3} + 4)\Big]\,\Lambda_{n+m}\\
&+(2n-m)\Big[\frac{512}{c}(T\Lambda) - \frac{192}{c}(T\partial^2 T) + \frac{48}{c}(W^2)\Big]_{n+m}\\
&-\frac{16}{3}(n^3 - n)(n^2 - 4)L_{n+m}\,. \qquad (24)
\end{aligned}$$

Similarly, S_{-5} also commutes with all the other generators in the set. It will be noticed that these generators can be found at the Laurent modes $-h+1$ in the special algebra under our consideration (where the indices vary from $-h+1$ to ∞). However, these commuting generators will not exist when $-h+1 = 0$ mod 3.

In the framework of NLR method, these generators correspond to the translation generators. The space translation generator is L_{-1} which corresponds to the first conserved quantity H_1. Rest of the conserved quantities correspond to the evolution directions on the coset space and they appear as "time" coordinates in the Boussinesq equations and their higher order hierarchies. For instance, equation (12) can be derived if we use the Poisson brackets (22) in the Hamilton equations of motion with the Hamiltonian as the conserved quantity H_2. For the derivations of the higher order Boussinesq hierarchies, one has to exploit H_4, H_5....... in the Hamilton equations of motion with the Poisson brackets (22). To establish the equivalence between conserved quantities and translation generators, one can first derive

$$\begin{aligned}
\frac{\partial u}{\partial t'} &= -\frac{1}{3}\frac{\partial^3 v}{\partial x^3} + 8\,u\,\frac{\partial v}{\partial x} + 8\,v\,\frac{\partial u}{\partial x},\\
\frac{\partial v}{\partial t'} &= \frac{1}{320}\Big(\frac{1}{5}\frac{\partial^5 u}{\partial x^5} - \frac{72}{5}\,u\,\frac{\partial^3 u}{\partial^3 x}\\
&- \frac{144}{5}\,\frac{\partial u}{\partial x}\,\frac{\partial^2 u}{\partial^2 x} + \frac{1152}{5}\,u^2\,\frac{\partial u}{\partial x} + 2560\,v\,\frac{\partial v}{\partial x}\Big), \qquad (25)
\end{aligned}$$

from the Hamiltonian $H_4 = c\int dx(u\,v)$ by using the second Hamiltonian structure (22), and can check that u and v fields form a closed set among themselves. This can be understood in the framework of NLR method *even without* quoting the appropriate stability subalgebra. The composite generators Φ_{-4}, S_{-5}.... could be taken into the coset manifold and could be linked with the new evolution parameters t', t''......, in addition to x and t which are linked with L_{-1} and W_{-2}, as follows [2]

$$g_1 = ...e^{t''S_{-5}}e^{\frac{t'}{320}\Phi_{-4}}e^{tW_{-2}}e^{xL_{-1}}e^{\psi_3 L_3}\Big(\Pi_{n\geq 4}e^{\psi_n L_n}e^{\xi_n W_n}\Big)e^{uL_2}e^{vW_3}......... \qquad (26)$$

Now any point on the coset manifold will be parametrized by coordinates x, t, t', t''... because the commutativity of the generators $L_{-1}, W_{-2}, \Phi_{-4}, S_{-5}$... ensures the linear independence of these directions. [3] The dots at the right end in (26) stand for the

[2]Even in the derivation of KdV equation under NLR scheme, there are some extra coset fields when composite generator Λ_{-3} is taken into the coset space due to correct choice of stability subalgebra. However, the extra essential field does not appear in the KdV equation [14].

[3]The factor $\frac{1}{320}$ in the second exponential has been taken for the convenience in the algebraic computations of the Cartan forms.

exponentials that contain extra coset fields associated with the composite generators. As discussed earlier in detail, only a few Cartan forms would be required for the derivation of equation (12) and (25) in the framework of NLR method. The analogue of (7) and (8), with the coset element (26) and commutators (24), are

$$
\begin{aligned}
\omega_{-1} &= dx + 2v\, dt' + ..., \quad \omega_0 = 12\xi_4 dt' + ..., \\
\omega_1 &= -3udx + 160vdt + 6\Big[7\,\xi_5 - 6u\,v\Big]dt' + ..., \\
\omega_2 &= du - 4\psi_3 dx + 320\xi_4 dt + 56\Big[2\,\xi_6 - 3\,u\,\xi_4 - v\,\psi_3\Big]dt' + ..., \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\theta_{-2} &= dt - \frac{3}{40}u\, dt' + ..., \quad \theta_{-1} = -\frac{3}{10}\psi_3 dt' + ..., \\
\theta_0 &= -6udt + \frac{3}{4}\Big[\frac{9}{10}u^2 - \psi_4\Big]dt' + .., \quad \theta_1 = -8\psi_3 dt + \frac{3}{2}\Big[\frac{9}{5}u\,\psi_3 - \psi_5\Big]dt' + .., \\
\theta_2 &= -5\,vdx + (12u^2 - 10\psi_4)dt \\
&+ \Big[\frac{69}{20}\psi_3^2 - \frac{21}{8}\psi_6 - \frac{12}{5}\,u^3 + \frac{9}{2}u\,\psi_4 - 25\,v^2\Big]dt' + ..., \\
\theta_3 &= dv - 6\,\xi_4 dx + (24u\psi_3 - 12\psi_5)dt \\
&+ \Big[\frac{147}{10}\psi_3\psi_4 + \frac{63}{10}u\,\psi_5 - \frac{21}{5}\psi_7 - \frac{189}{20}\,u^2\psi_3 - 84v\xi_4\Big]dt' + ..., \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (27)
\end{aligned}
$$

It will be noticed that the composite generators in the exponentials at the right end of (26) do not contribute to the above Cartan forms. The covariant reduction constraint on ω_2 and θ_3 leads to

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= -\,320\,\xi_4, \quad \frac{\partial v}{\partial t} = 12\,\psi_5 - 24\,u\,\psi_3, \\
\frac{\partial u}{\partial t'} &= -112\,\xi_6 + 168\,u\,\xi_4 + 56v\,\psi_3, \\
\frac{\partial v}{\partial t'} &= -\frac{147}{10}\psi_3\psi_4 - \frac{63}{10}u\,\psi_5 + \frac{21}{5}\psi_7 + \frac{189}{20}\,u^2\psi_3 + 84v\xi_4. \qquad (28)
\end{aligned}
$$

Both these sets of dynamical equations can be recognized as the Boussinesq equation (12) and the first Boussinesq hierarchy equation (25) if we exploit the inverse Higgs constraints appropriately from (10) and (11). This establishes the geometrical interpretation of commuting conserved quantities as the translation generators.

It is well known that the commuting conserved quantities for the Boussinesq equations obey Lenard recursion relations. It will be an interesting problem to provide a geometrical interpretation for these relations in the framework of NLR method. The other open question to be answered is to obtain the correct form of the stability subalgebras so that commuting composite generators could be taken into the coset manifold. It is also essential to apply NLR technique to obtain conserved quantities for the other well studied integrable equations such as : the Liouville equation [12], the super Boussinesq equations [9], the Toda field equation [13],

etc., and to provide geometrical origin for them in the framework of coset space construction. These are the issues for future investigations [15].

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