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τ DECAY AND e^+e^- ANNIHILATION AT LOW ENERGIES IN THE NONPERTURBATIVE APPROACH TO QCD

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I. Method

There are many problems of QCD requiring nonperturbative approaches. Here we use a method based on the ideas of the δ expansion and variational perturbation theory. The method leads to the so-called "floating" series, the convergence properties of which can be controlled by special parameters. In the simplest cases, there is a rigorous proof of the convergence of such an expansion [1]. The generalization of the method to the QCD case has been suggested in Ref. [2].

Within this method, the quantity under consideration, for example, the Green function can be approximated by a series different from the perturbative expansion and which can be used to go beyond the weak-coupling regime and allows one to deal with considerably lower energies than in the case of perturbation theory. The connection between the expansion parameter a and the original coupling constant g is given by the following equation

$$\lambda = \frac{g^2}{(4\pi)^2} \equiv \frac{\alpha_s}{4\pi} = \frac{1}{C}\frac{a^2}{(1-a)^3}\,. \tag{1}$$

As follows from this equation, at any values of the coupling constant g, the new expansion parameter a obeys the inequality $0 \le a < 1$. The parameter C is a positive constant which plays the role of a variational parameter. The original quantity approximated by this expansion does not depend on the auxiliary parameters C; however, any finite approximation depends on it on account of the truncation of the series. Here we fix this parameter using some further information which comes from the potential approach to meson spectroscopy.

The renormalization group analysis leads to the momentum dependence of a which is given by the transcendental equation

$$Q^2 = Q_0^2 \exp\left\{ \frac{C}{2\,b_0}\,[\,f(a) - f(a_0)\,] \right\}, \tag{2}$$

where b_0 is the first coefficient of the β function, Q_0 is a normalization point, and the function $f(a)$ has the following form

$$f(a) = \frac{2}{a^2} - \frac{6}{a} - 48\ln a - \frac{18}{11}\frac{1}{1-a} + \frac{624}{121}\ln(1-a) + \frac{5184}{121}\ln\left(1+\frac{9}{2}a\right). \tag{3}$$

An important feature of this approach is the fact that for sufficiently small α_s it reproduces the standard perturbative expansion. Therefore, all the high-energy physics described by perturbation theory is maintained by this method. However, in going to lower energies where the standard perturbation theory ceases to be valid, $\alpha_s(Q^2) \sim 1$, the expansion parameter $a(Q^2)$ remains small (moreover, this parameter has no infrared singularity), and it is possible to deal with considerably lower energies than in the case of perturbation theory.

There are several papers in which the method was described in detail, for example, Refs. [3-6]. Here, we will only consider its applications. We will consider the well-known ratio for electron-positron annihilation into hadrons

$$R_{e^+e^-} = \frac{\sigma_{tot}(\,e^+e^- \to \text{hadrons}\,)}{(\,e^+e^- \to \mu^+\mu^-)} \tag{4}$$

and also an analogous ratio for the inclusive decay rate of the τ lepton

$$R_\tau = \frac{\Gamma(\tau^- \to \text{hadrons} + \nu)}{\Gamma(\tau^- \to e\,\nu\bar{\nu})}. \quad (5)$$

II. R_τ ratio

The hadronic τ decay provides the possibility of determining the QCD coupling constant at a very low energy. To extract the value of this constant, it is necessary to estimate all the theoretical uncertainties. The starting point of the theoretical analysis is the expression (see, for example, [7])

$$R_\tau = 2\int_0^{M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left(1+\frac{2s}{M_\tau^2}\right)\tilde{R}(s), \quad (6)$$

where $M_\tau = 1.777$ GeV is the τ-lepton mass and

$$\tilde{R}(s) = \frac{\mathrm{N}}{2\pi \mathrm{i}}\left[\Pi(s+\mathrm{i}\epsilon) - \Pi(s-\mathrm{i}\epsilon)\right]. \quad (7)$$

In the framework of standard perturbation theory, the integral (6) cannot be evaluated directly since the integration region in (6) includes small values of the momentum for which perturbation theory is invalid. Instead of Eq. (6), the expression for R_τ may be rewritten, using Cauchy's theorem, as a contour integral in the complex s-plane with the contour running clockwise around the circle $|s| = M_\tau^2$. It seems that this trick allows one to avoid the problem of calculating the nonperturbative contribution which is needed if one uses Eq. (6). However, the application of Cauchy's theorem is based on specific analytic properties of $\Pi(s)$ or the Adler D function which is an analytic function in the complex q^2-plane with a cut along the positive real axis. It is clear that the approximation of the D-function by perturbation theory breaks these analytic properties. For example, the one-loop approximation for the QCD running coupling constant has a singularity at $Q^2 = \Lambda_{QCD}^2$, the existence of which prevents the application of Cauchy's theorem. Moreover, to define the running coupling constant in the timelike domain, one usually uses the dispersion relation for the D function derived on the basis of the above-mentioned analytic properties. In the framework of perturbation theory, this method gives the so-called π^2-term contribution.

In Ref. [5] it has been demonstrated that in the framework of this approach there exists a well-defined procedure for defining the running coupling in the timelike domain which does not conflict with the dispersion relation. We will use the following definitions: $\lambda^{\text{eff}} = \alpha_{\text{QCD}}/(4\pi)$ is the initial effective coupling constant in the t-channel (spacelike region) and λ_s^{eff} is the effective coupling constant in the s-channel (timelike region).

We can rewrite Eq. (6) in the form

$$R_\tau = 2\int_0^1 dx\,(1-x)^2(1+2x)\,\tilde{R}(M_\tau^2 x), \quad (8)$$

where $\tilde{R}(M_\tau^2 x)$ can be expanded as the series

$$\tilde{R}(M_\tau^2 x) = r_0\left[1 + r_1\,\lambda_s(M_\tau^2 x) + r_2\,\lambda_s^2(M_\tau^2 x) + \cdots\right] = r_0\left[1 + r_1\,\lambda_s^{\text{eff}}(M_\tau^2 x)\right]. \quad (9)$$

The effective coupling constant in the s-channel can be written as [5]

$$\lambda_s(s) = \frac{1}{2\pi\, b_0}\,\mathrm{Im}\,\phi(a_+), \quad (10)$$

where

$$\phi(a) = -4\ln a - \frac{72}{11}\frac{1}{1-a} + \frac{318}{121}\ln(1-a) + \frac{256}{363}\ln\left(1+\frac{9}{2}a\right). \quad (11)$$

and the value of a_+ obeys the following equation:

$$f(a_+) = f(a_0) + \frac{2\,b_0}{C}\left[\ln\frac{s}{Q_0^2} + \mathrm{i}\pi\right], \quad (12)$$

where a_0 is the value of the parameter a at some normalization point Q_0.

In the calculation of the integral (8) the number of active quarks is different in various regions of the integration. In the method under consideration we can ensure that both $\lambda_s(s)$ and its derivative $\lambda_s'(s)$ are continuous at various threshold points $\bar{s}_i$. The set of corresponding equations for the parameters $C^{(f)}$ and $a_0^{(f)}$ is as follows:

$$\frac{1}{b_0(f-1)}\,\mathrm{Im}\,\phi(a_+^{(f-1)}) = \frac{1}{b_0(f)}\,\mathrm{Im}\,\phi(a_+^{(f)}), \quad (13)$$
$$\frac{1}{C^{(f-1)}}\,\mathrm{Im}\left[\left(a_+^{(f-1)}\right)^2\left(1+3a_+^{(f-1)}\right)\right] = \frac{1}{C^{(f)}}\,\mathrm{Im}\left[\left(a_+^{(f)}\right)^2\left(1+3a_+^{(f)}\right)\right]$$

For the number of active flavours $f = 3$ we use the value $C^{(3)} = 4.1$ obtained from the phenomenology of meson spectroscopy [2]. Then for $f \neq 3$ we find the parameter $C^{(f)}$ using Eqs. (13).

In the alternative approach to the evaluation of R_τ using Cauchy's theorem one proceeds first by an integration by parts to convert $\tilde{R}$ into D, then represents the discontinuity as a contour integral and finally opens up the contour to the unit circle in the z plane. In this way R_τ is expressed in terms of λ^{eff} as

$$R_\tau = \frac{1}{2\pi i}\oint_{|z|=1}\frac{dz}{z}(1-z)^3(1+z)\,D(M_\tau^2 z), \quad (14)$$

where $D(M_\tau^2 z) = d_0\left[1 + d_1\,\lambda^{\text{eff}}(M_\tau^2 z)\right]$, and to order $O(a^3)$

$$\lambda^{\text{eff}}(Q^2) = \frac{1}{C}\,a^2(1+3a), \quad (15)$$

where $a = a(Q^2)$ is found from Eq. (2).

To check the consistency of our method we consider a fixed number of active quarks $f = 3$ and use $R_\tau = 3.56$ as an input. In this case, using Eqs. (8)-(12) we can find the parameter a_0 and then verify that Eqs. (8) and (14) give the same results for R_τ: $R_\tau = 3.560$ as should be the case.

To take into account the threshold effects, we have used Eqs. (13) to find the parameter $C^{(f)}$ for $f \neq 3$ and the conditions that the CKM matrix elements $V_{ud} = 0$ for $s < (m_u + m_d)^2$ and $V_{us} = 0$ for $s < (m_u + m_s)^2$. In this way, we obtain from Eq. (8) for R_τ the value 3.552 instead of 3.560. One can see that the threshold effects for R_τ are about 0.2 %. Now, taking the experimental value $R_\tau = 3.552$ [8] as an input, we obtain $\alpha_s(M_\tau^2) = 0.37$ and $\alpha(M_\tau^2) = 0.40$. The values of the coupling constant in the $s-$ and t-channels are clearly different from each other; the ratio is $\alpha_s(M_\tau^2)/\alpha(M_\tau^2) = 0.92$.

So, we have demonstrated that the initial expression for R_τ [Eq.(6)] can be calculated directly from QCD by using the VPT method. We have also demonstrated that the distinction between the functions $\alpha_s(s)$ and $\alpha(Q^2)$ is not simply a matter of the standard π^2 terms, which may be important for understanding certain discrepancies (see, for example, [9]) arising in the determination of the QCD coupling constant from various experiments.

III. $R_{e^+e^-}$ ratio

Now we consider the process of e^+e^- annihilation into hadrons using the renormalization scheme in which the quark masses are renormalized so that the value of m_q is the position of the pole in the quark propagator $S_q(p)$ (the corresponding consideration of this process in the MS-like renormalization scheme has been performed in Ref. [4]). In the framework of this scheme, the effective coupling constant depends on the quark masses, which provides a natural way to include the threshold effects without any additional matching procedure. We will consider the range of $Q = \sqrt{s}$ from 0 to 6 GeV (as in Ref. [10]) and compare with experiment by using the smearing method [11]. The renormalization scale dependence of the running expansion parameter $a = a(\mu^2)$ is defined by the following equation

$$C\,[U(a) - U(a_0)] = 11 \ln\frac{\mu^2}{\mu_0^2} - \frac{2}{3}\sum_f \left[I\left(\frac{\mu^2}{m_f^2}\right) - I\left(\frac{\mu_0{}^2}{m_f^2}\right)\right], \qquad (16)$$

where μ_0 is some normalization point, $a_0 = a(\mu_0^2)$, $I(\mu^2/m^2)$ is the well-known one-loop integral, and the function $U(a)$ has the following form

$$U(a) = \frac{1}{a^2} - \frac{3}{a} - 12\ln a + \frac{3}{4}\ln(1-a) + \frac{45}{4}\ln(1+3a). \qquad (17)$$

According to the smearing method [11], we consider the following smeared quantity

$$R_\Delta(Q) = \frac{\Delta}{\pi}\int_0^\infty ds \frac{R(s)}{(s-Q^2)^2+\Delta^2}, \qquad (18)$$

and also the smeared derivative

$$W_\Delta(s) \equiv \frac{d R_\Delta(s)}{ds} = -\frac{1}{2\mathrm{i}}\left[\frac{D(s+\mathrm{i}\Delta)}{s+\mathrm{i}\Delta} - \frac{D(s-\mathrm{i}\Delta)}{s-\mathrm{i}\Delta}\right], \qquad (19)$$

where $D(q^2)$ is the Adler function.

We find our parameters from the following considerations. We use the value $C = 39$ which can be found as the corresponding constant in the MS-like scheme (see Ref. [2]) taking into account information coming from meson spectroscopy [2]. Further, by using the fact that for sufficiently large $Q \simeq 5 \div 6$ GeV the quantity R_Δ must reproduce the experimental curve (see Fig. 1), we find the value of the parameter a_0. In Fig. 1, the smeared quantity (18) obtained in the first nontrivial order of our approximation is shown

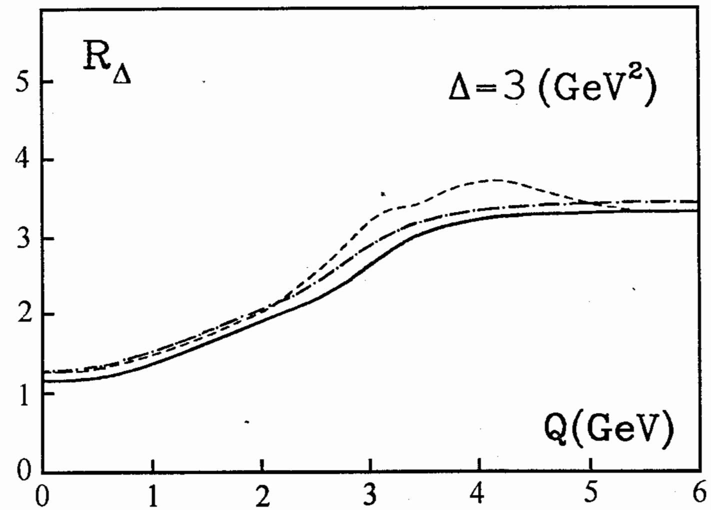

Fig. 1. The function R_Δ versus Q for $\Delta = 3$ GeV^2. The solid curve is obtained from Eq. (18), the dashed one from the smeared experimental data (taken from Ref. 10) and the dot-dashed one from applying the optimization procedure to the third-order calculation of $R_{e^+e^-}$ [10].

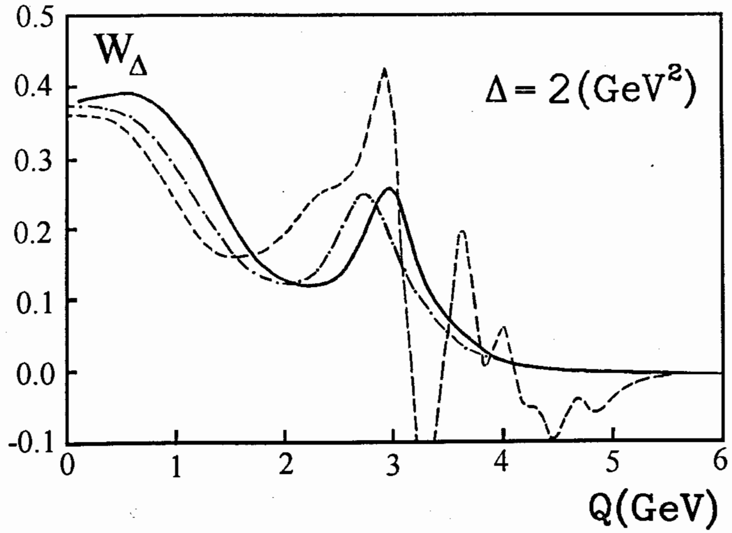

Fig. 2. The function W_Δ versus $Q = \sqrt{s}$ for $\Delta = 2$ GeV^2. The solid curve is obtained from Eq. (19), the dashed one from the smeared experimental data [10] and the dot-dashed one from applying the optimization procedure to the third-order calculation [10].

for $\Delta = 3\,\mathrm{GeV}^2$ (solid line). The experimental curve is taken from Ref. [10] (dot–dashed line); we also report the theoretical results from that paper (dashed line). In Fig. 2 we show the analogous result for the quantity (19) with $\Delta = 2\,\mathrm{GeV}^2$. As can be seen, our results obtained in first-order reproduces the experimental curve quite well and are close to the relevant result of [10] obtained on the basis of optimization of the third-order of standard perturbation theory. Fig. 2 shows that the value of $\Delta = 2\,\mathrm{GeV}^2$ is not sufficiently large to smooth the region of charm resonances (see the corresponding estimations in Ref. [11]); on increasing the value of Δ the experimental curve approaches the theoretical prediction shown in Fig. 2.

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