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QUANTIZATION OF A q-DEFORMED FREE RELATIVISTIC PARTICLE

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Symmetry groups and symmetry algebras are some of the firm pillars on which the whole edifice of the modern developments in theoretical physics rests. Mathematically, the q-deformed symmetry algebras (compact matrix pseudo-groups or quantum groups [1,2]) are examples of the quasi-triangular Hopf algebras [3]. These q-deformed groups and algebras have recently been the subject of considerable interest in the hope of developing some more general symmetries that might have profound implications in very sensitive physical theories (where q is very close to one) [4]. In spite of considerable progress in the mathematical direction, the key concepts of quantum groups have not penetrated into the realm of physical applications in an overwhelming and compelling manner. Some attempts have been made, however, to see the impact of these groups in the context of q-deformed gauge theories [5] as well as in few well-known physical examples [6]. These groups are also conjectured to provide a fundamental length in the context of space-time quantization [7] with a non-commutative underlying geometry of the space-time manifold [8]. These objects have manifested themselves in statistical systems, conformal field theories, knot theory, nuclear physics, etc.[see, e.g., Ref.9 and references therein].

Recently, a Lagrangian formulation has been developed to describe a q-deformed scalar as well as a spinning relativistic particle in a consistent and cogent way [10]. In this approach, the Lorentz invariance is respected throughout the discussion, which might turn out to be useful in the development of the Lorentz covariant q-deformed field theories. The main objective of the present paper is to develop the BRST formalism for the q-deformed scalar particle of Ref. [10] on a $GL_q(2)$ invariant quantum world-line defined on a *flat* but *q-deformed* cotangent manifold to the Minkowski space-time (configuration) manifold. We derive q-(anti)commutation relations for this system which are (graded)associative on the mass-shell and the on-shell. One of the key features of our work is the $GL_q(2)$ invariance of the solutions for the equations of motion on the *mass-shell* at any arbitrary value of the evolution parameter. The BRST quantization has been carried out by exploiting the local gauge symmetry and the reparametrization invariance of the starting q-deformed Lagrangian. The equivalence of the BRST charges corresponding to these symmetries requires that the deformation parameter q must be ± 1. This condition ($q = \pm 1$) also emerges from the conservation of the BRST charge on the unconstrained manifold and the requirement that the BRST algebra should be satisfied. We do not discuss here the q-deformed Hamiltonian formulation, q-deformed Dirac brackets, etc., for the above system. The q-deformed Hamiltonian formulation for a scalar as well as a spinning particle would be reported in a future publication [11].

We start off with three equivalent Lagrangians for an undeformed (classical) free relativistic particle [12] moving on a world-line embedded in a D-dimensional flat Minkowski manifold. The mass-shell condition ($p^2 - m^2 = 0$) is a common feature of the first-order Lagrangian ($L_F = p_\mu \dot{x}^\mu - \frac{e}{2}(p^2 - m^2)$), the second-order Lagrangian ($L_S = \frac{1}{2}e^{-1}\dot{x}^2 + \frac{1}{2}em^2$) and the Lagrangian with a square-root ($L_0 = m(\dot{x}^2)^{1/2}$).

Except the mass (cosmological constant) parameter (m), the target space canonically conjugate coordinates (x^μ) and momenta (p_μ) as well as the einbein field (e) are functions of an evolution parameter (τ) characterizing the trajectory of the free motion of a relativistic particle and $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$. All the above dynamical variables are the *even* elements of a Grassmann algebra. The first- and second-order Lagrangians are endowed with first-class constraints $\Pi_e \approx 0$ and $p^2 - m^2 \approx 0$, where Π_e is the conjugate momentum corresponding to the einbein field e. For the covariant canonical quantization of such systems, the most suitable approach is the BRST formalism [13]. The BRST invariant Lagrangian (L_{brst}) corresponding to the first-order Lagrangian (L_F) is [14]

$$L_{brst} = p_\mu \dot{x}^\mu - \frac{e}{2}(p^2 - m^2) + b\,\dot{e} + \frac{b^2}{2} + \dot{\bar{c}}\,\dot{c}, \tag{1}$$

where the even element b is the Nakanishi-Lautrup auxiliary field and (anti)ghost fields $(\bar{c})c$ are the *odd* elements of a Grassmann algebra ($c^2 = 0$, $\bar{c}^2 = 0$). In the BRST quantization procedure, the first-class constraints $\Pi_e = b \approx 0$ as well as $p^2 - m^2 \approx 0$ turn up as constraints on the physical states when one requires that the conserved and nilpotent BRST charge $Q_{brst} = \frac{c}{2}(p^2 - m^2) + b\dot{c}$ must annihilate the physical states in the quantum Hilbert space. The conservation of the BRST charge on any arbitrary unconstrained manifold is ensured by the equations of motion $\dot{p}_\mu = 0$, $\dot{b} = -\frac{1}{2}(p^2 - m^2)$, $\ddot{c} = \ddot{\bar{c}} = 0$, $b + \dot{e} = 0$, $\dot{x}_\mu = e p_\mu$.

To obtain the q-analogue of the above Lagrangian (L_{brst}), we follow the the prescription of Ref. [10] where the configuration space corresponding to the Minkowski space-time manifold is *flat* and *undeformed* ($x_\mu x_\nu = x_\nu x_\mu$) but the cotangent manifold (momentum phase space) is *q-deformed* ($x_\mu p_\nu = q\, p_\nu x_\mu$, $x_\mu x_\nu = x_\nu x_\mu$, $p_\mu p_\nu = p_\nu p_\mu$) in such a way that the Lorentz invariance is preserved for any arbitrary ordering of μ and ν. Here all the dynamical variables are taken as hermitian elements of an algebra in involution ($|q| = 1$) and q is a non-zero c-number. As a consequence of the above deformation, the following *on-shell* and *(graded)associative* q-(anti)commutation relations emerge [1]

$$\begin{array}{llll}
x_\mu\, x_\nu = x_\nu\, x_\mu, & \dot{x}_\mu\, \dot{x}_\nu = \dot{x}_\nu\, \dot{x}_\mu, & \dot{x}_\mu\, x_\nu = x_\nu \dot{x}_\mu, & x_\mu \dot{x}_\nu = \dot{x}_\nu x_\mu, \\
p_\mu\, p_\nu = p_\nu\, p_\mu, & x_\mu\, p_\nu = q\, p_\nu\, x_\mu, & \dot{x}_\mu\, p_\nu = q\, p_\nu\, \dot{x}_\mu, & e\, x_\mu = q\, x_\mu e, \\
e\, p_\mu = q\, p_\mu\, e, & e\, \dot{x}_\mu = q\, \dot{x}_\mu\, e, & e\, b = b\, e, & e\, c = c\, e, \quad e\, \bar{c} = \bar{c}\, e, \\
c\, \bar{c} = -\frac{1}{q}\bar{c}\, c, & c\, \dot{\bar{c}} = -\frac{1}{q}\dot{\bar{c}}\, c, & \dot{c}\, \bar{c} = -\frac{1}{q}\bar{c}\, \dot{c}, & \dot{c}\, \dot{\bar{c}} = -\frac{1}{q}\dot{\bar{c}}\, \dot{c},
\end{array}$$

[1]These on-shell q-(anti)commutation relations emerge from the basic (un)deformed relations on a deformed cotangent manifold, the equations of motion obtained from the (un)deformed BRST-invariant Lagrangians (1) *or* (8) and by exploiting the *mass-shell* condition $p^2 - m^2 = 0$. For instance, it can be readily seen that if we take the *on-shell* conditions only, there is a contradiction between the relations $\dot{b}\, p_\mu = q\, p_\mu\, \dot{b}$ and $p_\mu\, p_\nu = p_\nu\, p_\mu$ with $\dot{b} = -\frac{1}{2}(p^2 - m^2)$. Thus, in the computation of the q-(anti)commutation relations for the BRST invariant Lagrangians, the *mass-shell* as well as the *on-shell* conditions should be exploited together.

$$\begin{array}{lllll}
c\, x_\mu = q\, x_\mu\, c, & \bar{c}\, x_\mu = q\, x_\mu\, \bar{c}, & c\, p_\mu = q\, p_\mu\, c, & \bar{c}\, p_\mu = q\, p_\mu\, \bar{c}, & \\
b\, x_\mu = q\, x_\mu\, b, & b\, p_\mu = q\, p_\mu\, b, & b\, c = c\, b, & b\, \bar{c} = \bar{c}\, b, & \\
e\, m = q\, m\, e, & x_\mu\, m = q\, m\, x_\mu, & p_\mu\, m = m\, p_\mu, & c\, \dot{c} = -q\, \dot{c}\, c, & \\
b\, m = q\, m\, b, & c\, m = q\, m\, c, & \bar{c}\, m = q\, m\, \bar{c}, & c^2 = 0, & \bar{c}^2 = 0,
\end{array} \tag{2}$$

where the mass-shell condition $p^2 - m^2 = 0$, emerging from the equations of motion $\dot{b} = -\ddot{e} = -\frac{1}{2}(p^2 - m^2) = 0$ has been *imposed.* Mathematically, this restriction implies that the b field is τ-independent and the τ-dependence of the einbein field is at most linear. Physically, it just means that the mass-shell condition is *strongly* equal to zero on the quantum world-line even in the case of the q-deformed BRST formalism. It is straightforward to see that in the limit when the odd Grassmann variables $(c, \bar{c})$ and the even variable (b) are set equal to zero, we obtain the q-commutation relations for a q-deformed scalar free relativistic particle of Ref. [10] and in the limit $q \longrightarrow 1$ the usual (anti)commutation relations among the dynamical variables of the Lagrangian (1) emerge automatically.

Before obtaining the q-deformed Lagrangian, it is essential to define a q-deformed world-line for the free motion of a scalar relativistic particle on the cotangent manifold because the q-deformation is present in this manifold and the Lagrangian has to describe the motion on this specific quantum world-line. Such a $GL_q(2)$ invariant world-line, consistent with the q-(anti)commutation relations (2), can be defined in terms of the coordinate generator x^μ and the momentum generator p_μ as [10]

$$x_\mu(\tau)\, p^\mu(\tau) = q\, p_\mu(\tau)\, x^\mu(\tau), \tag{3}$$

where repeated indices are summed over (i.e., $\mu = 0, 1, 2........D-1$), and the world-line is parameterized by a real commuting variable τ. The following $GL_q(2)$ transformations

$$\begin{array}{lll}
x_\mu & \longrightarrow & A\, x_\mu + B\, p_\mu, \\
p_\mu & \longrightarrow & C\, x_\mu + D\, p_\mu,
\end{array} \tag{4}$$

are implied in the component pairs: $(x_0, p_0), (x_1, p_1)........(x_{D-1}, p_{D-1})$ of the phase variables in equation (3) and its form-invariance can be readily checked if we assume the commutativity of the phase variables with elements A, B, C, and D of a 2×2 $GL_q(2)$ matrix obeying the braiding relationship in rows and columns as:

$$\begin{array}{lll}
AB & = & q\, BA, \quad AC = q\, CA, \quad CD = q\, DC, \quad BD = q\, DB, \\
BC & = & CB, \quad AD - DA = (q - q^{-1})\, BC.
\end{array} \tag{5}$$

It will be noticed that there is another candidate, namely;

$$c(\tau)\, \bar{c}(\tau) = -\frac{1}{q}\, \bar{c}(\tau)\, c(\tau), \qquad c^2(\tau) = 0, \qquad \bar{c}^2(\tau) = 0, \tag{6}$$

which also remains form-invariant under the following transformations

$$\begin{pmatrix} c \\ \bar{c} \end{pmatrix} \rightarrow \begin{pmatrix} A, & B \\ C, & D \end{pmatrix} \begin{pmatrix} c \\ \bar{c} \end{pmatrix}, \tag{7}$$

if we assume the commutativity of the (anti)ghost fields $(\bar{c})c$ with the elements A, B, C, and D of the $GL_q(2)$ matrix obeying relations (5). However, it cannot be taken as the definition of the quantum world-line because these fields are totally decoupled from the rest of the theory and their on-shell conditions $\ddot{c} = \ddot{\bar{c}} = 0$ do not lead to anything interesting and substantial.

The BRST invariant first-order Lagrangian (L_f) that describes the free motion $(\dot{p}_\mu = 0)$ of a free q-deformed relativistic particle is

$$L_f = q^{1/2} p_\mu \dot{x}^\mu - \frac{e}{1+q^2}(p^2 - m^2) + b\,\dot{e} + \frac{b^2}{1+q^2} + \dot{\bar{c}}\,\dot{c}, \tag{8}$$

where the $q^{1/2}$ factor appears in the first term due to the Legendre transformation with q-symplectic metrices [10]

$$\Omega_{AB}(q) = \begin{pmatrix} 0, & -q^{-1/2} \\ q^{1/2}, & 0 \end{pmatrix} \quad \text{and} \quad \Omega^{AB}(q) = \begin{pmatrix} 0, & q^{-1/2} \\ -q^{1/2}, & 0 \end{pmatrix}. \tag{9}$$

In the third term of the Lagrangian (8), there is no $q^{1/2}$ factor because the canonically conjugate variables e and b commute $(e\,b = b\,e, \dot{e}\,b = b\,\dot{e})$. Therefore, the standard canonical symplectic metrices (i.e., $q = 1$ in equation (9)) have to be exploited for the Legendre transformations for these fields. Here the q-BRST Hamiltonian for a free relativistic particle has been taken to be: $H = \frac{e}{1+q^2}(p^2 - m^2) - \frac{b^2}{1+q^2} + \dot{\bar{c}}\dot{c}$. The equations of motion from the Lagrangian (8) on the mass-shell are

$$\begin{aligned} \dot{x}_\mu &= q^{1/2}\, e\, p_\mu, \quad \dot{p}_\mu = 0, \quad \ddot{c} = 0, \quad \ddot{\bar{c}} = 0, \\ \dot{b} &= -\frac{q^4}{1+q^2}(p^2 - m^2) = 0, \qquad \dot{e} = -\, b, \end{aligned} \tag{10}$$

which satisfy the *on-shell* and the *mass-shell* q-(anti)commutation relations (2). In the derivation of the equations of motion from the Lagrangian (8), the $GL_q(2)$ invariant differential calculus has been exploited [15]. For instance, for *even* dynamical variables obeying $xy = qyx$, any monomial in the Lagrangian (8) is arranged in the form $y^r\, x^s$, and then we use

$$\begin{aligned} \frac{\partial(y^r\, x^s)}{\partial x} &= y^r\, x^{s-1}\, q^r\, \frac{(1-q^{2s})}{(1-q^2)}, \\ \frac{\partial(y^r\, x^s)}{\partial y} &= y^{r-1}\, x^s\, \frac{(1-q^{2r})}{(1-q^2)}, \end{aligned} \tag{11}$$

where $r, s \in \mathcal{Z}$ are whole numbers (not fractions). For the differentiations with respect to the odd Grassmann variables $\dot{c}$ and $\dot{\bar{c}}$, these variables are first brought to the left side in the corresponding expressions by using q-(anti)commutation relations (2) and, only then, differentiation is carried out.

It is rather cumbersome to obtain general solutions for the equations of motion (10) for any arbitrary dependence of the dynamical variables on the evolution parameter τ. The mass-shell condition $(p^2 - m^2 = 0)$, however, emerging from the restriction $\ddot{e} = -\dot{b} = 0$ comes to our rescue. The $GL_q(2)$ invariant solutions for the equations of motion (10), under such restriction, are

$$\begin{aligned} x_\mu(\tau) &= x_\mu(0) + q^{1/2} e(0) p_\mu(0)\tau - \frac{1}{2}\, q^{1/2}\, b(0)\, p_\mu(0)\, \tau^2, \\ e\,(\tau) &= e(0) \,-\, b(0)\,\tau, \\ c\,(\tau) &= c(0) + f\,\tau, \\ \bar{c}\,(\tau) &= \bar{c}(0) + \bar{f}\,\tau, \\ b\,(\tau) &= b(0), \\ p_\mu(\tau) &= p_\mu(0), \end{aligned} \tag{12}$$

where f and $\bar{f}$, present in the solutions for $c(\tau)$ and $\bar{c}(\tau)$, are τ-independent *odd* elements of a Grassmann algebra $(f^2 = \bar{f}^2 = 0)$ and they obey the following q-anticommutation relations with the rest of the odd dynamical variables

$$\begin{aligned} f\,\bar{f} &= -\frac{1}{q}\,\bar{f}\,f, \quad f\,\bar{c} = -\frac{1}{q}\,\bar{c}\,f, \quad f^2 = \bar{f}^2 = 0, \\ c\,\bar{f} &= -\frac{1}{q}\bar{f}\,c, \quad c\,f = -\,f\,c, \quad \bar{c}\,\bar{f} = -\bar{f}\,\bar{c}. \end{aligned} \tag{13}$$

The q-commutation relations of f and $\bar{f}$ with the rest of the even dynamical variables are the same as that of c and $\bar{c}$ in equation (2). With equations (2), (13) and solutions (12), it is interesting to check that all these relations and the $GL_q(2)$ invariant quantum world-line (3) are invariant for any arbitrary value of the evolution parameter τ, if we assume the validity of these relations at initial "time" $\tau = 0$. The second-order Lagrangian (L_s), describing the motion of a scalar relativistic particle on the tangent manifold (velocity phase space), can be obtained from the first-order Lagrangian (8) by exploiting equations (2) and (10) as given below:

$$L_s = \frac{q^2}{1+q^2}\, e^{-1}\, (\dot{x}_\mu)^2 + \frac{e}{1+q^2}\, m^2 + b\,\dot{e} + \frac{b^2}{1+q^2} + \dot{\bar{c}}\,\dot{c}. \tag{14}$$

The consistent expression for the canonical momenta (p_μ) and the rest of the canonical momenta $(\Pi's)$ for the first- and second-order Lagrangians (8) and (14) are

$$\begin{aligned} p_\mu &= q^{-3/2}\Big(\frac{\partial L_{(f,s)}}{\partial \dot{x}^\mu}\Big) \equiv q^{-1/2}\, e^{-1}\, \dot{x}_\mu, \qquad \Pi_e = b, \\ \Pi_b &= 0, \qquad \Pi_c = -\,q\,\dot{\bar{c}}, \qquad \Pi_{\bar{c}} = \dot{c}. \end{aligned} \tag{15}$$

Due to the $GL_q(2)$ invariant differential calculus developed in Ref. [15], the differentiation of the Lagrangian L_s with respect to the einbein field e yields

$$\dot{b} = \frac{q^4}{1+q^2}\Big[m^2 - q^{-1}\, e^{-1}\,(\dot{x}_\mu)\, e^{-1}\,(\dot{x}^\mu)\Big], \tag{16}$$

which is consistent with the corresponding equation of motion derived from the first-order Lagrangian (L_f) and equation (15). In fact, the above second-order Lagrangian is equivalent to the first-order Lagrangian in all aspects.

It has been demonstrated in Ref. [10] that the first-order Lagrangian for the q-deformed scalar particle ($L_{\mathcal{F}} = q^{1/2} p_\mu \dot{x}^\mu - \frac{e}{1+q^2}(p^2 - m^2)$) is endowed with the q-deformed gauge and reparametrization symmetries which are found to be on-shell equivalent only for $q = \pm 1$. For an arbitrary value of q, the above symmetries are *not* equivalent. Thus, both of these symmetries can be exploited for the BRST quantization. For instance, it can be seen that the Lagrangian (8) is invariant under the following nilpotent BRST transformations

$$\begin{aligned} \delta_B x^\mu &= q^{1/2}\,\eta\, c\, p^\mu, \qquad \delta_B c = 0, \qquad \delta_B b = 0, \\ \delta_B p^\mu &= 0, \qquad \delta_B \bar{c} = q^2\,\eta\, b\,, \qquad \delta_B e = q^2\,\eta\,\dot{c}, \end{aligned} \tag{17}$$

because the Lagrangian transforms as

$$\delta_B L_f = \eta\,\frac{d}{d\tau}\Big[\frac{c\,(p^2 + q^2 m^2)}{(1+q^2)} + q^2\, b\,\dot{c}\Big], \tag{18}$$

where η is a τ-independent and a q-(anti)commutative *odd* element ($\eta^2 = 0$) of a Grassmann algebra (i.e., $\eta\; c = -q\; c\;\eta$, $\eta\;\bar{c} = -q\;\bar{c}\;\eta$) and it commutes with all the even fields (x_μ, p_μ, e, b) of the theory (i.e., $\eta\; x_\mu = x_\mu\;\eta$ etc.). It is the gauge symmetry of the first-order Lagrangian($L_{\mathcal{F}}$) that has been exploited for the BRST transformations (17). The reparametrization symmetry, corresponding to the one-dimensional diffeomorphism ($\tau \to \tau - \epsilon(\tau)$), can also be exploited for the BRST quantization. Such a first-order Lagrangian is

$$L_B^r = q^{1/2} p_\mu \dot{x}^\mu - \frac{e}{1+q^2}(p^2 - m^2) + \mathcal{B}\,\dot{e} + \frac{\mathcal{B}^2}{1+q^2} + \dot{\bar{\lambda}}\frac{d}{d\tau}(\lambda\; e), \tag{19}$$

where $\mathcal{B}$ is the Nakanishi-Lautrup auxiliary field obeying the same q-commutation relations as b in (2) and $\bar{\lambda}(\lambda)$ are the (anti)ghost fields corresponding to the diffeomorphism transformations. These (anti)ghost fields are odd elements of a Grassmann algebra ($\bar{\lambda}^2 = \lambda^2 = 0, \lambda\;\bar{\lambda} = -\bar{\lambda}\;\lambda$) and they commute with all the *even* elements of a Grassmann algebra. It can be checked that under the following nilpotent BRST transformations

$$\begin{aligned} \delta_B^r x^\mu &= \eta\;\lambda\;\dot{x}_\mu, \qquad \delta_B^r p^\mu = \eta\;\lambda\;\dot{p}_\mu, \qquad \delta_B^r \mathcal{B} = 0, \\ \delta_B^r\; e &= \eta\;\frac{d}{d\tau}(\lambda\; e), \qquad \delta_B^r \bar{\lambda} = \eta\;\mathcal{B}, \qquad \delta_B^r \lambda = \eta\lambda\dot{\lambda}, \end{aligned} \tag{20}$$

the Lagrangian (19) transforms as

$$\delta_B^r L_B^r = \eta\;\frac{d}{d\tau}\Big[q^{1/2}\lambda\; p_\mu \dot{x}^\mu - \frac{\lambda e}{1+q^2}\,(p^2 - m^2) + \mathcal{B}\frac{d}{d\tau}(\lambda\; e)\Big]. \tag{21}$$

The equations of motion that emerge from (19) for $e \neq 0$ on the mass-shell are:

$$\begin{aligned} \dot{x}_\mu &= q^{1/2}\; e\; p_\mu, \quad \frac{d^2}{d\tau^2}(\lambda\; e) = 0, \quad \dot{p}_\mu = 0, \quad \ddot{\bar{\lambda}} = 0, \\ \dot{\mathcal{B}} &= -\frac{q^4}{1+q^2}(p^2 - m^2) = 0, \qquad \dot{e} = -\,\mathcal{B}. \end{aligned} \tag{22}$$

The analogue of the Euler-Lagrange equations (10) and (22) can be obtained from the least action principle in the form of the Hamilton equations. As a bonus, we can also derive the expressions for the conserved charges as illustrated below:

$$\begin{aligned} \delta S = 0 \;\doteq\; & \int d\tau \Big(\delta[q^{1/2} p_\mu \dot{x}^\mu + b\dot{e} + \dot{c}\Pi_c + \dot{\bar{c}}\Pi_{\bar{c}} \\ & - H(x_\mu, p_\mu, e, b, c, \bar{c}, \dot{c}, \dot{\bar{c}})] - \frac{dg(\tau)}{d\tau}\Big), \end{aligned} \tag{23}$$

where S is the action corresponding to the Lagrangian (8), H is the most general expression for the BRST Hamiltonian function for a q-deformed free relativistic particle and the expressions for $g(\tau)$ are:

$$\begin{aligned} g(\tau) &= \frac{c\,(p^2 + q^2 m^2)}{1+q^2} + q^2\; b\;\dot{c}, \\ g(\tau) &= q^{1/2}\lambda p_\mu \dot{x}^\mu - \frac{\lambda e}{1+q^2}(p^2 - m^2) + \mathcal{B}\frac{d}{d\tau}(\lambda\; e), \end{aligned} \tag{24}$$

for the (gauge) BRST transformations (17) and the (diffeomorphism) BRST symmetry transformations (20), respectively. Now, using the q-(anti)commutation relations $\dot{\bar{c}}\;\delta\dot{c} = -q\;\delta\dot{c}\;\dot{\bar{c}}$ and $\delta\dot{x}^\mu p_\mu = q\; p_\mu \delta\dot{x}^\mu$, all the variations can be taken to the left in the corresponding terms of (23). For the validity of the following Hamilton equations [2]

$$\begin{aligned} \dot{x}^\mu &= q^{-1/2}\frac{\partial H}{\partial p^\mu}, \quad \dot{p}^\mu = -\,q^{1/2}\frac{\partial H}{\partial x^\mu}, \quad \dot{e} = \frac{\partial H}{\partial b}, \qquad \dot{b} = -\,\frac{\partial H}{\partial e}, \\ \dot{c} &= \frac{\partial H}{\partial \dot{\bar{c}}}, \qquad \ddot{c} = -\frac{\partial H}{\partial \bar{c}}, \qquad \dot{\bar{c}} = -q^{-1}\frac{\partial H}{\partial \dot{c}}, \qquad \ddot{\bar{c}} = q^{-1}\frac{\partial H}{\partial c}, \end{aligned} \tag{25}$$

we obtain the most general expression for the conserved charge (Q) as:

$$Q = q^{-1/2}\delta x^\mu p_\mu + b\;\delta e + \delta\bar{c}\;\dot{c} - q\;\delta c\;\dot{\bar{c}} - g(\tau). \tag{26}$$

[2] In the variation of $\delta(\dot{c}\;\Pi_c + \dot{\bar{c}}\Pi_{\bar{c}})$ which is equivalent to $\delta(\dot{\bar{c}}\;\dot{c} + \dot{\bar{c}}\;\dot{c})$, we have taken $\delta\dot{\bar{c}}\;\dot{c} - q\;\delta\dot{c}\;\dot{\bar{c}}$ from the first-term and the second-term is expressed as $\frac{d}{d\tau}(\delta\bar{c}\;\dot{c}) - q\delta\bar{c}\;\ddot{c} - q\;\frac{d}{d\tau}(\delta c\;\dot{\bar{c}}) + q\;\delta c\;\ddot{\bar{c}}$ to yield the equations of motion $\ddot{c} = \ddot{\bar{c}} = 0$. In analogy with equations (23),(24) and (25), it is straightforward to derive the Hamilton equations corresponding to (22).

The Hamilton equations of motion (25) with the BRST Hamiltonian

$$H = \frac{e}{1+q^2}(p^2 - m^2) - \frac{b^2}{1+q^2} + \dot{\bar{c}}\,\dot{c}, \tag{27}$$

turn out to be consistent with the Euler-Lagrange equations (10) and the contravariant metric (9). For the global version of the BRST symmetry transformations (17) and (20), equation (26) yields the following charges

$$Q_B = \frac{q^2\, c\,(p^2 - m^2)}{1+q^2} + q^2\, b\dot{c} \quad \text{and} \quad Q_r = \frac{\lambda e\,(p^2 - m^2)}{1+q^2} + \mathcal{B}\frac{d}{d\tau}(\lambda\, e), \tag{28}$$

which are found to be equivalent under the identifications $b = \mathcal{B}$ and $c = \lambda e$ *only* for $q = \pm 1$. In fact, this requirement ($q = \pm 1$) for the above equivalence is a manifestation of the on-shell equivalence of the gauge and reparametrization symmetries in the case of the deformed Lagrangian ($L_{\mathcal{F}}$). It is interesting to check that $(\dot{c}, e)\, p_\mu = q\, p_\mu\, (\dot{c}, e)$, (10) and (22) lead to:

$$\dot{Q}_B = \frac{q^2\, \dot{c}\,(p^2 - m^2)}{1+q^2}(1 - q^2) \quad \text{and} \quad \dot{Q}_r = \frac{d}{d\tau}(\lambda\, e)\, \frac{p^2 - m^2}{1+q^2}(1 - q^2). \tag{29}$$

To have an analogy with the undeformed case ($q = 1$), where the BRST charge is conserved on any arbitrary (unconstrained) manifold, it is essential that the deformation parameter (q) must be ± 1 for the conservation of the above q-deformed BRST charge (28). However, even for an arbitrary value of q, the BRST charge (28) is conserved on the constrained submanifold where $p^2 - m^2 = 0$.

To obtain the BRST quantization scheme, the dynamical variables are first changed to the hermitian operators and then we require that the physical Hilbert space must be annihilated by the BRST operator. This, in turn, implies that the constraint operators should annihilate the physical states. In the q-deformed BRST approach, it is essential to invoke various consistency conditions e.g. hermitian properties and the BRST algebra to obtain a precise expression for the q-(anti)commutators. To illustrate this point, we first demonstrate the correctness of $\dot{Q}_B$ of (29) in terms of the q-analogue of the Heisenberg equations, namely;

$$\dot{Q}_B \;=\; -\frac{i}{\hbar}\,[\, Q_B, H\,]_q, \tag{30}$$

where first we define the q-commutators $[A, B]_q = A\,B - f(q)B\,A$ in terms of an arbitrary q-dependent function $f(q)$ ($f(q) \to 1$ when $q \to 1$ *or* $A = B$) and do the ordering by exploiting q-(anti)commutation relations (2) to obtain the desired q-(anti)commutators. For instance, using equations (27) and (28), we obtain

$$\dot{Q}_B = -\frac{i}{\hbar}\frac{q^2}{1+q^2}\Big(\,[\, c\,(p^2 - m^2), \dot{\bar{c}}\,\dot{c}\,]_q + [\, b\,\dot{c}, e\,(p^2 - m^2)\,]_q\Big). \tag{31}$$

Now, using the above definition of the q-commutator and exploiting the relations $\dot{c}\ (p_\mu, m) = q\ (p_\mu, m)\ \dot{c}$, $\dot{\bar{c}}\ (p_\mu, m) = q\ (p_\mu, m)\ \dot{\bar{c}}$ and $\dot{c}\ c = -\frac{1}{q}\ c\ \dot{c}$, the first q-commutator in (31) can be converted into a q-anticommutator and the second commutator can be reordered using $b\ (p_\mu, m) = q\ (p_\mu, m)\ b$, $\dot{c}\ (p_\mu, m) = q\ (p_\mu, m)\ \dot{c}$, and $\dot{c}\ e = e\ \dot{c}$ to yield the right hand side of (31) as

$$-\frac{i}{\hbar}\frac{q^2}{1+q^2}\left[\,\frac{\{c, \dot{\bar{c}}\}_q}{q^4} + [\, b, e\,]_q\,\right]\dot{c}\,(p^2 - m^2), \tag{32}$$

where the q-(anti)commutators are

$$\{c, \dot{\bar{c}}\}_q = c\ \dot{\bar{c}} + q^3\ f(q)\ \dot{\bar{c}}\ c, \qquad [b,\ e]_q = b\ e - \frac{g(q)}{q^4}\ e\ b. \tag{33}$$

Here arbitrary q-dependent functions $g(q)$ and $f(q)$ reduce to one as $q \to 1$. Comparison and consistency with (29) yields *one* of the solutions as:

$$\{c, \dot{\bar{c}}\}_q\ = i\,\hbar\, q^4, \qquad [b, e]_q = -i\,\hbar\, q^2. \tag{34}$$

The hermiticity requirement on the above q-(anti)commutators leads to

$$\begin{aligned} \Big(|q|^6\ |f(q)|^6 - 1\Big)\ \dot{\bar{c}}\ c \;&=\; i\,\hbar\ q^{*3}\ \Big(q^4\ f^*(q) - q^*\Big), \\ \Big(|q|^8 - |g(q)|^2\Big)\ e\ b \;&=\; i\,\hbar\ q^4\ \Big(q^{*6} - q^2\ g^*(q)\Big). \end{aligned} \tag{35}$$

as the general restriction on $g(q)$ and $f(q)$ (see, e.g., Aref'eva and Volovich Ref.[6]). One of the trivial solutions ($g(q) = q^4,\ f(q) = q^{-3},\ q^2 = q^{*2},\ q^4 = q^{*4}$) implies that q^2 and q^4 are real parameters. Under such restrictions, the (anti)commutators (34)

$$\{c, \dot{\bar{c}}\}_q \equiv c\,\dot{\bar{c}} + \dot{\bar{c}}\,c = i\,\hbar\ q^4, \qquad [b, e]_q \equiv b\,e - e\,b = -i\,\hbar\ q^2, \tag{36}$$

reduce to the corresponding undeformed BRST (anti)commutators for $q = \pm 1$. The nilpotency of the q-BRST charge $Q_B^2 = \frac{1}{2}\{Q_B, Q_B\}_q = 0$ is trivially satisfied because of the absence of the canonically conjugate variables in the expression for Q_B. To complete the BRST algebra, we further require the validity of the relation ($-\frac{i}{\hbar}[Q_c, Q_B]_q = Q_B$) where the ghost charge $Q_c = c\ \dot{\bar{c}} + \bar{c}\ \dot{c}$, emerging due to the global scale invariance, is conserved only for $q = 1$ (and $[Q_c, Q_c]_q = 0$). This q-commutator is succinctly expressed as:

$$-\frac{i}{\hbar}\,[Q_c, Q_B]_q = -\frac{i}{\hbar}\,\frac{q^2}{1+q^2}[c\,\dot{\bar{c}}, c\,(p^2 - m^2)]_q - \frac{i}{\hbar}\,q^2[\bar{c}\,\dot{c}, b\,\dot{c}]_q. \tag{37}$$

In the computation of the first q-commutator in (37), we choose the arbitrary function such that we are consistent with the q-anticommutator (36). For instance, after reordering, this q-commutator becomes $-\frac{i}{\hbar}\,\frac{q^2}{1+q^2}\ c(\dot{\bar{c}}\ c + \frac{F(q)}{q^4}\ c\ \dot{\bar{c}}\ )(p^2 - m^2)$. Now, choosing $F(q) = q^4$, we obtain this q-commutator as $\frac{q^2}{1+q^2}\ q^4\ c\ (p^2 - m^2)$. We exploit an analogous procedure for the computation of the second q-commutator which ultimately reduces to $\frac{i}{\hbar}\, q^2\, b\, \{\bar{c}, \dot{c}\}_q\, \dot{c}$ where $\{\bar{c}, \dot{c}\}_q = \bar{c}\, \dot{c} + G(q)\dot{c}\, \bar{c}$ with an arbitrary function $G(q)$. For the sum of these two q-commutators to yield Q_B, we require:

$$\{\bar{c}, \dot{c}\}_q = -i\hbar, \qquad q^4 = 1. \tag{38}$$

The hermiticity requirement on the above q-anticommutator implies $G(q) = 1$. A definite and sensible expression for Q_B, however, requires that q must be ± 1.

To compute the q-commutator between x_μ and p_ν, we first define a relationship between the basic q-commutator $[x_\mu, p_\nu]_q$ and a q-Poisson bracket $\{x_\mu, p_\nu\}_q^{PB}$ as

$$[\, x_\mu, p_\nu\,]_q = i\, \hbar\, M(q)\, \{x_\mu, p_\nu\}_q^{PB}, \tag{39}$$

where $\{x_\mu, p_\nu\}_q^{PB} = q^{-1/2}\, \eta_{\mu\nu}$ due to symplectic metric (9) and $[x_\mu, p_\nu]_q = x_\mu\, p_\nu - N(q)\, p_\nu\, x_\mu$. Here q-dependent functions $M(q)$ and $N(q)$ go to one as $q \to 1$. The hermitian condition on (39) yields one of the solutions as:

$$|N(q)|^2 = 1, \quad \text{and} \quad M(q)\, q^{-1/2} = \frac{M^*(q)}{N^*(q)}\, q^{*-1/2}. \tag{40}$$

With the basic definition (39), we obtain

$$[x_\mu, p^2]_q = x_\mu\, p^2 - N^2(q)\, p^2\, x_\mu \equiv i\, \hbar\, M(q)\, \{x_\mu, p^2\}_q^{PB}, \tag{41}$$

where $\{x_\mu, p^2\}_q^{PB} = q^{-1/2}\,(1 + q^2)\, p_\mu$ fixes $N(q)$ to be q^2 and, therefore, $|q|^4 = 1$. Now, we require the validity of equation (10) by exploiting (anti)commutators (36), (38) and (41) in the Heisenberg equations of motion. For instance, the "time" derivative of x_μ can be expressed in terms of the q-BRST Hamiltonian H as:

$$\dot{x}_\mu = -\frac{i}{\hbar}[x_\mu, H]_q \equiv q^{1/2}\, e\, p_\mu. \tag{42}$$

In the computation of $[x_\mu, e\, p^2]_q = x_\mu\, e\, p^2 - h(q)\, e\, p^2\, x_\mu$, we do the reordering using $e\, x_\mu = q\, x_\mu\, e$ and require the consistency with (41) which fixes $h(q) = q^3$. Finally equality in (42) leads to $M(q) = q^2$. This, in turn, yields $q = \pm 1$ due to the requirement (40) (for the real value of q). Similarly, rest of the equations of motion (10) can be checked to be satisfied only for $q = \pm 1$ if we use the q-(anti)commutators (36) and (38) in the Heisenberg equations of motion.

The key ingredients in our quantization scheme are hermitian condition on q-(anti)commutators, validity of the q-BRST algebra, conservation of the BRST charge on an unconstrained manifold and the requirement that the on-shell condition should remain intact under q-deformed Heisenberg equations of motion. In the limit when $q \to 1, \hbar \to 0$, we obtain classical relations and in the limit $q \to 1$ the usual quantum mechanical (anti)commutators emerge automatically. We hope the q-deformed Hamiltonian formulation for this system with q-deformed Dirac brackets, q-deformed constraint analysis, etc., would be able to shed more light on the quantization scheme for any arbitrary value of q [11].

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Квантование свободной релятивистской q-деформированной частицы

В рамках БРСТ-формализма рассматривается q-деформированная скалярная релятивистская частица. При БРСТ-квантовании этой системы на $GL_q(2)$-инвариантных траекториях учитывается q-деформированная (локальная) калибровочная симметрия и репараметризационная инвариантность лагранжиана первого порядка. Эквивалентность соответствующих БРСТ-зарядов на массовой поверхности фиксирует параметр деформации $q = \pm 1$. То же условие на параметр деформации накладывается требованиями сохранения БРСТ-заряда вне массовой поверхности и сохранения БРСТ-алгебры. Решения уравнений движения обладают $GL_q(2)$-симметрией при любых значениях параметра эволюции.

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Malik R.P. E2-95-96

Quantization of a q-Deformed Free Relativistic Particle

A q-deformed free scalar relativistic particle is discussed in the framework of the BRST formalism. The q-deformed local gauge symmetry and reparametrization invariance of the first-order Lagrangian have been exploited for the BRST quantization of this system on a $GL_q(2)$ invariant quantum world-line. The on-shell equivalence of these BRST charges requires the deformation parameter to be ± 1 under certain identifications. The same restriction ($q = \pm 1$) emerges from the conservation of the q-deformed BRST charge on an arbitrary (unconstrained) manifold and the validity of the BRST algebra. The solutions for the equations of motion respect $GL_q(2)$ invariance on the mass-shell at any arbitrary value of the evolution parameter characterizing the quantum world-line.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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