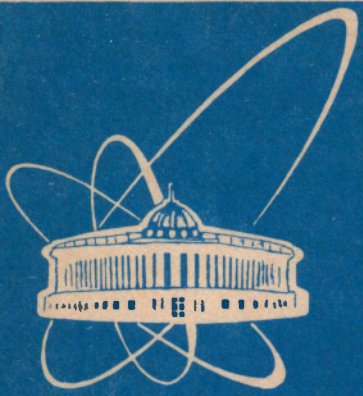

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MINIMIZATION OF THE SCALAR HIGGS POTENTIAL IN THE FINITE SUPERSYMMETRIC GRAND UNIFIED THEORY

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1 Introduction

Some years ago a possibility was discovered to construct $N = 1$ supersymmetric gauge theories with vanishing β-functions of the gauge and Yukawa couplings in all orders of perturbation theory (finite theories) [1]. Following the algorithm suggested there, a finite $SU(5)$ grand unification theory was constructed [2]. Compared to the Minimal SUSY $SU(5)$ model, this model has three additional pairs of Higgs multiplets. Peculiar features of the theory are that each generation of matter interacts with its pair of Higgs fields and each type of Yukawa interactions is degenerate with reference to generations of fermions or, in other words, only three different Yukawa couplings y_U, y_D, y_E in accordance with three types of interaction between Higgses and matter fields exist [2]. Testing finite $SU(5)$ GUT for the compatibility with modern precise experimental data for $sin^2\theta_W$ and proton decay was performed in [3]. As it was shown there, these data could be naturally reproduced within finite $SU(5)$ GUT due to proper choosing the mass splitting of additional multiplets in the Higgs sector of the model.

Recently, it has been noted that this model can possibly explain hierarchy observed in the fermionic mass spectrum by the hierarchy of vacuum expectation values of the Higgs fields [4]. To check this suggestion, it is necessary to solve minimization conditions of the scalar Higgs potential, written on the scale of quark masses. From the mathematical point of view we must solve the system of nonlinear equations, and it is a nontrivial problem owing to a large number of Higgs fields. Nevertheless, it appears that this system can be solved analytically. Below we find an exact mathematical solution of minimization conditions and analyze types of extrema.

The paper is organized as follows. The next section is devoted to a review of the main ideas and results of [[4],[6]], and formulae, necessary for what follows, are written. Also, we consider the origin and special features of the scalar Higgs potential on the M_Z scale. The exact solution of the nonlinear minimization conditions for this potential with respect to the neutral $SU(2)$ components of Higgs fields is offered in section 3. The domains of the quantities of the potential for which the extrema, presenting interest for us from the physical point of view, exist are outlined there too. Further, in chapter 4, we analyze the types of those extremal solutions which are gauge equivalent to real field configurations. And, finally, we pick out the extremum which is physically acceptable as the only candidate for a nontrivial absolute minimum of the potential necessary for giving a fermion sector of the theory masses. In conclusion, we resume peculiar features of the potential allowing us to find the solutions of the minimization conditions analytically and to derive explicitly masses of the new Higgs particles arising after the spontaneous breaking of electroweak symmetry. The obvious phenomenological consequences for relations between quark masses are discussed briefly.

2 Higgs potential: origin and special features

The multiplet contents of the unified theory has been described in [2]. For our purpose only its Higgs part is important. It consists of four pairs of chiral superfields Φ_k and $\overline{\Phi}_k$, $k = 1,2,3,4$, in $\mathbf{5}$ and $\overline{\mathbf{5}}$ representations of $SU(5)$, respectively, and one chiral

superfield Σ in 24 representation which breaks $SU(5)$ down to $SU(3)\times SU(2)\times U(1)$. In addition to Higgs superfields, the unified theory includes chiral matter superfields usual for the supersymmetric $SU(5)$ theory of grand unification [7]. They are Ψ_i, $i=1,2,3$, in $\overline{5}$ representation and Λ_i, $i=1,2,3$, in **10** representation of $SU(5)$, where i is generation index. The contents of these superfields have standard form like the SUSY $SU(5)$ model [7]. Reviewing the main ideas and basic formulae of [6] and conserving the notation used there, we write down the Higgs and Yukawa parts of the unified finite theory Lagrangian as

$$\begin{aligned}\mathcal{L}_{Higgs+Yukawa} &= y_1\Psi_iK_{ij}\overline{\Phi}_i\Lambda_j+y_1'\Psi_i\overline{\Phi}_4\Lambda_i+\frac{y_2}{8}\Phi_i\Lambda_i\Lambda_i+\frac{y_2'}{8}\Phi_4\Lambda_i\Lambda_i\\ &+ y_3\overline{\Phi}_iS_{ij}\Sigma\Phi_j+y_3'\overline{\Phi}_4\Sigma\Phi_4+\frac{y_4}{3}\Sigma^3\\ &+ \overline{\Phi}_iM_{ij}\Phi_j+\overline{\Phi}_4M\Phi_4+\frac{M_0}{2}\Sigma^2. \qquad (1)\end{aligned}$$

In all the terms of (1) with repeating generation indices we imply sum on them. The $SU(5)$ indices are omitted here but they can easily be restored in a covariant manner. The Yukawa constants $y_1,y_1',y_2,y_2',y_3,y_3',y_4$ are expressed in polynomial functions of the gauge coupling g through the finiteness conditions [[2], [6]]. The Lagrangian is written in such a way that only the fourth Higgs pair couples with matter like the Higgs pair in the minimal SUSY $SU(5)$ model while other pairs interact with each generation separately. The orthogonal matrix S mixing them in the Higgs generation space will play a very important role in this model. Its presence is not in conflict with the finiteness conditions [2] and a possibility to introduce it in the theory always exists [[4], [6]]. The unitary matrix K_{ij} is the usual CKM matrix [2].

The mass parameters of the Lagrangian (1) are not fixed by the finiteness conditions and by doing fine-tuning we can choose M_0 and M_{ij} so that $SU(5)$ should be broken in such way that we will have only three pairs light superHiggs $SU(2)$-doublets

$$\hat{\overline{H}}_i=\begin{pmatrix}\hat{\overline{H}}_i^0\\ \hat{\overline{H}}_i^-\end{pmatrix},\quad \hat{H}_i=\begin{pmatrix}\hat{H}_i^+\\ \hat{H}_i^0\end{pmatrix},\quad i=1,2,3, \qquad (2)$$

with opposite hypercharges (-1 and 1, respectively) below the unification scale while the fourth pair Higgs $SU(2)$-doublets and all colour Higgs $SU(3)$-triplets remain heavy with masses having an order of the unification scale magnitude [[4], [6]]. At this mechanism of the $SU(5)$ violation, these three pairs of light Higgs doublets come from those Higgs $SU(5)$-quintets that were coupled with matter generations separately. In view of this, the Higgs part of the Lagrangian (1) after the $SU(5)$ symmetry violation takes the form

$$\mu_0S_{ij}\hat{\overline{H}}_i\epsilon\hat{H}_j,\quad i,j=1,2,3,\quad \mu_0\sim 10^{2-3}Gev.$$

We introduce here the following notation for brevity:

$$\overline{H}_i\epsilon H_i=\overline{H}_i^{\alpha}\epsilon_{\alpha\beta}H_i^{\beta},$$

where α,β are the $SU(2)$ indices [2].

Excluding the auxiliary components of the gauge and Higgs superfields and adding soft supersymmetry breaking terms, we get the scalar Higgs potential on the unification scale M_X [6]:

$$\begin{aligned}V &= (\mu_0^2+m_0^2)|\overline{H}_i|^2+(\mu_0^2+m_0^2)|H_i|^2+(R_0\overline{H}_i\epsilon H_i+T_0S_{ij}\overline{H}_i\epsilon H_j+h.c.)\\ &+ \frac{g^2+g'^2}{8}\left(|\overline{H}_i|^2-|H_i|^2\right)^2+\frac{g^2}{4}\left[\left(\overline{H}_i^{\dagger}\overline{H}_j\right)^*\left(\overline{H}_i^{\dagger}\overline{H}_j\right)-\left(\overline{H}_i^{\dagger}\overline{H}_i\right)^*\left(\overline{H}_j^{\dagger}\overline{H}_j\right)\right.\\ &+ \left.\left(H_i^{\dagger}H_j\right)^*\left(H_i^{\dagger}H_j\right)-\left(H_i^{\dagger}H_i\right)^*\left(H_j^{\dagger}H_j\right)+2\left(\overline{H}_i^{\dagger}H_j\right)^*\left(\overline{H}_i^{\dagger}H_j\right)\right], \qquad (3)\end{aligned}$$

where R_0, T_0 and m_0 are the soft breaking parameters. Here and below we use the notation $\overline{H}_i,H_i$ for the low scalar components of the Higgs superfields $\hat{\overline{H}}_i,\hat{H}_i$. The gauge coupling constants g and g' correspond to the $SU(2)$ and $U(1)$ gauge group of the Standard Model, respectively. Also we denoted for brevity

$$|\overline{H}_i|^2=\sum_i|\overline{H}_i^{\dagger}\overline{H}_i|,\quad |H_i|^2=\sum_i|H_i^{\dagger}H_i|.$$

In other terms of (3) we imply the convolution of the Higgs generation indices as well. The quartic terms in the Higgs scalar potential arise after re-expression of highest components of the $SU(2)$ and $U(1)$ gauge supermultiplets through their lowest dynamical components. In this sense, the situation is completely equivalent to that we have in the Minimal Supersymmetric Standard Model (MSSM) [5]. The difference is in that we have three pairs of Higgs doublets instead of one in the MSSM. It slightly complicates the form of the potential but does not result in principal distinctions.

Below the unification scale, the finiteness property is absent and all quantities start to renormalize while we are evolving our theory to low energies. The remarkable property of the theory is that the quartic terms in (3), dictated by supersymmetry invariance, maintain their form from high to low energies, apart from the usual renormalization of the gauge coupling constants [8]. The soft breaking of supersymmetry does not alter this persistency property shown by the exactly supersymmetric Lagrangian. On the contrary, the quadratic terms in (3) are slightly renormalized from their original form, and on the M_Z scale we get

$$\begin{aligned}V &= m_1^2|\overline{H}_i|^2+m_2^2|H_i|^2+(R\overline{H}_i\epsilon H_i+TS_{ij}\overline{H}_i\epsilon H_j+h.c.)+\frac{g^2+g'^2}{8}\left(|\overline{H}_i|^2-|H_i|^2\right)^2\\ &+ \frac{g^2}{4}\left[\left(\overline{H}_i^{\dagger}\overline{H}_j\right)^*\left(\overline{H}_i^{\dagger}\overline{H}_j\right)-\left(\overline{H}_i^{\dagger}\overline{H}_i\right)^*\left(\overline{H}_j^{\dagger}\overline{H}_j\right)+\left(H_i^{\dagger}H_j\right)^*\left(H_i^{\dagger}H_j\right)\right.\\ &- \left.\left(H_i^{\dagger}H_i\right)^*\left(H_j^{\dagger}H_j\right)+2\left(\overline{H}_i^{\dagger}H_j\right)^*\left(\overline{H}_i^{\dagger}H_j\right)\right]. \qquad (4)\end{aligned}$$

For the spontaneous symmetry breaking to occur, this potential should have non-trivial minimum. The vacuum expectation values of neutral components of the $SU(2)$ doublets H_i^0 and $\overline{H}_i^0$ will generate masses of fermions. The beauty of this model is in

[2] we imply that $\epsilon_{12}=1$

the minimal influence of the Yukawa constants on the mass spectrum of the theory. The main load in the explanation of the observed hierarchy in it lies on the vacuum expectation values of the Higgs fields [4]. This can be shown in the following way. From the Lagrangian (1) of the unified theory we can get supersymmetric Yukawa Lagrangian on the M_X scale:

$$\mathcal{L}_{Yukawa} = y_D K_{ij}(Q_j \epsilon \overline{H}_i)D_i + y_L(L_i \epsilon \overline{H}_i)E_i + y_U(Q_i \epsilon H_i)U_i, \qquad (5)$$

where $\hat{Q}$, $\hat{D}$, $\hat{U}$, $\hat{L}$, and $\hat{E}$ are usual matter superfields like ones in the MSSM [5], being $SU(2)$ doublets. If we reexpress (5) in terms of the superfield components, Yukawa interactions do not change their form. One loop radiative corrections do not destroy the degeneracy of the Yukawa constants with reference to fermionic generations [[9], [6]]. Thus, on the M_Z scale quarks and leptons will gain masses [4]:

$$m_{D_i} = y_D \overline{v}_i, \quad m_{U_i} = y_U v_i, \quad m_{L_i} = y_L \overline{v}_i, \qquad (6)$$

where $\overline{v}_i$, v_i are VEVs of $\overline{H}^0$, H^0, respectively. In view of this, it is especially important to have the exact solution of the minimization conditions of potential (4). We shall solve this problem in the next section.

3 Solution of the minimization conditions

For our purpose, it is convenient to rewrite our $SU(2)$ invariant potential in terms of the $SU(2)$ components of scalar Higgs doublets:

$$\begin{aligned} V &= m_1^2(\overline{H}_i^{0*}\overline{H}_i^0 + \overline{H}_i^+\overline{H}_i^-) + m_2^2(H_i^{0*}H_i^0 + H_i^+H_i^-) + \mu_{ij}(\overline{H}_i^0 H_j^0 + \overline{H}_i^{0*}H_j^{0*}) \\ &- \mu_{ij}(\overline{H}_i^- H_j^+ + \overline{H}_i^+ H_j^-) + \frac{g^2+g'^2}{8}\left(|\overline{H}_i^0|^2 + \overline{H}_i^+\overline{H}_i^- - |H_i^0|^2 - H_i^+H_i^-\right)^2 \\ &+ \frac{g^2}{2}\left(\overline{H}_i^0\overline{H}_j^{0*}\overline{H}_i^+\overline{H}_j^- - \overline{H}_i^0\overline{H}_i^{0*}\overline{H}_j^+\overline{H}_j^- + H_i^0H_j^{0*}H_j^+H_i^- - H_i^0H_i^{0*}H_j^+H_j^- \right. \\ &+ \left. \overline{H}_i^0\overline{H}_i^{0*}H_j^+H_j^- + \overline{H}_i^0H_j^0\overline{H}_i^+H_j^- + \overline{H}_i^{0*}H_j^{0*}\overline{H}_i^-H_j^+ + H_i^{0*}H_i^0\overline{H}_j^+\overline{H}_j^-\right), \end{aligned} \qquad (7)$$

where $\overline{H}_i^+ = (\overline{H}_i^-)^*$, $H_i^- = (H_i^+)^*$ and $\mu_{ij} = R\delta_{ij} + TS_{ij}$.

It is necessary for us to find the nontrivial extremum of this potential with reference to neutral components, and conditions which must be satisfied for its existence. For this aim, we need to solve the system of nonlinear equations

$$\begin{aligned} \frac{1}{2}\frac{\delta V}{\delta \overline{H}_i} &= m_1^2\overline{H}_i + \mu_{ij}H_j + \frac{g^2+g'^2}{4}\left(\overline{H}_i^2 + \overline{h}_i^2 - H_i^2 - h_i^2\right)\overline{H}_i = 0 \\ \frac{1}{2}\frac{\delta V}{\delta H_i} &= m_2^2 H_i + \mu_{ji}\overline{H}_j - \frac{g^2+g'^2}{4}\left(\overline{H}_i^2 + \overline{h}_i^2 - H_i^2 - h_i^2\right)H_i = 0 \\ \frac{1}{2}\frac{\delta V}{\delta \overline{h}_i} &= m_1^2\overline{h}_i - \mu_{ij}h_j + \frac{g^2+g'^2}{4}\left(\overline{H}_i^2 + \overline{h}_i^2 - H_i^2 - h_i^2\right)\overline{h}_i = 0 \\ \frac{1}{2}\frac{\delta V}{\delta h_i} &= m_2^2 h_i - \mu_{ji}\overline{h}_j - \frac{g^2+g'^2}{4}\left(\overline{H}_i^2 + \overline{h}_i^2 - H_i^2 - h_i^2\right)h_i = 0, \end{aligned} \qquad (8)$$

where we introduced the new notation for brevity:

$$\overline{H}_i = Re\overline{H}_i^0, \quad \overline{h}_i = Im\overline{H}_i^0, \quad H_i = ReH_i^0, \quad h_i = ImH_i^0.$$

Further, we shall also denote

$$\mathcal{H}_i = H_i + \imath h_i, \quad \overline{\mathcal{H}}_i = \overline{H}_i + \imath\overline{h}_i.$$

Although these equations are written in terms of the real and imaginary parts of the neutral components of $SU(2)$ doublets, it can easily be seen that they are invariant under the abelian gauge transformations

$$\mathcal{H}_i \to e^{\imath\alpha}\mathcal{H}_i, \quad \overline{\mathcal{H}}_i \to e^{-\imath\alpha}\overline{\mathcal{H}}_i.$$

As we can see, this system contains nonlinearity as a quadratic combination, whose square was in the potential (7). It is the key property of system allowing us to solve it analytically. As a first step, let us rewrite (8) in the matrix form denoting the quadratic combination by x:

$$\begin{aligned} &(m_1^2 + x)\overline{H} + \mu H = 0 \\ &(m_2^2 - x)H + \mu^T\overline{H} = 0 \\ &(m_1^2 + x)\overline{h} - \mu h = 0 \\ &(m_2^2 - x)h - \mu^T\overline{h} = 0 \\ &x = \frac{g^2+g'^2}{4}\left(\overline{H}^2 + \overline{h}^2 - H^2 - h^2\right). \end{aligned} \qquad (9)$$

where H, $\overline{H}$, h, and $\overline{h}$ are the real vectors in the Higgs generation space:

$$\overline{H} = \begin{pmatrix} \overline{H}_1 \\ \overline{H}_2 \\ \overline{H}_3 \end{pmatrix}, \quad H = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix}, \quad \overline{h} = \begin{pmatrix} \overline{h}_1 \\ \overline{h}_2 \\ \overline{h}_3 \end{pmatrix}, \quad h = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}. \qquad (10)$$

and μ is matrix with elements μ_{ij} operating in the generation space. It can be found that

$$det\mu = (R+T)(R^2 + T^2 + RT(trS - 1)).$$

Below we shall suggest $R \neq 0$, $T \neq 0$, $det\mu \neq 0$. Now let us reduce (9) to an equivalent system

$$\begin{aligned} &\mu\mu^T H = (m_1^2 + x)(m_2^2 - x)H \\ &\overline{H} = -(\mu^T)^{-1}(m_2^2 - x)H \\ &\mu\mu^T h = (m_1^2 + x)(m_2^2 - x)h \\ &\overline{h} = (\mu^T)^{-1}(m_2^2 - x)h. \end{aligned} \qquad (11)$$

It is obvious that if system (11) has a nontrivial solution, the condition

$$det(\mu\mu^T - (m_1^2 + x)(m_2^2 - x)I) = 0 \qquad (12)$$

should be satisfied. It is equivalent to

$$(m_1^2+x)(m_2^2-x)=\lambda RT+(R^2+T^2), \tag{13}$$

where λ is the eigenvalue of the matrix $S+S^T$:

$$det(S+S^T-\lambda I)=-(\lambda-2)(\lambda-(trS-1))^2. \tag{14}$$

We shall have two variants of the solution of the system, depending on what eigenvalue is taken into account. Let us consider both the cases.

3.1 Solution in the $\lambda=2$ case

In this case, the system (11) has the form

$$\begin{aligned}
&(S+S^T)H=2H\\
&(S+S^T)h=2h\\
&\overline{H}=-(\mu^T)^{-1}(m_2^2-x)H\\
&\overline{h}=(\mu^T)^{-1}(m_2^2-x)h\\
&(m_1^2+x)(m_2^2-x)=(R+T)^2.
\end{aligned} \tag{15}$$

Solutions of the first and second equations are:

$$H=k_1\begin{pmatrix}s_{23}-s_{32}\\ s_{31}-s_{13}\\ s_{12}-s_{21}\end{pmatrix},\qquad h=k_2\begin{pmatrix}s_{23}-s_{32}\\ s_{31}-s_{13}\\ s_{12}-s_{21}\end{pmatrix}. \tag{16}$$

The fifth equation of (15) puts the restriction on k_1 and k_2. In fact, from (15) we get

$$x=\frac{1}{2}\left(m_2^2-m_1^2\pm\sqrt{(m_1^2+m_2^2)^2-4(R+T)^2}\right). \tag{17}$$

On the other side

$$x=\frac{g^2+g'^2}{4}\left(\overline{H}^2+\overline{h}^2-H^2-h^2\right)=\frac{g^2+g'^2}{4}\left(H^2+h^2\right)\left(\frac{(m_2^2-x)^2}{(R+T)^2}-1\right).$$

Hence,

$$H^2+h^2=\left(m_1^2+m_2^2\pm\sqrt{(m_1^2+m_2^2)^2-4(R+T)^2}\right)F_\pm\left((R+T)^2\right), \tag{18}$$

$$\overline{H}^2+\overline{h}^2=\left(m_1^2+m_2^2\mp\sqrt{(m_1^2+m_2^2)^2-4(R+T)^2}\right)F_\pm\left((R+T)^2\right), \tag{19}$$

where

$$F_\pm(\kappa)=\frac{1}{g^2+g'^2}\frac{\pm(m_1^2-m_2^2)-\sqrt{(m_1^2+m_2^2)^2-4\kappa}}{\sqrt{(m_1^2+m_2^2)^2-4\kappa}}.$$

From (18) we can easily get

$$k_1^2+k_2^2=\frac{\left(m_1^2+m_2^2\pm\sqrt{(m_1^2+m_2^2)^2-4(R+T)^2}\right)F_\pm\left((R+T)^2\right)}{4-(trS-1)^2}.$$

Using the parametrization $k_1=k\cos\phi, k_2=k\sin\phi$, we can write the solution of (15) for H and h as

$$\begin{aligned}
H &= \cos\phi\sqrt{\frac{\left(m_1^2+m_2^2\pm\sqrt{(m_1^2+m_2^2)^2-4(R+T)^2}\right)F_\pm\left((R+T)^2\right)}{4-(trS-1)^2}}\begin{pmatrix}s_{23}-s_{32}\\ s_{31}-s_{13}\\ s_{12}-s_{21}\end{pmatrix},\\
h &= \sin\phi\sqrt{\frac{\left(m_1^2+m_2^2\pm\sqrt{(m_1^2+m_2^2)^2-4(R+T)^2}\right)F_\pm\left((R+T)^2\right)}{4-(trS-1)^2}}\begin{pmatrix}s_{23}-s_{32}\\ s_{31}-s_{13}\\ s_{12}-s_{21}\end{pmatrix}.
\end{aligned} \tag{20}$$

It is obvious that if $\mathcal{H}_i=H_i+\imath h_i$ is a solution of the system (15), $\mathcal{H}_i'=e^{\imath\alpha}\mathcal{H}_i$ will be its solution as well. It is a consequence of the abelian symmetry of equations (8). We need to replace only the angle ϕ in (20) on the angle $\phi+\alpha$. As it can be seen, all solutions (20) are gauge equivalent to the real ones. However, from the physical point of view, we are interested in the solutions equivalent to the real positive field configurations. It constrains the matrix S, which is a parameter of the theory, because to attain it we must choose such S that all components of the vector $\epsilon_{ijk}S_{jk}$, around which the matrix S rotates all other vectors of the three dimensional generation space, have the same signs. In addition to these constraints there are others. In order to get real and positive right-hand sides of (18) and (19) and to have the potential bounded from below in the direction of vanishing quartic terms in (7), the following conditions should be satisfied :

$$m_1^2+m_2^2>2|R+T|, \tag{21}$$

$$m_1^2m_2^2<(R+T)^2. \tag{22}$$

Arbitrariness in choosing signs in (20) originating from (17) is fixed in the following way : we take up sign if $m_1^2>m_2^2$ and low sign in the opposite case. Knowing (20) we can get from (15)

$$\begin{aligned}
\overline{H} &= -\cos\phi\ sign(R+T)\sqrt{\frac{\left(m_1^2+m_2^2\mp\sqrt{(m_1^2+m_2^2)^2-4(R+T)^2}\right)}{4-(trS-1)^2}}\times\\
&\quad\times\sqrt{F_\pm\left((R+T)^2\right)}\begin{pmatrix}s_{23}-s_{32}\\ s_{31}-s_{13}\\ s_{12}-s_{21}\end{pmatrix},\\
\overline{h} &= \sin\phi\ sign(R+T)\sqrt{\frac{\left(m_1^2+m_2^2\mp\sqrt{(m_1^2+m_2^2)^2-4(R+T)^2}\right)}{4-(trS-1)^2}}\times
\end{aligned}$$

$$\times\sqrt{F_\pm((R+T)^2)}\begin{pmatrix} s_{23}-s_{32} \\ s_{31}-s_{13} \\ s_{12}-s_{21}\end{pmatrix}. \qquad (23)$$

It is necessary to have such R and T that $(R+T)<0$. Otherwise we can not make solutions (20) and (23) real and positive simultaneously.

3.2 Solution in the $\lambda = trS-1$ case

In the same manner we can analyze the second variant of the extremal solution when $\lambda = trS-1$. Instead of (15) we get:

$$\begin{aligned}
&(S+S^T)H = (trS-1)H \\
&(S+S^T)h = (trS-1)h \\
&\overline{H} = -(\mu^T)^{-1}(m_2^2-x)H \\
&\overline{h} = (\mu^T)^{-1}(m_2^2-x)h \\
&(m_1^2+x)(m_2^2-x) = R^2+T^2+RT(trS-1). \qquad (24)
\end{aligned}$$

It is easy to show that the solution of the first equation is the vector H satisfying the equation

$$(s_{23}-s_{32})H_1+(s_{31}-s_{13})H_2+(s_{12}-s_{21})H_3 = 0. \qquad (25)$$

This is true for the second equation of (24) too. The general solution of the first two equations is

$$H = \begin{pmatrix} -K_1(s_{31}-s_{13})-K_2(s_{12}-s_{21}) \\ K_1(s_{23}-s_{32}) \\ K_2(s_{23}-s_{32})\end{pmatrix}, \qquad (26)$$

$$h = \begin{pmatrix} -k_1(s_{31}-s_{13})-k_2(s_{12}-s_{21}) \\ k_1(s_{23}-s_{32}) \\ k_2(s_{23}-s_{32})\end{pmatrix}, \qquad (27)$$

where K_1, K_2, k_1, and k_2 are some quantities. The fifth equation of (24) puts a constraint on them. In fact, from it we find

$$x = \frac{1}{2}\left(m_2^2-m_1^2 \pm \sqrt{(m_1^2+m_2^2)^2-4(R^2+T^2+RT(trS-1))}\right) \qquad (28)$$

and

$$\begin{aligned}
H^2+h^2 &= \left(m_1^2+m_2^2 \pm \sqrt{(m_1^2+m_2^2)^2-4(R^2+T^2+RT(trS-1))}\right)\times \\
&\quad \times F_\pm\left(R^2+T^2+RT(trS-1)\right), \qquad (29) \\
\overline{H}^2+\overline{h}^2 &= \left(m_1^2+m_2^2 \mp \sqrt{(m_1^2+m_2^2)^2-4(R^2+T^2+RT(trS-1))}\right)\times \\
&\quad \times F_\pm\left(R^2+T^2+RT(trS-1)\right). \qquad (30)
\end{aligned}$$

To get the real and positive right-hand sides of (29) and (30) and to have the potential bounded from below in the direction of vanishing quartic terms in (7), the following conditions should be satisfied :

$$m_1^2+m_2^2 > 2\sqrt{R^2+T^2+RT(trS-1)}, \qquad (31)$$

$$m_1^2 m_2^2 < R^2+T^2+RT(trS-1). \qquad (32)$$

Arbitrariness in choosing the signs originating from (28) must be fixed in complete analogy with the previous case. Introducing parametrization

$$K_1 = \omega\cos\phi\cos\theta_1, \quad K_2 = \omega\sin\phi\cos\theta_2, \quad k_1 = \omega\cos\phi\sin\theta_1, \quad k_2 = \omega\sin\phi\sin\theta_2,$$

we get from (29)

$$\begin{aligned}
\omega(\phi,\theta_1,\theta_2) &= \sqrt{\frac{\left(m_1^2+m_2^2 \pm \sqrt{(m_1^2+m_2^2)^2-4(R^2+T^2+RT(trS-1))}\right)}{(s_{23}-s_{32})^2+\cos^2\phi(s_{13}-s_{31})^2+\sin^2\phi(s_{12}-s_{21})^2 \\ +(s_{31}-s_{13})(s_{12}-s_{21})\sin 2\phi\cos(\theta_1-\theta_2)}}\times \\
&\quad \times\sqrt{F_\pm(R^2+T^2+RT(trS-1))}, \qquad (33)
\end{aligned}$$

$$H = \omega(\phi,\theta_1,\theta_2)\begin{pmatrix} -\cos\phi\ \cos\theta_1(s_{31}-s_{13})-\sin\phi\ \cos\theta_2(s_{12}-s_{21}) \\ \cos\phi\cos\theta_1(s_{23}-s_{32}) \\ \sin\phi\cos\theta_2(s_{23}-s_{32})\end{pmatrix}, \qquad (34)$$

$$h = \omega(\phi,\theta_1,\theta_2)\begin{pmatrix} -\cos\phi\ \sin\theta_1(s_{31}-s_{13})-\sin\phi\ \sin\theta_2(s_{12}-s_{21}) \\ \cos\phi\sin\theta_1(s_{23}-s_{32}) \\ \sin\phi\sin\theta_2(s_{23}-s_{32})\end{pmatrix}. \qquad (35)$$

We have three free parameters that are the angles θ_1, θ_2, ϕ. The gauge symmetry manifests itself in the following way. If $\mathcal{H}(\phi,\theta_1,\theta_2)$ is the solution of (24), $\mathcal{H}' = e^{i\alpha}\mathcal{H}(\phi,\theta_1,\theta_2) = \mathcal{H}(\phi,\theta_1+\alpha,\theta_2+\alpha)$ is its solution too.

Unlike the $\lambda = 2$ case, this case contains extremal configurations which are not gauge equivalent to the real ones. Gauge equivalence to the real field configurations takes place only if $\theta_1 = \theta_2 \equiv \theta$:

$$\begin{aligned}
\mathcal{H}(\phi,\theta,\theta) &= \omega(\phi,\theta,\theta)\cos\theta\begin{pmatrix} -\cos\phi(s_{31}-s_{13})-\sin\phi(s_{12}-s_{21}) \\ \cos\phi(s_{23}-s_{32}) \\ \sin\phi(s_{23}-s_{32})\end{pmatrix}. \\
&+ i\omega(\phi,\theta,\theta)\sin\theta\begin{pmatrix} -\cos\phi(s_{31}-s_{13})-\sin\phi(s_{12}-s_{21}) \\ \cos\phi(s_{23}-s_{32}) \\ \sin\phi(s_{23}-s_{32})\end{pmatrix}. \qquad (36)
\end{aligned}$$

Let us note that like in the $\lambda = 2$ case, for the solution to have physical interest it is necessary that all real parts in (36) have the same signs. This contradicts an analogous constraint in the $\lambda = 2$ case. Indeed, as it can be seen from (25), if all components of the vector $\epsilon_{ijk}S_{jk}$ have the same signs, the coordinates of every point on the plane,

orthogonal to it and containing zero point, have different signs. Vice versa, if the coordinates of any point belonging to this plane have the same signs, the coordinates of the vector $\epsilon_{ijk}S_{jk}$ have different signs. This situation is illustrated Fig.1, where the real solutions of the minimization conditions are depicted. The solution corresponding to the $\lambda = 2$ case lies on the axis in the Higgs generation space, around which the matrix S performs rotations. The solution corresponding to the $\lambda = trS - 1$ case lies on the circle in the plane orthogonal to this axis. The angle ϕ in (36) determines the position of the extremum on this circle. Knowing the expressions for H and h we can obtain from (24) the expressions for $\overline{H}$ and $\overline{h}$. Let us note that after the gauge transformation of (36) to the real configuration the extremal solution for $\overline{\mathcal{H}}$ will become real too. It can be found by using the following formula

$$\overline{H} = \sqrt{\frac{\left(m_1^2+m_2^2 \mp \sqrt{(m_1^2+m_2^2)^2 - 4(R^2+T^2+RT(trS-1))}\right)}{(s_{23}-s_{32})^2+\cos^2\phi(s_{13}-s_{31})^2+\sin^2\phi(s_{12}-s_{21})^2 \\ +(s_{31}-s_{13})(s_{12}-s_{21})\sin 2\varphi}} \times$$
$$\times\sqrt{F_{\pm}(R^2+T^2+RT(trS-1))}\times$$
$$\times\frac{R+TS}{\sqrt{R^2+T^2+RT(trS-1)}}\begin{pmatrix} -\cos\phi(s_{31}-s_{13})-\sin\phi(s_{12}-s_{21}) \\ \cos\phi(s_{23}-s_{32}) \\ \sin\phi(s_{23}-s_{32})\end{pmatrix}.$$

If (21), (22) and (31), (32) are satisfied, both variants of the solution can occur. To decide finally which extremum is suitable for us from the physical point of view, we need to determine its type. However, yet now we can say that the solution corresponding to the case $\lambda = trS - 1$ has an additional global symmetry. Indeed, if $\mathcal{H}_i$ and $\overline{\mathcal{H}}_i$ are the solutions of (24), the field configurations $O_{ij}\mathcal{H}_j$ and $O_{ij}\overline{\mathcal{H}}_j$, where O is some orthogonal matrix commuting with S, will be solutions of (24) as well. Breaking this symmetry generates additional Goldstone bosons, what will be demonstrated explicitly in the next section.

4 Higgs masses and types of extrema

As it has been noted, we are interested in the extremal field configurations which are gauge equivalent to the real ones. In this case, the phases of $\mathcal{H}$ and $\overline{\mathcal{H}}$ can be put equal to zero simultaneously, and we get

$$\mathcal{H}_i = v_i + \imath 0, \quad \overline{\mathcal{H}}_i = \overline{v}_i + \imath 0. \tag{37}$$

Let us determine the type of extrema at these points. To do this, we need to find the eigenvalues of the matrices of second derivatives of the potential (7) at this point. The matrix of second derivatives of (7) with respect to the real parts of the neutral $SU(2)$

components H_i and $\overline{H}_i$

$$\begin{pmatrix} \frac{1}{2}\frac{\delta^2 V}{\delta\overline{H}_i\delta\overline{H}_j} & \frac{1}{2}\frac{\delta^2 V}{\delta\overline{H}_i\delta H_j} \\ \frac{1}{2}\frac{\delta^2 V}{\delta H_i\delta\overline{H}_j} & \frac{1}{2}\frac{\delta^2 V}{\delta H_i\delta H_j}\end{pmatrix}$$

at the point $H_i = v_i,\ h_i = 0, H_i^+ = 0, \overline{H}_i = \overline{v}_i,\ \overline{h}_i = 0, \overline{H_i^-} = 0$, has the form [6]

$$\begin{pmatrix} (m_1^2+x)\delta_{ij}+\frac{1}{2}(g^2+g'^2)\overline{v}_i\overline{v}_j & \mu_{ij}-\frac{1}{2}(g^2+g'^2)\overline{v}_i v_j \\ \mu_{ji}-\frac{1}{2}(g^2+g'^2)\overline{v}_j v_i & (m_2^2-x)\delta_{ij}+\frac{1}{2}(g^2+g'^2)v_i v_j\end{pmatrix}. \tag{38}$$

The matrix of second derivatives of (7) with respect to the imaginary parts of the neutral $SU(2)$ components h_i and $\overline{h}_i$

$$\begin{pmatrix} \frac{1}{2}\frac{\delta^2 V}{\delta\overline{h}_i\delta\overline{h}_j} & \frac{1}{2}\frac{\delta^2 V}{\delta\overline{h}_i\delta h_j} \\ \frac{1}{2}\frac{\delta^2 V}{\delta h_i\delta\overline{h}_j} & \frac{1}{2}\frac{\delta^2 V}{\delta h_i\delta h_j}\end{pmatrix}$$

at the same point has the form [6]

$$\begin{pmatrix} (m_1^2+x)\delta_{ij} & -\mu_{ij} \\ -\mu_{ji} & (m_2^2-x)\delta_{ij}\end{pmatrix}. \tag{39}$$

And, finally, the matrix of second derivatives of (7) with respect to the charged $SU(2)$ components $\overline{H^-}$ and $\overline{H^+}$

$$\begin{pmatrix} \frac{\delta^2 V}{\delta\overline{H_i^+}\delta\overline{H_j^-}} & \frac{\delta^2 V}{\delta\overline{H_i^+}\delta H_j^-} \\ \frac{\delta^2 V}{\delta H_i^+\delta\overline{H_j^-}} & \frac{\delta^2 V}{\delta H_i^+\delta H_j^-}\end{pmatrix}$$

has the form [6]:

$$\begin{pmatrix} (m_1^2+z)\delta_{ij}+\frac{1}{2}g^2\overline{v}_i\overline{v}_j & -\mu_{ij}+\frac{1}{2}g^2\overline{v}_i v_j \\ -\mu_{ji}+\frac{1}{2}g^2\overline{v}_j v_i & (m_2^2-z)\delta_{ij}+\frac{1}{2}g^2 v_i v_j\end{pmatrix}, \tag{40}$$

where $z = \frac{g'^2-g^2}{4}(\overline{v}_i^2 - v_i^2)$. In complete analogy with the MSSM [5], the eigenvalues of these matrices are the masses of CP-even, CP-odd and charged Higgses [6]. For instance we shall find the eigenvalues of the mass matrix (40) which are the squares of masses of charged Higgses. At this moment, we do not fix which variant of the extremum is realized. We have only in mind that these vectors v and $\overline{v}$ obey the equations

$$\begin{aligned} -\mu v &= (m_1^2+x)\overline{v} \\ -\mu^T\overline{v} &= (m_2^2-x)v. \end{aligned} \tag{41}$$

Now we introduce the matrices

$$u = \begin{pmatrix} v_1 & 0 & 0 \\ v_2 & 0 & 0 \\ v_3 & 0 & 0 \end{pmatrix}, \quad \overline{u} = \begin{pmatrix} \overline{v}_1 & 0 & 0 \\ \overline{v}_2 & 0 & 0 \\ \overline{v}_3 & 0 & 0 \end{pmatrix}. \tag{42}$$

and write the following system of linear equations:

$$\left[(m_1^2 + z - \xi)I + \frac{g^2}{2}\overline{u}\overline{u}^T\right] f - \left[\mu - \frac{g^2}{2}\overline{u}u^T\right] \overline{f} = 0$$
$$\left[(m_2^2 - z - \xi)I + \frac{g^2}{2}uu^T\right] \overline{f} - \left[\mu^T - \frac{g^2}{2}u\overline{u}^T\right] f = 0, \tag{43}$$

where f and $\overline{f}$ are some three dimensional vectors. The system (43) has nontrivial solution if corresponding matrix has determinant equal to zero. To avoid the cumbersome formulae we denote:

$$m_1 \equiv m_1^2 + x, \quad m_2 \equiv m_2^2 - x, \quad A \equiv -\frac{g^2}{2}(\overline{v}^2 - v^2)$$

and absorb the factor $\sqrt{\frac{g^2}{2}}$ in v and $\overline{v}$. Then, (43) take the following form:

$$\left[(m_1 - A - \xi)I + \overline{u}\overline{u}^T\right] f - \left[\mu - \overline{u}u^T\right] \overline{f} = 0$$
$$\left[(m_2 + A - \xi)I + uu^T\right] \overline{f} - \left[\mu^T - u\overline{u}^T\right] f = 0. \tag{44}$$

It can be shown that if the conditions (21), (22) and (31), (32) are satisfied, we have

$$det(\mu - \overline{u}u^T) \neq 0.$$

Taking this into account we get from (44)

$$\left[\left((m_2 + A - \xi)I + uu^T\right)\left(\mu - \overline{u}u^T\right)^{-1}\left((m_1 - A - \xi)I + \overline{u}\overline{u}^T\right)\right] f - \left[\mu^T - u\overline{u}^T\right] f = 0. \tag{45}$$

The condition for a nontrivial solution for f to exist in (45) is

$$det\left[\left((m_2 + A - \xi)I + uu^T\right)\left(\mu - \overline{u}u^T\right)^{-1}\left((m_1 - A - \xi)I + \overline{u}\overline{u}^T\right) - \left(\mu^T - u\overline{u}^T\right)\right] = 0. \tag{46}$$

Using (41), after some transformations we get from (46)

$$det\left[(m_2 + A - \xi)(m_1 - A - \xi)I + \frac{uu^T}{v^2}(m_1 m_2 - (m_1 - A)(m_2 + A)) - \right.$$
$$\left. - uu^T\left(\frac{(2m_1 + v^2)\xi^2 - \xi(m_1 + m_2)(m_1 + v^2)}{(m_1 + v^2)^2}\right) - \mu\mu^T\right] = 0. \tag{47}$$

It can be calculated that

$$det\left(aI + buu^T - RT(S + S^T)\right) = (a - 2RT)(a - RT(trS - 1))^2 +$$
$$+b\left(a^2v^2 - 2aRTB_1 + R^2T^2(trS - 1)B_2\right), \tag{48}$$
$$B_1 \equiv v^2 trS - \frac{1}{2}tr\left(\left(S + S^T\right)uu^T\right),$$
$$B_2 \equiv v^2(trS + 1) - tr\left(\left(S + S^T\right)uu^T\right).$$

Here, we have the distinctions between the first and second variants of the extremum. In the $\lambda = 2$ case, we get $B_1 = B_2 = v^2(trS - 1)$ while in the $\lambda = (trS - 1)$ case $B_1 = \frac{1}{2}v^2(trS + 1), \quad B_2 = 2v^2$. Using (48) we can see that a characteristic equation can be factored in both the cases and following eigenvalues can be found: in the $\lambda = 2$ case

$$\xi_1 = \xi_2 = \frac{1}{2}(m_1^2 + m_2^2) + \left[R^2 + T^2 + RT(trS - 1)\right.$$
$$\left. + \left(\frac{g^2}{g^2 + g'^2}(m_2^2 - m_1^2) \pm \frac{1}{2}\frac{g^2 - g'^2}{g^2 + g'^2}\sqrt{(m_1^2 + m_2^2)^2 - 4(R + T)^2}\right)^2\right]^{\frac{1}{2}},$$
$$\xi_3 = \xi_4 = \frac{1}{2}(m_1^2 + m_2^2) - \left[R^2 + T^2 + RT(trS - 1)\right.$$
$$\left. + \left(\frac{g^2}{g^2 + g'^2}(m_2^2 - m_1^2) \pm \frac{1}{2}\frac{g^2 - g'^2}{g^2 + g'^2}\sqrt{(m_1^2 + m_2^2)^2 - 4(R + T)^2}\right)^2\right]^{\frac{1}{2}},$$
$$\xi_5 = m_1^2 + m_2^2 + \frac{g^2}{2}(v^2 + \overline{v}^2) = m_1^2 + m_2^2 +$$
$$+ \frac{g^2}{g^2 + g'^2}(m_1^2 + m_2^2)\frac{\pm(m_1^2 - m_2^2) - \sqrt{(m_1^2 + m_2^2)^2 - 4(R + T)^2}}{\sqrt{(m_1^2 + m_2^2)^2 - 4(R + T)^2}},$$
$$\xi_6 = 0 \tag{49}$$

and in the $\lambda = (trS - 1)$ case

$$\xi_1 = \frac{1}{2}(m_1^2 + m_2^2) + \left[(R^2 + T^2 + RT(trS - 1))\right.$$
$$\left. + \left(\frac{g^2}{g^2 + g'^2}(m_2^2 - m_1^2) \pm \frac{1}{2}\frac{g^2 - g'^2}{g^2 + g'^2}\sqrt{(m_1^2 + m_2^2)^2 - 4(R^2 + T^2 + RT(trS - 1))}\right)^2\right]^{\frac{1}{2}},$$
$$\xi_2 = \frac{1}{2}(m_1^2 + m_2^2) - \left[(R^2 + T^2 + RT(trS - 1))\right.$$
$$\left. + \left(\frac{g^2}{g^2 + g'^2}(m_2^2 - m_1^2) \pm \frac{1}{2}\frac{g^2 - g'^2}{g^2 + g'^2}\sqrt{(m_1^2 + m_2^2)^2 - 4(R^2 + T^2 + RT(trS - 1))}\right)^2\right]^{\frac{1}{2}},$$
$$\xi_3 = \frac{1}{2}(m_1^2 + m_2^2) + \left[(R + T)^2\right.$$
$$\left. + \left(\frac{g^2}{g^2 + g'^2}(m_2^2 - m_1^2) \pm \frac{1}{2}\frac{g^2 - g'^2}{g^2 + g'^2}\sqrt{(m_1^2 + m_2^2)^2 - 4(R^2 + T^2 + RT(trS - 1))}\right)^2\right]^{\frac{1}{2}},$$

$$\xi_4 = \frac{1}{2}(m_1^2+m_2^2) - \left[(R+T)^2 + \left(\frac{g^2}{g^2+g'^2}(m_2^2-m_1^2) \pm \frac{1}{2}\frac{g^2-g'^2}{g^2+g'^2}\sqrt{(m_1^2+m_2^2)^2-4(R^2+T^2+RT(trS-1))}\right)^2\right]^{\frac{1}{2}},$$

$$\xi_5 = m_1^2+m_2^2+\frac{g^2}{2}(v^2+\overline{v}^2) = m_1^2+m_2^2 + \frac{g^2}{g^2+g'^2}(m_1^2+m_2^2)\frac{\pm(m_1^2-m_2^2)-\sqrt{(m_1^2+m_2^2)^2-4(R^2+T^2+RT(trS-1))}}{\sqrt{(m_1^2+m_2^2)^2-4(R^2+T^2+RT(trS-1))}},$$

$$\xi_6 = 0. \qquad (50)$$

Acting in the same manner we can find the eigenvalues for the mass matrix of CP-even Higgses (38) in the $\lambda = 2$ case

$$\xi_1 = \frac{\pm(m_1^2-m_2^2)(m_1^2+m_2^2)}{2\sqrt{(m_1^2+m_2^2)^2-4(R+T)^2}} + \left[\frac{(m_1^2-m_2^2)^2(m_1^2+m_2^2)^2}{4((m_1^2+m_2^2)^2-4(R+T)^2)} + (m_1^2+m_2^2)^2-4(R+T)^2 \mp (m_1^2-m_2^2)\sqrt{(m_1^2+m_2^2)^2-4(R+T)^2}\right]^{\frac{1}{2}},$$

$$\xi_2 = \frac{\pm(m_1^2-m_2^2)(m_1^2+m_2^2)}{2\sqrt{(m_1^2+m_2^2)^2-4(R+T)^2}} - \left[\frac{(m_1^2-m_2^2)^2(m_1^2+m_2^2)^2}{4((m_1^2+m_2^2)^2-4(R+T)^2)} + (m_1^2+m_2^2)^2-4(R+T)^2 \mp (m_1^2-m_2^2)\sqrt{(m_1^2+m_2^2)^2-4(R+T)^2}\right]^{\frac{1}{2}},$$

$$\xi_3 = \xi_4 = \frac{1}{2}\left((m_1^2+m_2^2)+\sqrt{(m_1^2+m_2^2)^2-4RT(3-trS)}\right).$$

$$\xi_5 = \xi_6 = \frac{1}{2}\left((m_1^2+m_2^2)-\sqrt{(m_1^2+m_2^2)^2-4RT(3-trS)}\right) \qquad (51)$$

and in the $\lambda = trS-1$ case:

$$\xi_1 = \frac{\pm(m_1^2-m_2^2)(m_1^2+m_2^2)}{2\sqrt{(m_1^2+m_2^2)^2-4(R^2+T^2+RT(trS-1))}} + \left[\frac{(m_1^2-m_2^2)^2(m_1^2+m_2^2)^2}{4((m_1^2+m_2^2)^2-4(R^2+T^2+RT(trS-1)))} + (m_1^2+m_2^2)^2-4(R^2+T^2+RT(trS-1)) \mp (m_1^2-m_2^2)\sqrt{(m_1^2+m_2^2)^2-4(R^2+T^2+RT(trS-1))}\right]^{\frac{1}{2}},$$

$$\xi_2 = \frac{\pm(m_1^2-m_2^2)(m_1^2+m_2^2)}{2\sqrt{(m_1^2+m_2^2)^2-4(R^2+T^2+RT(trS-1))}} - \left[\frac{(m_1^2-m_2^2)^2(m_1^2+m_2^2)^2}{4((m_1^2+m_2^2)^2-4(R^2+T^2+RT(trS-1)))} + (m_1^2+m_2^2)^2-4(R^2+T^2+RT(trS-1)) \mp (m_1^2-m_2^2)\sqrt{(m_1^2+m_2^2)^2-4(R^2+T^2+RT(trS-1))}\right]^{\frac{1}{2}},$$

$$\xi_3 = \frac{1}{2}\left((m_1^2+m_2^2)+\sqrt{(m_1^2+m_2^2)^2+4RT(3-trS)}\right),$$

$$\xi_4 = \frac{1}{2}\left((m_1^2+m_2^2)-\sqrt{(m_1^2+m_2^2)^2+4RT(3-trS)}\right),$$

$$\xi_5 = m_1^2+m_2^2,$$

$$\xi_6 = 0, \qquad (52)$$

and eigenvalues for the mass matrix of CP-odd Higgses (39) in the $\lambda = 2$ case:

$$\xi_1 = 0,$$

$$\xi_2 = m_1^2+m_2^2,$$

$$\xi_3 = \xi_4 = \frac{1}{2}\left((m_1^2+m_2^2)+\sqrt{(m_1^2+m_2^2)^2-4RT(3-trS)}\right),$$

$$\xi_5 = \xi_6 = \frac{1}{2}\left((m_1^2+m_2^2)-\sqrt{(m_1^2+m_2^2)^2-4RT(3-trS)}\right) \qquad (53)$$

and in the $\lambda = trS-1$ case

$$\xi_1 = \xi_2 = 0,$$

$$\xi_3 = \xi_4 = m_1^2+m_2^2,$$

$$\xi_5 = \frac{1}{2}\left((m_1^2+m_2^2)+\sqrt{(m_1^2+m_2^2)^2+4RT(3-trS)}\right),$$

$$\xi_6 = \frac{1}{2}\left((m_1^2+m_2^2)-\sqrt{(m_1^2+m_2^2)^2+4RT(3-trS)}\right). \qquad (54)$$

Let us consider the expressions for the eigenvalues of CP-even Higgses mass matrix (38). It is easy to show that if the conditions (21), (22) and (31), (31) are satisfied, type of the extremum depends on $sign(RT)$. In fact, if $RT > 0$, the matrix (38) has all non-negative eigenvalues (51) in the $\lambda = 2$ case while another extremum is the saddle point. Otherwise, if $RT < 0$, the matrix (38) has all non-negative eigenvalues (52) in the $\lambda = trS-1$ case while extremum corresponding to the $\lambda = 2$ case appears saddle point. We can observe the same situation considering eigenvalues of the matrices (39) and (40). Having written the matrix of second derivatives of the potential (7) at zero, we can see that zero is the saddle point for any sign of RT (if the conditions (21), (22) and (31), (32) are satisfied). Goldstone bosons, which we have in the $\lambda = 2$ case, are the results of electroweak symmetry breaking. They generate masses of gauge Z-boson (CP-odd Goldstone boson) and $W^{\pm}$ bosons (charged Goldstone bosons). The additional zero

eigenvalues in the $\lambda = trS - 1$ case correspond to the global symmetry breaking. As we have written in the previous section, this global symmetry is the symmetry of the potential (7) with respect to the rotation of fields in the Higgs generation space which is performed by the orthogonal matrices commuting with S. Taking into account the presence of these additional Goldstone bosons, we may conclude that this extremum is not suitable from the physical point of view. For final conclusion to make, let us calculate the significance of the potential (7) on the extremal configurations. In both the cases we get

$$V_{ext} = -\frac{2}{g^2 + g'^2}x^2. \tag{55}$$

Then, using (17) and (28), we calculate

$$V_{ext}^{\lambda=2} = -\frac{1}{2(g^2+g'^2)}\left(|m_1^2 - m_2^2| - \sqrt{(m_1^2+m_2^2)^2 - 4(R+T)^2}\right)^2, \tag{56}$$

$$V_{ext}^{\lambda=trS-1} = -\frac{1}{2(g^2+g'^2)}\left(|m_1^2 - m_2^2| - \sqrt{(m_1^2+m_2^2)^2 - 4(R^2+T^2+RT(trS-1))}\right)^2. \tag{57}$$

Thus, for $RT > 0$ the absolute minimum of the potential (7) is the extremum corresponding to $\lambda = 2$, while zero and extremum corresponding to $\lambda = trS - 1$ are the saddle points. In the opposite case, for $RT < 0$ the absolute minimum is the extremum corresponding to $\lambda = trS - 1$ while zero and extremum corresponding to the $\lambda = 2$ case are the saddle points. We must discard the second extremum due to additional Goldstone bosons. Taking into account the afore-mentioned arguments and conclusions of the previous section, we summarize that the potential (7) has an absolute minimum with respect to neutral components of the $SU(2)$ scalar Higgs doublets, interesting physically, on the field configurations (20) and (23) under the following restrictions on the quantities of the potential (7):

$$\begin{aligned}
&R + T < 0,\\
&\epsilon_{ijk}S_{jk} > 0 \text{ for any } i,\\
&m_1^2 + m_2^2 > 2|R+T|,\\
&m_1^2 m_2^2 < (R+T)^2,\\
&RT > 0.
\end{aligned}$$

Finally, let us make some remarks regarding extremal field configurations in (34), (35) discarded by us in view of their inequivalency to the real ones. The potential (7) on these configurations equals the significance (57), which is greater than the absolute minimum (56). Moreover, if our system is in the vicinity of this extremum, the afore-mentioned global symmetry in the generation space is broken and additional Goldstone bosons appear.

5 Summary

The main reason why we have succeeded in the exact solution of the system (8) is that this system includes nonlinearity as a whole having the form of a quadratic combination

of unknowns. This type of nonlinearity is generated by quartic terms in the potential (7), which, in their turn, arise after excluding auxiliary non-dynamical components of gauge supermultiplets. Therefore, this form of quartic terms in the potential is typical of the $N = 1$ supergravity GUT's with enlarged Higgs sector [10]. The quadratic part of the potential (7) has a specific form dictated by finiteness. This fact allowed us to find the nonlinear combination in (8) explicitly. Let us note also that, as it is not difficult to see, our result for absolute minimum configurations (20) and (23) is a simple generalization of the analogous result in the MSSM [5].

In conclusion, we would like to attract attention to the interesting phenomenological predictions for the quark mass spectrum. After the transition of the system to the absolute minimum (56) of the potential on the field configurations (20) and (23), we fix the phases of these configurations to equal zero, and the following relations between up and down quark masses can be observed:

$$\frac{m_u}{m_d} = \frac{m_c}{m_s} = \frac{m_t}{m_b}.$$

Quark masses in these relations are running masses and must be taken on the M_Z scale [6]. At the same time, the hierarchy between quark generations is completely controlled by the matrix S that is the parameter of the theory. Parametrizing this orthogonal matrix by three Euler angles $\theta_1, \theta_2, \theta_3$, we can get the following hierarchy relations between up quarks:

$$m_u : m_c : m_t = \cos\frac{\theta_1}{2}\ \sin\frac{\theta_2+\theta_3}{2} : \sin\frac{\theta_1}{2}\ \sin\frac{\theta_2-\theta_3}{2} : \sin\frac{\theta_1}{2}\ \cos\frac{\theta_2-\theta_3}{2}.$$

It is clear that we can fit these angles in order to guarantee any hierarchy. Unfortunately, we have not succeeded connecting S with other parameters of the theory. A complete analysis of this model with numerical results for masses of all particles of the theory is in preparation [6] and will be published elsewhere.

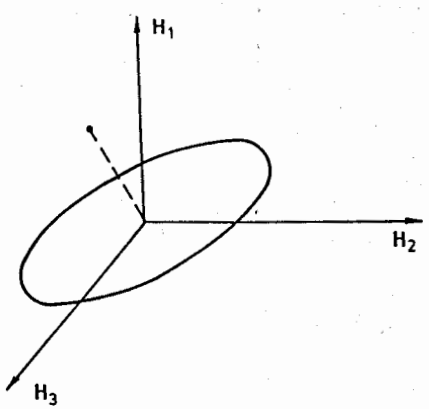

Figure 1: In this figure the real solutions of the minimization conditions are depicted. The solution corresponding to the $\lambda = 2$ case lies on the axis in the Higgs generation space, around which the matrix S performs rotations. The solution corresponding to the $\lambda = trS - 1$ case lies on the circle in the plane orthogonal to this axis. The angle ϕ in (36) determines the position of the extremum on this circle.

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References

[1] D.I. Kazakov, Phys.Lett.179B (1986) 352
D.I. Kazakov, Mod.Phys.Lett.A, Vol.2, No.9 (1987) 663

[2] A.V. Ermushev, D.I. Kazakov, O.V. Tarasov, Nucl.Phys.B281 (1987) 72

[3] D.I. Kazakov and I.N. Kondrashuk, Int.J.Mod.Phys.A, Vol.7, No.16 (1992) 3869

[4] D.I. Kazakov, *On a possible explanation of the origin of the quark mass spectrum*, preprint JINR E2-94-162

[5] H.E. Haber, *Introductory low-energy supersymmetry*, preprint SCIPP 92/33

[6] D.I. Kazakov, M.Yu. Kalmykov, I.N. Kondrashuk, A.V. Gladyshev, *Low energy predictions of supersymmetric finite $SU(5)$ model*, in preparation

[7] S. Dimopoulos, H. Georgi, Nucl.Phys.B193 (1981) 150
N. Sakai, Zeit. für Phys.C11 (1982) 153

[8] R. Barbieri, Rivista del Nuovo Cimento, Vol.11, No.4, (1988) 1

[9] J.E. Björkman, D.R.T. Jones, Nucl.Phys.B259 (1985) 533
J.E. Björkman, D.R.T. Jones, and S. Raby, Nucl.Phys.B259 (1985) 503

[10] L.E. Ibañez, Phys.Lett.139B (1984) 363

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Кондрашук И.Н. E2-95-63

Минимизация скалярного хиггсовского потенциала
в конечной суперсимметричной теории великого объединения

Найдено точное математическое решение уравнений минимизации скалярного хиггсовского потенциала в конечной суперсимметричной теории великого объединения. Аналитически исследован тип экстремумов потенциала и определены массы хиггсовских частиц, возникающих после спонтанного нарушения электрослабой симметрии. Даны условия существования абсолютного минимума потенциала, приемлемого с физической точки зрения. Оказалось, что этот минимум является простым обобщением аналогичного решения в минимальной суперсимметричной стандартной модели. Феноменологические следствия найденного решения кратко обсуждаются.

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Kondrashuk I.N. E2-95-63

Minimization of the Scalar Higgs Potential
in the Finite Supersymmetric Grand Unified Theory

Exact mathematical solution of the minimization conditions of scalar the Higgs potential of the Finite Supersymmetric Grand Unification Theory is proposed and extremal field configurations are found. Types of extrema are investigated and masses of the new Higgs particles arisen after electroweak symmetry breaking are derived analytically. The conditions for existing of physically acceptable minimum are given. As it appears, this minimum is simple generalization of the analogous solution in the Minimal Supersymmetric Standard Model. Phenomenological consequences are discussed briefly.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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