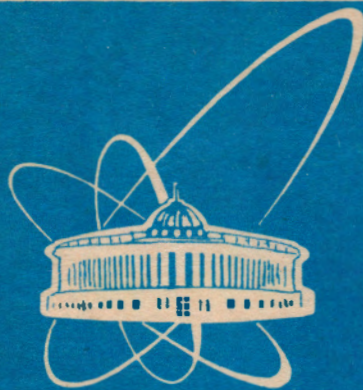


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L.V.Avdeev¹

RECURRENCE RELATIONS
FOR THREE-LOOP PROTOTYPES
OF BUBBLE DIAGRAMS WITH A MASS²

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¹E-mail: avdeevL@thsun1.jinr.dubna.su or avdeevL@hrz.uni-bielefeld.de

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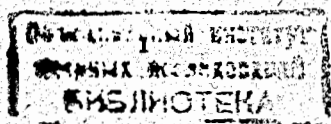
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Vacuum (bubble) Feynman integrals (without external momenta) appear as low-energy limits of certain physical amplitudes or as Taylor coefficients of multipoint Green functions. The coefficients can then be used to recover the functions in the whole complex plane of momenta [1]. The presence of virtual heavy particles, like the top quark in the quantum chromodynamics (QCD), generates an effective high-energy scale, which makes perturbation theory applicable owing to asymptotic freedom. In quadratically and linearly divergent diagrams the contributions of heavy particles are enhanced by the power of the mass, so that light particles can well be considered as massless. This leaves us with an important special case of just one nonzero mass.

After performing the Dirac and Lorentz algebra, any vacuum diagram can be reduced to some linear combinations of scalar bubble integrals. In the three-loop case with the full tetrahedron topology of the diagram, all scalar products in the numerator can be expressed through the quadratic combinations in the denominators. Thus, in the most general case, only a product of some powers of the denominators should be integrated. A prototype defines the arrangement of massive and massless lines in a diagram. Individual integrals are specified by the powers of the denominators, called indices of the lines.

In the dimensional regularization with $N = 4 - 2\epsilon$, any massless bubbles are trivially equal to zero, so that at least one massive line should be present. Fig. 1 displays all possible three-loop prototypes. It was convenient also to distinguish some reduced prototypes with a line missing, E_{2-4} , since they are generated in evaluating several different full prototypes. The B_M and B_N types have been completely analyzed in Ref. [2] and need not be discussed here.

The method of recurrence relations [3, 2] connects integrals of the same prototype but with different values of the indices. Using these relations ingeniously enough, one can reduce any integral to a limited set of so-called master integrals. However, that remains still a kind of art without any strict assertions as to the minimal set of the master integrals or the most efficient strategy. Let us derive a relation [4] for a generic triangle subgraph with masses on its lines m_1 , m_2 , and m_3 , line momenta p_1 , $p_2 = p_1 - p_{12}$, and



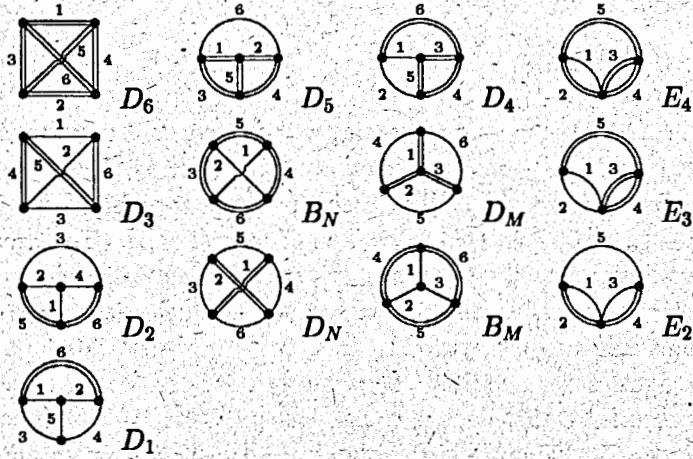


Figure 1: The three-loop scalar bubble prototypes with one mass. Double (single) lines refer to massive (massless) propagators in the momentum representation. Any line may have an integer index, a power of the denominator. The indices stay as arguments of the corresponding functions. Numbers define the ordering of the arguments.

$p_3 = p_1 - p_{13}$, and indices j_1 , j_2 , and j_3 , respectively:

$$0 = \int d^N p_1 \frac{\partial}{\partial p_1^\mu} \frac{p_1^\mu}{c_1^{j_1} c_2^{j_2} c_3^{j_3}} = \int \frac{d^N p_1}{c_1^{j_1} c_2^{j_2} c_3^{j_3}} (N - 2j_1 - j_2 - j_3 + j_1 \frac{2m_1^2}{c_1} + j_2 \frac{m_1^2 + m_2^2 - m_{12}^2 + c_{12} - c_1}{c_2} + j_3 \frac{m_1^2 + m_3^2 - m_{13}^2 + c_{13} - c_1}{c_3}), \quad (1)$$

where $c_k = p_k^2 + m_k^2$. Dividing or multiplying by c_k just increments or decrements index j_k of the k th line. The corresponding operator can be denoted by K^\pm [2]. For an arbitrary L -loop diagram, integrating by parts the first derivatives provides us with L^2 linearly independent relations in total. It is convenient to refer to triangle recurrence relations by specifying only the line numbers: {123} for Eq. (1), line 1 being the base. Half sum of the relations for three faces of a tetrahedron as their base lines form a triangle, like {124}, {534}, and {623} in D_3 , gives a relation {dim} that is evident on dimensional grounds:

$$[\frac{3}{2}N - j_1 - \dots - j_6 + m^2(j_4 4^+ + j_5 5^+ + j_6 6^+)]D_3(j_1, \dots, j_6) = 0. \quad (2)$$

Now follows a brief description of evaluating various prototypes. In Fig. 1 they have been ordered from the 'most difficult' to the 'simplest'. Whenever in D_6 a denominator is absent ($j_k \leq 0$) the diagram is immediately reduced to the 'simpler' D_5 type by expanding the power of the polynomial. A typical trick to bring any positive index down to 1 is as follows. A combination is sought, like $3\{146\} - \{416\} - \{614\}$ for $j_1 > 1$ in D_6 , in which only one 'highest' term $m^2 j_1 1^+$ is present, all others having the sum of the indices less by one. The highest term can be expressed through the others until the index on the line reaches 1. Thus we arrive at the master integral $D_6(1, \dots, 1)$.

If D_5 has $j_5 \leq 0$, it is reduced to B_N ; $j_{1-4} \leq 0$ to D_4 . As $j_6 > 0$, it is profitable to solve {612} (no m^2 for a massless exchange between particles of unchanged masses [4]) with respect to the free term. Eventually, this brings $j_{3,4}$ or j_6 to zero. To get rid of the numerator $j_6 < 0$, a denominator on an adjacent line is typically used: if for example $j_1 > 1$, {315} can be solved relative to $1^+ 6^-$. Otherwise, as $j_{1-4} = 1$, the quadratic denominators can be created by solving {126} with respect to the free term. The irreducible master integral is $D_5(1, 1, 1, 1, 1, 0)$.

In D_4 , $j_6 \leq 0$ leads to B_M ; $j_5 \leq 0$ to D_M ; $j_{3,4} \leq 0$ to D_3 ; and $j_{1,2} = 0$ to E_4 . The numerator on line 1 can be eliminated by {246} if $j_6 > 1$; by $2\{\text{dim}\} - \{534\}$ if $j_3 > 1$; by $\{\text{dim}\} - \{345\}$ if $j_5 > 1$; by $\{\text{dim}\} - \{624\}$ if $j_2 \neq 1$; or by {215} otherwise. Solving {215} relative to $m^2 1^+$ diminishes the denominator $j_1 > 1$; {512} reduces $j_5 > 1$; {435} - {534} reduces $j_3 > 1$; {136} + {246} reduces $j_6 > 1$, leading to $D_4(1, \dots, 1)$.

As $j_{4,6} \leq 0$ in D_3 , this is D_2 ; $j_5 \leq 0 \Rightarrow D_N$; $j_2 \leq 0 \Rightarrow B_N$. The numerator $j_1 < 0$ is normally reduced by $2\{\text{dim}\} - \{534\}$ as $j_4 > 1$; by {623} as $j_2 > 1$; by {236} as $j_6 > 1$; by $2\{\text{dim}\} - \{435\}$ as $j_5 > 1$; and {dim} creates $j_{4-6} > 1$ otherwise. The special case $j_1 = j_3 = 0$ is more efficiently worked out as E_4 with $j_5 = 0$. The remaining denominator on line 5 in D_3 is brought down to 1 by {516}; $j_2 > 1$ by $(1 - j_2)\{236\} + j_3 3^+ 2^- \{326\} + j_6 6^+ 2^- \{623\}$; and $j_6 > 1$ by {156} - {534} + $(2 + 1^-/m^2)\{\text{dim}\}$. However, that may revive massless numerators $j_{1,3} < 0$. To avoid infinite loops on recursive application of the relations, we eliminate single numerators by a general projection-operator

method [3]:

$$\int d^N p_1 d^N p_2 f_1[p_1^2, (p_1 - q)^2] f_2[p_2^2, (p_2 - q)^2] A^{2n}(p_1, p_2, q) = \frac{\Gamma(n + \frac{1}{2}) \Gamma[\frac{1}{2}(N - 1)]}{\Gamma(\frac{1}{2}) \Gamma[n + \frac{1}{2}(N - 1)]} \prod_{j=1}^2 \int d^N p_j f_j[p_j^2, (p_j - q)^2] A^n(p_j, p_j, q), \quad (3)$$

where $A(p_1, p_2, q) = 4 p_1^\mu (g_{\mu\nu} - q_\mu q_\nu / q^2) p_2^\nu$, and $f_{1,2}$ are arbitrary functions of their scalar arguments. A numerator $[(p_1 - p_2)^2]^n$ can be re-expanded:

$$(p_1 - p_2)^2 = \frac{1}{2} \{ p_1^2 + (p_1 - q)^2 + p_2^2 + (p_2 - q)^2 - [p_1^2 - (p_1 - q)^2] [p_2^2 - (p_2 - q)^2] / q^2 - q^2 - A(p_1, p_2, q) \}. \quad (4)$$

Odd powers of $A(p_1, p_2, q)$ fall out after integration, and for even powers Eq. (3) yields $A(p, p, q) = 2[p^2 + (p - q)^2] - [p^2 - (p - q)^2]^2 / q^2 - q^2$. For D_3 we identify the left-hand side of Eq. (4) with c_1 , and q^2 with c_3 . If we deal with $j_1 = -1$, $j_3 = 0$, $j_4 = j_6$, then q^2 on the right-hand side of Eq. (4) generates the original integral which can then be eliminated.

The denominator $j_3 > 1$ is reduced by {156}–{236} while $j_1 < 0$. At $j_1 = 0$ with $j_{2,4-6} = 1$, we apply {236}, eliminate $j_1 = -1$ by Eqs. (3) and (4), reduce $j_6 = 2$ as usual, and expand the first numerator again. The original integral with the same value of j_3 can be expressed by solving the resulting equation.

The case $D_3(0, 1, 1, 1, 1, 1)$ can be transformed as follows. The differential equation for the two-loop subgraph without line 2, Eq. (43) of Ref. [5], is divided by $q^2(q^2 + m^2)$, and integrated over q . Derivatives with respect to m^2 are taken via {dim}, and after substituting simple integrals we get

$$D_3(0, 1, 1, 1, 1, 1) = \frac{3N - 8}{2m^2} D_3(0, 1, 0, 1, 1, 1) + \frac{8(m^2)^{3N/2-5}}{(N-2)(N-3)(N-4)^3} \left[\frac{\Gamma(\frac{1}{2}N - 1) \Gamma(5 - N)}{\Gamma(3 - \frac{1}{2}N)} + 4 \right], \quad (5)$$

where each loop integral was divided by $\pi^{N/2} \Gamma(3 - \frac{1}{2}N)$; in one loop, this modification agrees with the standard \overline{MS} definition but is more convenient in higher-loop massive calculations. As a result of the transformations, any D_3 -type diagram is reduced to simpler types and two master integrals $D_3(0, 1, 0, 1, 1, 1)$ and $D_3(1, \dots, 1)$.

With $j_{3,4} \leq 0$, E_4 is a particular case of B_M . The numerator on line 5 can be taken off by {120} as $j_2 > 1$ (massless line 0 with index 0 is assumed to connect 1.3 and 2.4); by {210} as $j_1 \neq 1$; by {430} as $j_3 > 1$; or {dim} should be applied otherwise. The $j_1 < 0$ numerator is eventually reduced to $j_1 = 0$ (hence, to the two-loop massive bubbles) by {120} if $j_2 > 1$, or by {210}. The denominators $j_{1,2} > 1$ can be brought down to 1 by solving {120} relative to 2^+5^- , or {210} – $2m^25^+$ + {120} relative to 1^+5^- . That always increases j_5 . The latter helps to reduce the denominators in the massive one-loop subgraph by $2\{340\}$ – $\{430\}$. The extra denominator $j_5 > 0$ can be integrated off by applying {dim} and reducing $j_2 = 2$ by {120}. Iteratively, we arrive at the master integral $E_4(1, 1, 1, 1, 0) = D_3(0, 1, 0, 1, 1, 1)$.

Further prototypes are quite simple indeed. The numerators are manageable by adjacent-triangle relations, and the denominators can be reduced by combining three relations for a triangle that contains the line. The master integrals are $D_M(1, \dots, 1)$, $E_3(1, \dots, 1)$, and $D_N(1, \dots, 1)$. In D_2 the massless relation {125} solved with respect to the free term allows one to cancel out a denominator. Thus, in the end D_2 is reduced to Γ functions, just as D_1 does. For E_2 a massless relation can be constructed as {524}–{dim}.

The described algorithms are implemented as a package of procedures in the symbolic-manipulation language FORM [6] well suited to evaluating the Feynman diagrams as well as any polynomial-like expressions with a large number of terms. However, the efficiency of the essentially recursive programs is restricted by some features of the existing FORM translator. In particular, 'infinitely' iterative substitutions are only allowed without any intermediate sorting of terms. On the other hand, the recurrence relations generate rather many equal terms, and after exceeding certain machine-dependent limits on the size of the scratch expression generated in a module, the sorting becomes extremely slow. The only way out is to use step-by-step sorting inside the preprocessor #do loops with a pre-estimated number of repetitions. But sometimes the number is rather difficult to guess at beforehand, while any misjudgement spoils the program performance.

Therefore, it would be highly desirable to implement a kind of the preprocessor #repeat/endrepeat construct into FORM, which would terminate as soon as no actual transformations are applicable in any module inside its

body³. Also of use would be any means of redefining preprocessor variables, based on global tests on the terms of sorted expressions. An invariable essential inconvenience for structured packages is the global scope of all names in FORM.

The first application of the described package was the evaluation of the three-loop QCD correction to the electroweak ρ parameter [7]:

$$\delta^{QCD} = -\frac{2}{3}[1 + 2\zeta(2)]\frac{\alpha_s}{\pi} + \left\{\frac{157}{648} - \frac{3313}{162}\zeta(2) - \frac{308}{27}\zeta(3) + \frac{143}{18}\zeta(4)\right. \\ \left. - \frac{4}{3}\zeta(2)\ln 2 + \frac{441}{8}S_2 - \frac{1}{9}B_4 - \frac{1}{18}D_3 - \left[\frac{1}{18} - \frac{13}{9}\zeta(2) + \frac{4}{9}\zeta(3)\right]n_f\right. \\ \left. - \left(\frac{11}{6} - \frac{1}{9}n_f\right)[1 + 2\zeta(2)]\ln(\mu^2/m_t^2)\right\}\left(\frac{\alpha_s}{\pi}\right)^2, \quad (6)$$

where α_s is the QCD coupling constant at the renormalization scale μ in the \overline{MS} scheme with the total number of quark flavors n_f ($=6$); m_t is the pole mass of the top quark; B_4 has been introduced in Ref. [2]; S_2 determines the finite part of the two-loop massive bubble master integral [5]; and D_3 is the finite part of $D_3(1, 1, 1, 1, 1)$ which has been evaluated numerically by the momentum-expansion method [1] and independently in Ref. [8]. An error in the coefficient of $\zeta(4)$ in the original publication has been fixed, so that Eq. (6) completely agrees with the independent calculation of Ref. [8].

Acknowledgments

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³ I thank Timo van Ritbergen for informing me that in an wxperimental version of FORM 2.2 it is possible to terminate #do loops as nothing changes. However, this undocumented feature is unavailable in public versions and, as Jos A.M. Vermaseren communicates, liable to changes in FORM 3

References

- [1] J. Fleischer and O.V. Tarasov, *Z. Phys. C* **64**, 413 (1994); in *Proc. of the 1994 Zeuthen Workshop on Elementary Particles, Nucl. Phys. B (Proc. Suppl.)* **37 B**, 115 (1994).
- [2] D. Broadhurst, *Z. Phys. C* **54**, 599 (1992).
- [3] K.G. Chetyrkin and F.V. Tkachov, *Nucl. Phys. B* **192**, 159 (1981); F.V. Tkachov, *Phys. Lett. B* **100**, 65 (1981).
- [4] A.V. Kotikov, *Phys. Lett. B* **254**, 185 (1991); **259**, 314 (1991); **267**, 123 (1991).
- [5] D.J. Broadhurst, J. Fleischer and O.V. Tarasov, *Z. Phys. C* **60**, 287 (1993).
- [6] J.A.M. Vermaseren, *Symbolic Manipulation with FORM*, Version 2, Tutorial and Reference Manual (Computer Algebra Nederland, Amsterdam, 1991).
- [7] L. Avdeev, J. Fleischer, S. Mikhailov and O. Tarasov, *Phys. Lett. B* **336**, 560 (1994); Erratum **B 349**, 597 (1995); hep-ph/9406363.
- [8] K.G. Chetyrkin, J.H. Kühn and M. Steinhauser, *Phys. Lett. B* **351**, 331 (1995); hep-ph/9502291.

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