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ON HIDDEN SYMMETRY IN GENERAL RELATIVITY

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1 Symmetry Groups and Gauge Principle

There are two symmetry groups that are intimately connected with General Relativity. One group of this kind is a group of transformations of the space-time manifold M itself, and the other is a group of transformations acting in tangent vector spaces $T_p(M)$. The last concept is clearly expounded in the treatise by Misner, Thorne, Wheeler, (1973).

A diffeomorphism of a manifold M onto itself is a homeomorphism φ such that φ and φ^{-1} are differentiable (Kobayashi and Nomizu, 1963). A diffeomorphism is a transformation on M. Transformations on M form a group denoted by $Diff(M)$. The group of diffeomorphisms is often called the group of general transformations of coordinates. The transformation φ on M induces an automorphism $\tilde{\varphi}$ of the algebra of tensor fields that preserves the type of tensor fields and is transposable with tensor contractions. Let $\tilde{T} = \tilde{\varphi}T$ for any tensor field T; the tensor field $\tilde{T}$ is called equivalent to T with respect to the group $Diff(M)$. The physical meaning of the diffeomorphism group is that it is a group of symmetry of gravitational interactions in Einstein theory of gravity. A systematic and deep thorough consideration of the questions connected with space-time symmetry of General Relativity may be found in ref.(Anderson, 1967). We emphasize only that the diffeomorphism group is evidently the widest group of space-time symmetry.

The other group of symmetry can be characterized as follows: tensor fields of the type (1,1) on M are called affinor fields. Let S be an affinor field on M with $\det(S^i_j) \neq 0$. In a coordinate patch with local coordinates x^0, x^1, x^2, x^3 a nondegenerate linear transformation of vector field has the form $\bar{V}^i(x) = S^i_j(x)V^j(x)$, where $\bar{V}^i(x)$, $V^i(x)$ are components of vector fields with respect to x^0, x^1, x^2, x^3. A set of nondegenerate affinor fields is a group with an associative binary operation P=ST , where

$$P^i_j(x) = S^i_k(x)T^k_j(x). \tag{1}$$

In what follows this group of symmetry underlying the very notion of space-time manifold will be called the gauge group. It is evident, this is the natural and widest gauge group tightly connected with General Relativity.

Now we are ready to formulate the gauge principle in the framework of General Relativity. The tangent space $T_p(M)$ is identified here with the so-called gauge or internal space. The group defined by equation (1) is a gauge-symmetry group. To complete the gauge principle, it is to be added with an important concept of a polarized particle.

Any pair (p, X), where p is a point of spacetime M and X is a vector tangent to the manifold M at the point p, will be called the polarized particle. Polarization of a particle is associated with direction of the related vector X.

2 Gauge Fields

To establish the type of fields defined by the gauge principle, we derive equations of motion for polarized particles. The gauge group act on tangent vectors and do not act on coordinates. Therefore, it is important to determine the laws of change of polarization. Our aim is then to derive the simplest equations of motion of the polarization vector when a particle is moving along a given curve.

Let us take points p and q on the curve $\gamma(t)$ corresponding to the moments of time t and $\bar{t} = t + dt$. To determine an infinitesimal change in the vector $X(t)$ in time dt, the vector $X(\bar{t})$ at point q should be transported along the curve $\gamma(t)$ to the point p and compared there with the vector $X(t)$ at point p. In the general case the infinitesimal change of the vector field on curve $\gamma(t)$ is given by the expression (Schrödinger, 1950; Anderson, 1967)

$$\delta V^i = dV^i + \Gamma^i_{jk}\dot{x}^j V^k dt,$$

where V^i are components of the vector $X(t)$. To derive equations for $V(t)$, we apply to symmetry considerations. We assume that at every moment of time t an infinitesimal change of a vector along the curve $\gamma(t)$ is equal to an infinitesimal linear transformation of the vector induced by the gauge group. If $S^i_j = \delta^i_j + B^i_j dt$ is an infinitesimal gauge transformation, we have

$$\delta V^i = B^i_j V^j dt,$$

from which we obtain a system of ordinary linear homogeneous differential equations

$$\frac{dV^i}{dt} + \Gamma^i_{jk}\frac{dx^j}{dt}V^k = B^i_j V^j, \qquad (2)$$

defining the law of change of the polarization vector in the course of motion of a polarized particle along the curve $\gamma(t)$.

Let us show that eqs. (2) allow us to establish the laws of change of fields Γ and B under gauge transformations in a natural way. Let vector fields $\bar{X}$ and X be equivalent with respect to the gauge group, then $\bar{V}^i = S^i_j V^j$. If the components V^i obey the equation (2), it is not difficult to verify that the components $\bar{V}^i$ will be a solution of the equation

$$\frac{d\bar{V}^i}{dt} + \bar{\Gamma}^i_{jk}\frac{dx^j}{dt}\bar{V}^k = \bar{B}^i_j \bar{V}^j,$$

where

$$\bar{\Gamma}^i_{jk} = S^i_l \Gamma^l_{jm} T^m_k + S^i_l \partial_j T^l_k, \quad \bar{B}^i_j = S^i_k B^k_l T^l_j,$$

and T^i_j are components of the affinor field S^{-1} inverse to S . Thus, the transformation laws of fields Γ and B, under gauge transformations are determined.

Let us mention some characteristic properties of the basis fields Γ and B. For brevity, we will use the matrix notation

$$B = (B^i_j), \quad \Gamma_j = (\Gamma^i_{jk}), \quad E = (\delta^i_j), \quad TrB = B^i_i, \quad ST = (S^i_k T^k_j),$$

in which the transformation law of the affine connection Γ is of the form

$$\bar{\Gamma}_i = S\Gamma_i S^{-1} + S\partial_i S^{-1} = \Gamma_i + S\nabla_i S^{-1}, \qquad (3)$$

where ∇_i stands for the covariant derivative with respect to the connection Γ

$$\nabla_i S = \partial_i S + \Gamma_i S - S\Gamma_i = \partial_i S + [\Gamma_i, S].$$

As $S\nabla_i S^{-1}$ is a tensor field of the type (1,2), then $\bar{\Gamma}$ is the affine connection together with Γ. Let

$$(R^k_{ijl}) = R_{ij} = \partial_i\Gamma_j - \partial_j\Gamma_i + [\Gamma_i, \Gamma_j] \qquad (4)$$

be components of the Riemann tensor of the affine connection Γ, then from (3) and (4) we obtain

$$\bar{R}_{ij} = SR_{ij}S^{-1}. \qquad (5)$$

The relations (3), (4), and (5) clearly show that the affine connection is really a gauge field with respect to the gauge group defined above.

The affinor field B transforms under gauge transformations by the following law

$$\bar{B} = SBS^{-1}. \qquad (6)$$

3 Gauge- Invariant Equations

As it is noted above, the diffeomorphism group is responsible for gravitational interactions, and thus, the gauge group we have defined is a symmetry group of new interactions. All that is required is to derive the simplest equations for fields Γ and B invariant under transformations of the gauge group. To simplify computations and to write equations in a symmetric and manifestly gauge-invariant form, we introduce the gauge-covariant derivative. We will say that a tensor field T of the type (m,n) is of the gauge type (p,q) if under the transformations of the gauge group there is the correspondence

$$T \Rightarrow \bar{T} = \underbrace{S\cdots S}_{p}T\underbrace{S^{-1}\cdots S^{-1}}_{q},$$

where

$$0 \le p \le m \text{ and } 0 \le q \le n.$$

The Riemann tensor is a tensor field of the type (1,3) and according to (5) has the gauge type (1,1). From (6) it follows that the affinor field being a tensor field of the type (1,1) has the gauge type (1,1). As it follows from the consideration of the left-hand side of the Einstein equations, the Einstein potential g_{ij} being a tensor field of the type (0,2) is to be assigned the gauge type (0,0).

Let now T be components of the tensor field (tensor density) of the **gauge type (1,1)**, then by definition

$$D_iT = \partial_iT + [\Gamma_i, T]$$

is the gauge- covariant derivative. For instance, for the Riemann tensor

$$D_iR_{jk} = \partial_iR_{jk} + [\Gamma_i, R_{jk}].$$

For the affinor field the gauge- covariant derivative coincides with the standard covariant derivative,

$$D_iB = \partial_iB + [\Gamma_i, B] = \nabla_iB.$$

In the general case the operator D_i is not covariant since D_iT will not always be components of the tensor field together with T. However, the commutator $[D_i, D_j]$ is covariant, since

$$[D_i, D_j]T = [R_{ij}, T].$$

Hence we obtain the important relation for the Riemann tensor

$$[D_i, D_j]R_{kl} = [R_{ij}, R_{kl}]. \quad (7)$$

The basic property of the gauge- covariant derivative follows from its definition

$$\bar{D}_i\bar{T} = S(D_iT)S^{-1},$$

where $\bar{T} = STS^{-1}$, and $\bar{D}_i$ is the gauge- covariant derivative with respect to the connection

$$\bar{\Gamma} = \Gamma + S\nabla S^{-1}.$$

So, the tensor fields B and D_iB are of the same gauge type.

As it is known, the determinator $|g_{ij}| \neq 0$ that actually allows us to obtain, for the tensor field g_{ij}, the equations invariant under the transformations of diffeomorphism group. By analogy, let us consider the case when the determinator $|B^i{}_j| \neq 0$. Under this condition the affinor field B has the inverse one for which the nonlinear gauge-invariant equations can be suggested. The simplest gauge-invariant Lagrangian has the form

$$L = -\frac{1}{2}Tr(D_iBD^iB^{-1}) + \lambda TrB - \frac{1}{4}Tr(R_{ij}R^{ij}), \quad (8)$$

where λ is a constant,

$$D^i = g^{ij}D_j \text{ and } R^{ij} = g^{ik}g^{jl}R_{kl}.$$

Taking into account that

$$\delta B = -B(\delta B^{-1})B$$

and varying (8) with respect to B and Γ we obtain the following equations for Γ

$$D_i(\sqrt{|g|}R^{ij}) = \sqrt{|g|}J^j, \quad (9)$$

where

$$J^i = [B^{-1}, D^i B]$$

and for B

$$D_i(\sqrt{|g|}B^{-1}D^iB) = \lambda\sqrt{|g|}B. \qquad (10)$$

The tensor current J has to satisfy the equation

$$D_i(\sqrt{|g|}J^i) = 0$$

as in accordance with (7),

$$D_iD_j(\sqrt{|g|}R^{ij}) = 0.$$

Since this is really so, the system of equations (9),(10) is consistent. Varying the Lagrangian (8) with respect to g_{ij} we obtain the gauge-invariant metric tensor of energy-momentum

$$T_{ij} = Tr(D_iBD_jB^{-1}) + Tr(R_{ik}R^k_j) + g_{ij}L$$

which satisfies the equation

$$T^{ij}{}_{;i} = 0. \qquad (11)$$

The semicolon denotes the covariant derivative with respect to the Levi-Civita connection belonging to the field g_{ij}. When deriving (11), besides equations of fields, one should use the standard relations of tensor analysis (Schouten,1954) and the identity

$$D_iR_{jk} + D_jR_{ki} + D_kR_{ij} = 0$$

which can easily be obtained with the help of the relation

$$[D_i, D_j]B = [R_{ij}, B].$$

From (11) and the gauge invariance of the metric tensor of energy-momentum it follows that the complete system of

equations derived from the Lagrangian $L_F = L_g + L$, where L_g is the Einstein-Hilbert Lagrangian, will be consistent.

Some very interesting gauge-invariant quantities can be constructed from the fields B and R_{ij}. We dwell here on some of them. The invariants, in particular, are

$$\varphi = TrB, \; \triangle = |B^i_j|.$$

If B obeys equation (10), then taking the trace of both sides of this equation we obtain that the invariants φ and $\triangle$ satisfy the equation

$$\partial_i(\sqrt{|g|}g^{ij}\partial_j ln|\triangle|) = \lambda\sqrt{|g|}\varphi.$$

A state (B, R_{ij}) is said to be singlet if it is invariant under all the symmetry transformations. In our case a singlet state is given by the equations

$$B = SBS^{-1}, \quad R_{ij} = SR_{ij}S^{-1}$$

to be satisfied at any S. The first equation has the solution

$$B = e^\alpha E, \quad B^{-1} = e^{-\alpha}E,$$

where α is a scalar field. As in this case

$$\varphi = ne^\alpha, \quad \triangle = e^{n\alpha},$$

from the equation for invariants φ and $\triangle$ we obtain that α will obey the Liouville like equation

$$\partial_i(\sqrt{|g|}g^{ij}\partial_j\alpha) = \lambda\sqrt{|g|}e^\alpha$$

which establishes the link with the Liouville field theory.

In conclusion: what is the Einstein problem? If you look into Einstein, you would find that Einstein had a very clear statement about the left- hand side and the right- hand side of his equations

$$G_{ij} = kT_{ij}.$$

He said, "The left- hand side is built up of gold, and the right-hand side is built up of garbage" (Yang, 1980). Einstein said this because the left- hand side is defined by the symmetry and the field g_{ij} with deep geometrical meaning, but this was not so with respect to the right- hand side of his equations .

From our consideration it follows that the Gauge Principle is such an integral part of General Relativity as the Principle of General Covariance. In the framework of these two fundamental principles, both left- hand side and right-hand sides of the Einstein equations are in fact defined uniquely. Besides, according to (11), the right hand side of the Einstein equations as well as the left-hand side are defined by the quantities with deep geometrical meaning. Looking at equations (2) we see that the composite geometrical object (Γ, B) defines a general law of parallel displacement.

Thus, General Relativity harbours the Gauge Principle that gives the complete and unique solution to the Einstein problem.

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