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ZERO-CURVATURE REPRESENTATION FOR HARMONIC-SUPERSPACE EQUATIONS OF MOTION IN $N=1$, $D=6$ SUPERSYMMETRIC GAUGE THEORY

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1 Introduction

The harmonic superspace (HS) has been introduced in Refs[1, 2] for a consistent description of the supergravity, supersymmetric gauge and matter theories with $N = 2, D = 4$ supersymmetry. The harmonic approach is a covariant version of the twistor methods and it has been intensively used for the construction of self-dual solutions in the ordinary and supersymmetric Yang-Mills and gravity theories [3, 4].

We shall use the standard notation SYM_D^N for supersymmetric gauge theories with (D, N)-supersymmetry in the space-time dimension D.

It should be noted that the twistor-harmonic methods have been applied for the integrability interpretation of the non-self-dual theories SYM_4^3 [5]-[11], SYM_5^4 [12] and SYM_{10}^1 [13, 14], however, the corresponding harmonic superspaces are very complicated. The simple harmonic $SU(2)/U(1)$ formalism allows to construct a general solution of the 3-dimensional SYM_3^6 equations [15].

The notion of *HS-integrability* is connected with a reformulation of the SYM-equations as the conditions of zero-harmonic-superfield curvatures constructed by means of the covariant harmonic derivatives and the harmonized spinor or vector covariant derivatives. These conditions can be interpreted as the covariant conservation of harmonic analyticity [1]. The harmonic coordinates in the superfield formalism of HS-theories are the analogues of the auxiliary (spectral) parameters. The final construction of HS-solutions can be reformulated in terms of the ordinary coordinates.

We propose the HS-integrability interpretation of the supersymmetric $N = 1, D = 6$ gauge theory SYM_6^1 connected via a dimensional reduction with the SYM_4^2-theory. The HS-formalism of SYM_6^1 has been considered in Refs[14, 16, 17] by analogy with [1]. A review of the standard harmonic approach is presented in Section 2. We call this version of harmonic formalism *a V-frame of the analytic basis*, because it uses the analytic prepotential V^{++} [1] as a basic field variable. Section 3 contains a new version of the harmonic formalism (A-frame) using the nonanalytic harmonic connection A^{--} as an independent variable. It will be shown that the SYM_6^1 superfield constraints and equations of motion can be reformulated as an dynamical zero-curvature equation plus a linear solvable constraint in this frame. The HS-approach produces also an infinite number of conservation laws and equations for the Bäcklund-transformation matrix in SYM_6^1. Section 5 is devoted to the analysis of SYM_6^1-solutions in the V-frame for the gauge group $SU(2)$. We use a special harmonic representation of the $SU(2)$-prepotential and the simplest harmonic gauge [15]. This gauge simplifies a study of the spontaneous-breaking phase of SYM_6^1. The A-frame analysis of $SU(2)$-solutions is considered in Section 6.

In conclusion the HS-integrability of the SYM_4^2-theory is discussed briefly. In particular, self-dual and anti-self-dual solutions and a duality transformation have the simple representations in the A-frame. We hope that the HS-integrability of SYM_4^2 can help to understand the remarkable quantum properties of this theory [18].

2 Harmonic formalism of SYM_6^1

Different versions of superfield formalism produce the manifestly supersymmetric description of the off-shell SYM_6^1-theory [14, 17, 20, 21]. Consider $N=1, D=6$ superspace $M(6,8)$ [20, 21] with the 6 vector coordinates x^{ab} and 8 spinor coordinates θ_i^a where $a, b\ldots$ are the 4-spinor indices of the Lorentz group $SU^*(4) \sim SO(5,1)$ and $i, k\ldots$ are the 2-spinor indices of $SU(2)$ group. Let $z=(x^{ab}, \theta_i^a)$ be the short notation for the coordinates in this superspace.

The plane spinor derivatives in $M(6,8)$ satisfy the basic relation

$$\{D_a^k, D_b^l\} = i\varepsilon^{kl}\partial_{ab} \tag{2.1}$$

where $\partial_{ab} = \partial/\partial x^{ab}$. We shall use the following combinations of the spinor derivatives [20]

$$(D_2)_{ab} = (1/2)\varepsilon_{ik}\, D^i_{(a}\, D^k_{b)} \tag{2.2}$$

$$(D_4)^{iklm} = (1/24)\varepsilon^{abcd}\, D_a^i\, D_b^k\, D_c^l\, D_d^m \tag{2.3}$$

where parentheses denote a symmetrization of indices. Write the useful identities for these quantities [20]

$$D_a^{(n}(D_4)^{iklm)} = 0, \qquad (D_2)_{ab}(D_4)^{iklm} = 0 \tag{2.4}$$

By analogy with the SYM_4^2-theory [19] the superfield constraints of SYM_6^1 can be written in the following form [20, 21]:

$$\{\nabla_a^i, \nabla_b^k\} + \{\nabla_a^k, \nabla_b^i\} = 0 \tag{2.5}$$

where $\nabla_a^i = D_a^i + A_a^i(z)$ is the covariant spinor derivative and A_a^i is the spinor gauge superfield in a central basis (CB).

The SYM_6^1 superfield equation of motion has a dimension $d=-2$ in units of length

$$\nabla_a^i W^{ak} + \nabla_a^k W^{ai} = 0 \tag{2.6}$$

where W^{ai} is the covariant superfield-strength of SYM_6^1

$$W^{ak} = (i/12)\varepsilon^{abcd}\varepsilon_{jl}[\nabla_b^k, \{\nabla_c^j, \nabla_d^l\}] \tag{2.7}$$

The integrable superfield constraints (2.5) can be solved in the harmonic approach, and this solution generates a covariant off-shell description of the SYM_6^1 theory. An integrability of the whole SYM_6^1-system including Eqs (2.5) and (2.6) will be discussed below.

We shall use the standard notation for $SU(2)/U(1)$ harmonics $u_i^\pm$ and the partial harmonic derivatives

$$\begin{gathered}[\partial^{++}, \partial^{--}] = \partial^0 \\ \partial^{++}\, u_i^+ = 0, \quad \partial^{++}\, u_i^- = u_i^+ \\ \partial^{--}\, u_i^- = 0, \quad \partial^{--}\, u_i^+ = u_i^- \end{gathered} \tag{2.8}$$

where ∂^0 is the operator corresponding to $U(1)$-charge q. These harmonics and derivatives have simple representations in terms of the real $U(1)$-variable φ and the complex spectral variable λ (see e.g.[1, 24])

$$\begin{pmatrix} u_1^- & u_1^+ \\ u_2^- & u_2^+ \end{pmatrix} = \frac{1}{\eta(\lambda)}\begin{pmatrix} 1 & -\lambda \\ \bar\lambda & 1 \end{pmatrix}\begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix} \tag{2.9}$$

where $\eta(\lambda) = \sqrt{1+\lambda\bar\lambda}$. The convenient representation of the harmonic derivatives has the following form:

$$\partial^{++} = e^{2i\varphi}[\eta^2(\lambda)\bar\partial_\lambda - (i/2)\lambda\partial_\varphi] \tag{2.10}$$

$$\partial^{--} = -\, e^{-2i\varphi}[\eta^2(\lambda)\partial_\lambda + (i/2)\bar\lambda\partial_\varphi], \qquad \partial^0 = -i\partial_\varphi \tag{2.11}$$

where corresponding partial derivatives are introduced.

Consider the harmonic (twistor) transform from the central basis of SYM_6^1 to the analytic basis (AB)

$$u_i^+\nabla_a^i = u_i^+ h^{-1} D_a^i h \tag{2.12}$$

where $h(z,u)$ is a bridge matrix satisfying the basic harmonic equation [1]

$$(\partial^{++} + V^{++})h(z,u) = \nabla^{++}h = 0 \qquad (2.13)$$

Discuss briefly a terminology of the harmonic approach used in this paper. A notion of *the basis* (CB or AB) includes the choice of the gauge group representation (τ-group or Λ-group [1]) and the complete set of relations between covariant derivatives. We use also a notion of *the frame* in the analytic basis that means the choice of independent field variables and basic equations generating the complete system of equations.

The analytic connection V^{++} with $q = +2$ (prepotential) determines the off-shell structure of SYM^1_6 in the V-frame. It 'lives' in an analytic harmonic superspace with the coordinates $\zeta = (x^{ab}_A,\ \theta^c_+)$

$$x^{ab}_A = x^{ab} + (i/4)(\theta^a_+\theta^b_- - \theta^b_+\theta^a_-). \qquad (2.14)$$

$$\theta^a_+ = u^i_+\theta^a_i, \quad \theta^a_- = u^i_-\theta^a_i \qquad (2.15)$$

The plane differential operators in the analytic coordinates $(\zeta,\ \theta^a_-)$ have the following form:

$$\partial^{++}_A = \partial^{++} + (i/2)\theta^a_+\theta^b_+\partial_{ab} + \theta^a_+\partial^+_a \qquad (2.16)$$

$$\partial^{--}_A = \partial^{--} + (i/2)\theta^a_-\theta^b_-\partial_{ab} + \theta^a_-\partial^-_a \qquad (2.17)$$

$$D^+_a = \partial^+_a = \partial/\partial\theta^a_- = u^+_i\ D^i_a \qquad (2.18)$$

$$D^-_a = -\partial^-_a - i\theta^b_-\partial_{ab} = u^-_i\ D^i_a \qquad (2.19)$$

Note that the AB-superfields can be described in terms of the central coordinates z, u, too. Write the useful relations and definitions for spinor derivatives:

$$\{\ D^-_a, D^+_b\ \} = i\partial_{ab}, \quad (D_2)_{ab} = D^+_{(a}\ D^-_{b)} \qquad (2.20)$$

$$(D^+)^4 = u^+_i u^+_k u^+_l u^+_m (D_4)^{iklm} = (1/24)\varepsilon^{abcd}\ D^+_a\ D^+_b\ D^+_c\ D^+_d \qquad (2.21)$$

$$(D^{+3})^a = (1/6)\varepsilon^{abcd}\ D^+_b\ D^+_c\ D^+_d \qquad (2.22)$$

The V-system of equations in the SYM^1_6-theory contains off-shell constraints and an equation of motion. The basic V-frame constraints are:

1) The harmonic zero-curvature (HZC) equation [14, 22]

$$[\nabla^{++},\ \nabla^{--}] - \partial^0 = \partial^{++}_A A^{--} - \partial^{--}_A V^{++} + [V^{++}, A^{--}] = 0 \qquad (2.23)$$

where $A^{--}(\zeta,\ \theta^a_-, u)$ is the harmonic connection with $q = -2$. The harmonic connections can be expressed via the bridge matrix h but we treat the HZC-equation as an independent basic equation. The general perturbative and non-perturbative solutions of the basic harmonic equations (2.13) and (2.23) have been discussed in Refs[2, 14, 15, 22].

2) The 'kinematic' V-analyticity condition (VZC-equation) [1]

$$[\nabla^+_a,\ \nabla^{++}] = D^+_a V^{++} = 0 \qquad (2.24)$$

3) The conventional spinor constraint [14, 22]

$$[\nabla^{--},\ \nabla^+_a] = \nabla^-_a = D^-_a + A^-_a = D^-_a - D^+_a A^{--} \qquad (2.25)$$

This constraint allows us to write the spinor connection A^-_a in terms of the harmonic connection A^{--}.

4) The initial CB-integrability condition is solved trivially by the transition to AB [1]

$$u^+_i u^+_k \{\nabla^i_a, \nabla^k_b\} = 0 \Rightarrow \{\nabla^+_a,\ \nabla^+_b\} = \{D^+_a,\ D^+_b\} = 0 \qquad (2.26)$$

Secondary constraints follow from the basic constraints 1)-4)

$$[\nabla^{--},\ \nabla^-_a] = 0, \quad \{\nabla^-_a,\ \nabla^-_b\} = 0 \qquad (2.27)$$

$$\{\nabla^+_a,\ \nabla^-_b\} + \{\nabla^-_a,\ \nabla^+_b\} = 0 \qquad (2.28)$$

The V-frame SYM^1_6-equation of motion has been obtained in Ref [14] by the use of a corresponding nonpolynomial action

$$F^{++} = (1/4)D^+_a\ W^{a+}(V) = (D^+)^4 A^{--}(V) = 0 \qquad (2.29)$$

A perturbative solution for A^{--} has the following form

$$A^{--}(V) = \sum_{n=1}^{\infty}(-1)^n \int du_1 \ldots du_n \frac{V^{++}(z, u_1)\ldots V^{++}(z, u_n)}{(u^+u^+_1)\ldots(u^+_n u^+)} \qquad (2.30)$$

where the harmonic distributions $(u^+_1 u^+_2)^{-1}$ [2] are used. Eq(2.29) is equivalent to the analyticity condition on the AB-superfield-strength $W^{a+}(V)$.

The harmonic distributions have a simple complex representation

$$\frac{1}{(u^+_1 u^+_2)} = e^{-i(\varphi_1+\varphi_2)}\frac{\eta(\lambda_1)\ \eta(\lambda_2)}{\lambda_1 - \lambda_2} \qquad (2.31)$$

Using this representation, Eq(2.10) and the known formula for complex distributions

$$\frac{\partial}{\partial\lambda_1}\frac{1}{\lambda_1-\lambda_2}=\pi\delta(\lambda_1-\lambda_2) \qquad (2.32)$$

one can reproduce the differential relation [2]

$$\partial_1^{++}\frac{1}{(u_1^+u_2^+)}=\delta^{(1,-1)}(u_1,u_2)=\pi e^{i(\varphi_1-\varphi_2)}\eta^4(\lambda_1)\delta(\lambda_1-\lambda_2) \qquad (2.33)$$

The equation of motion simplifies in the normal V-gauge [1] which is an analogue of the WZ-gauge of the simplest superfield theories

$$V_N^{++}=(1/2)\theta_+^a\theta_+^b A_{ab}(x_A)+(\theta^{+3})_a u_i^-\psi_i^a(x_A)+(\theta_+)^4\, u_i^-u_k^- D^{ik}(x_A) \quad (2.34)$$

$$(\theta^{+3})_a=(1/6)\varepsilon_{abcd}\theta_+^b\theta_+^c\theta_+^d, \quad (\theta_+)^4=(1/24)\varepsilon_{abcd}\theta_+^a\theta_+^b\theta_+^c\theta_+^d$$

where A_{ab},ψ_i^a and D^{ik} are the component fields of a gauge supermultiplet.

SYM_6^1-action in the normal gauge is the 4-th order polynomial

$$S_N=\sum_{n=2}^{4}\int d^{12}z du_1\ldots du_n\frac{\mathrm{Tr}V_N^{++}(z,u_1)\ldots V_N^{++}(z,u_n)}{(u_1^+u_2^+)\ldots(u_n^+u_1^+)} \qquad (2.35)$$

This action generates the 3-d order V-equation of motion equivalent to the component SYM_6^1 equations of motion. An analysis of the nonlinear equation (2.29) is a difficult problem even in the normal gauge. Thus, the V-frame is useful for the solution of the off-shell constraints (2.5) and quantization [2] but is not very convenient for the search of the classical solutions.

The original works on harmonic superspaces [1, 2] and Refs [14, 15] use *the regular harmonic functions* V^{++} and $h(z,u)$ treated as the convergent or formal harmonic series. Regular harmonic functions $f(u)$ correspond to globally defined functions on the sphere S^2, and irregular functions can contain poles and other singularities. The assumption of regularity is natural for the perturbation theory (e.g. in the normal gauge) but it leads to unreasonable restrictions on the nonperturbative solutions. We shall discuss the irregular bridge functions in Section 3.

3 New harmonic frame for the SYM_6^1 equations of motion

Now we shall consider a new harmonic representation of the SYM_6^1 equations which allows to prove the HS-integrability of this theory and to solve the equation with a dimension $d=-2$. Only the complete system of covariant equations in the analytic basis has an invariant meaning, however, one can change the choice of field variables and independent equations. A basic field variable of the V-frame is $V^{++}=h\partial^{++}h^{-1}$. It is clear that one can use other functions of the bridge h as the field variables of AB.

Let us treat the harmonic connection $A^{--}=h\partial^{--}h^{-1}$ as a basic superfield of the classical SYM_6^1 theory in *the A-frame*. The complete A-system of SYM_6^1-equations for covariant derivatives with $d\geq-2$ is identical to the corresponding V-system, however we change *the interpretation, the basic set and the order of the dynamical equations and the auxiliary field structure* of the harmonic formalism in the new frame. The HZC-equation (2.23) in this frame is treated as an integrable equation for the connection $V^{++}(A^{--})$. A basic A-bridge equation contains the covariant derivative ∇^{--}, and Eq(2.13) becomes a secondary equation. Harmonic equations with $d=0$ do not guarantee the conservation of analyticity. We shall preserve the standard transform between CB and AB (2.12), the basic AB constraints (2.23),(2.25) and (2.26) but treat analyticity in the A-frame as a new dynamic zero-curvature equation instead of the 'kinematic' analyticity constraint of the V-frame (2.24).

It should be underlined that the nonlinear in V^{++} equation (2.29) transforms to a linear kinematic constraint of the new A-frame:

$$(D^+)^4A^{--}(z,u)=0 \qquad (3.1)$$

Using a nilpotency of D_a^+ we can obtain the following general solution of this constraint:

$$A^{--}(z,u)=D_a^+\,A^{a(-3)}(z,u) \qquad (3.2)$$

where $A^{a(-3)}$ is the on-shell SYM_6^1 prepotential.

Now the whole SYM_6^1-system reduces to the dynamic analyticity (zero-spinor-curvature) condition which we shall call AZC-equation

$$[\nabla_a^-,\,\nabla^{--}]=D_a^-\,A^{--}+\partial^{--}\,D_a^+\,A^{--}-[D_a^+\,A^{--},\,A^{--}]=0 \qquad (3.3)$$

where the constraint (2.25) is used.

This condition and the representation (3.2) generate a nonlinear equation for the superfield $A^{a(-3)}$

$$D^-_a\, D^+_b\, A^{b(-3)} + \partial^{--}\, D^+_a\, D^+_b\, A^{b(-3)} - [D^+_a\, D^+_b\, A^{b(-3)},\; D^+_c\, A^{c(-3)}] = 0 \quad (3.4)$$

This equation has the following gauge invariance:

$$\delta A^{a(-3)} = R^{a(-3)}\Lambda + [\Lambda, A^{a(-3)}] + D^+_b \Lambda^{ab(-4)} \qquad (3.5)$$

where a general symmetrical spinor $\Lambda^{ab(-4)}$ and an analytic scalar Λ are the Lie-algebra valued superfield gauge parameters. The spinor derivative of $\delta A^{a(-3)}$ produces the standard AB-gauge transformation $\delta A^{--} = \nabla^{--}\Lambda$.

$$\left\{D^+_a, R^{a(-3)}\right\} = \partial^{--}_A \qquad (3.6)$$

$$R^{a(-3)} = \theta^a_- \partial^{--} + \frac{i}{4}\theta^a_- \theta^b_- \theta^c_- \partial_{bc} + \frac{1}{2}\theta^a_- \theta^b_- \partial^-_b \qquad (3.7)$$

Let us consider a regular harmonic functions $A^{a(-3)}(z,u)$ and choose a normal A-gauge for the on-shell superfield $A^{--} = D^+_a A^{a(-3)}$

$$A^{--}_N = \theta^a_- \beta^-_a(\zeta,u) + \frac{1}{2}\theta^a_- \theta^b_- \alpha_{ab}(\zeta,u) + (\theta^{-3})_a \psi^{a+}(\zeta,u) \qquad (3.8)$$

$$(\theta^{-3})_a = D^+_a(\theta^-)^4 = (1/6)\varepsilon_{abcd}\theta^b_- \theta^c_- \theta^d_-$$

where β, α and ψ are the analytic functions. This gauge has a residual gauge invariance with restricted parameters $\partial^{--}\Lambda = 0$

$$\delta\beta^-_a = \partial^-_a \Lambda + \ldots, \quad \delta\alpha_{ab} = \partial_{ab}\Lambda + \ldots \qquad (3.9)$$

Note that $(\theta_-)^4$ term vanishes due to the constraint (3.2).

The superfield A^{--}_N contains a physical part A^{--}_P and an auxiliary-field part H^{--}. All auxiliary harmonic component fields vanish as a consequence of Eq(3.4) so the physical harmonic connection be

$$A^{--}_P = (1/2)\theta^a_- \theta^b_- [A_{ab} + \varepsilon_{abcd}\theta^c_+ u^i_- \psi^d_i] + (\theta^{-3})_a [u^i_+ \psi^a_i + \theta^b_+ F^a_b] \qquad (3.10)$$

where $A_{ab}(x_A)$ and $\psi^a_i(x_A)$ are the vector and spinor fields and $F^a_b(x_A)$ is an independent field-strength.

Eq(3.4) generates the usual connection between F^a_b and A_{ab} and the component SYM^1_6 equations. Thus, the A-frame corresponds to the first-order component SYM^1_6 formalism. It is evident that all frames of AB are equivalent on-shell and have identical component solutions for the physical fields.

It should be underlined that an alternative equivalent form of the HS-integrability condition in the A-frame can be written as a dynamical VZC-equation

$$[\nabla^+_a, \nabla^{++}] = D^+_a\, V^{++}(A^{--}) = 0 \qquad (3.11)$$

where $V^{++}(A^{--})$ is a solution of Eq(2.23) . A perturbative form of this solution is an analogue of the solution (2.30) but contains the new harmonic distribution

$$\frac{1}{(u^-_1 u^-_2)} = e^{i(\varphi_1+\varphi_2)}\frac{\eta(\lambda_1)\,\eta(\lambda_2)}{\lambda_1 - \lambda_2} \qquad (3.12)$$

satisfying the relation

$$\partial^{--}_1 \frac{1}{(u^-_1 u^-_2)} = \delta^{(-1,1)}(u_1, u_2) \qquad (3.13)$$

The third equivalent form of the dynamical A-frame equation can be written as

$$[\nabla^{++}, \nabla^-_a] = \nabla^+_a \Rightarrow [\nabla^{++}, [\nabla^{--}, \nabla^-_a]] = 0 \qquad (3.14)$$

The bridge matrix $h_A = h(A^{--})$ of the A-frame is a solution of the following harmonic equation

$$\nabla^{--} h_A = (\partial^{--} + D^+_a\, A^{a(-3)}\,)\, h_A = 0 \qquad (3.15)$$

This equation on the sphere $SU(2)/U(1)$ is the harmonic part of the linear problem for the HS-integrable SYM^1_6-system. A key point of the harmonic approach is the integrability of the bridge harmonic equation. If we restrict ourselves by regular solutions for h_A, then the consistency conditions on the regular harmonic connections appear [23, 15]. The explicit solutions for the $SU(2)$ gauge group be considered in sections 5 and 6.

Consider a typical example of the linear harmonic differential equation that arises in an analysis of the bridge equation

$$\partial^{++} f^{(-2)} = f^0(u) = c + c^{(ik)} u^+_i u^+_k + \ldots \qquad (3.16)$$

where f^0 is a regular harmonic function. For $c \neq 0$ this equation has no regular solution in terms of the harmonic expansion. However, Eq(3.16) in the complex coordinates (2.9) has the integral solution with a simple complex pole kernel. The harmonic analogue of this integral representation is

$$f^{(-2)}(u) = \int du_1 G^{(-2,0)}(u, u_1) f^0(u_1) \tag{3.17}$$

$$\partial^{++} G^{(-2,0)}(u, u_1) = \delta^{(0,0)}(u, u_1) = \pi\eta^4(\lambda)\delta(\lambda - \lambda_1) \tag{3.18}$$

In contrast to the standard harmonic distributions [2] $G^{(-2,0)}(u, u_1)$ has not any illustrative harmonic expansion, but it has the simple complex representation

$$G^{(-2,0)}(\lambda, \lambda_1) = e^{-2i\varphi} \frac{\eta^2(\lambda)}{(\lambda - \lambda_1)} \tag{3.19}$$

Thus, one can admit the appearance of isolated harmonic singularities in the bridge function and even in the harmonic connections. As a rule we shall use regular initial data and choose the gauge freedom to obtain the solutions with a minimal number of singularities.

The CB-gauge superfield does not depend on the harmonics

$$A^i_a(z) = h^{-1}\, D^i_a h - u^{+i} h^{-1}\, (D^+_a A^{--})\, h \tag{3.20}$$

This superfield satisfies the relations $\partial^{\pm\pm} A^i_a(z) = 0$ and also the equations (2.5) and (2.6) which are equivalent to the component SYM^1_6-equations.

Thus, one can consider irregular harmonic superfields if the following general rule is used[1]: *The physical component fields 'live' in the central basis.*

4 Conservation laws and Bäcklund transformations in SYM^1_6

The most attractive feature of integrable field theories is an infinite number of conservation laws. The explicit construction of the conserved quantities follows immediately from the zero-curvature representation and has a clear geometric interpretation in terms of the contour variables [28]. Analogous constructions arise also for the integrable SYM^3_4 equation [10].

The HS-integrable theories possess the specific properties. The corresponding zero-curvature equations contain covariant spinor and harmonic

[1] This rule for the harmonic method has been formulated by V.I.Ogievetsky

derivatives and mean a conservation of the analyticity in HS [1]. Now we shall try to show that ordinary conservation laws follow from the dynamic harmonic-spinor analyticity equation of SYM^1_6-theory. Consider a vector covariant derivative in the A-frame

$$\nabla_{ab} = -i\{\nabla^+_a, \nabla^-_b\} = \partial_{ab} + iD^+_a\, D^+_b\, A^{--} \tag{4.1}$$

The basic equation (3.4) generates the relation

$$[\nabla^{--}, \nabla_{ab}] = 0 \tag{4.2}$$

Let us choose a time variable $t = x^{12}$

$$\nabla_t = \nabla_{12} = \partial_t + A_{12}, \quad A_{12}(z, u) = iD^+_1\, D^+_2\, A^{--} \tag{4.3}$$

It is evident that ∇_t commutes on-shell with the covariant harmonic derivatives (4.2).

It should be stressed that the bridge is a natural harmonic analogue of the contour variables of integrable theories in the zero-curvature representation . The transformation law of the bridge has the following form:

$$\delta h_A \doteq \Lambda(\zeta, u)\, h_A - h_A\, \tau(z) \tag{4.4}$$

where Λ and τ are the gauge parameters in AB and CB, correspondingly. The covariant constancy of the bridge in the all spinor and vector directions is a consistency condition for the dynamic analyticity equations (3.4) or (3.11), for instance

$$\nabla_t\, h_A = \partial_t\, h_A + A_{12}\, h_A - h_A\, A_t(z) = 0 \tag{4.5}$$

where $A_t(z)$ is a time component of the gauge CB-superfield.

One can choose a special τ-gauge for the SYM^1_6-theory

$$A_t(z) = 0, \qquad \partial_t \tau(z) = 0 \tag{4.6}$$

The A-frame covariant derivative ∇_{12} commutes with D^+_1 and D^+_2. The simplest conserved quantities in the τ-gauge can be constructed as Λ-invariant functions of h_A, for example

$$C^{++}(z, u) = \mathrm{Tr}(D^+_1 h_A D^+_2 h^{-1}_A), \qquad \partial_t C^{++} = 0 \tag{4.7}$$

It is not difficult to built the conserved quantities invariant under the τ- and Λ-gauge transformations

$$P^{ab(\pm\pm)} = \mathrm{Tr} W^{a\pm} W^{b\pm}, \qquad \partial_{ab} P^{ab(\pm\pm)} = 0 \tag{4.8}$$

where $W^{a\pm}$ are components of the on-shell superfield-strength

$$W^{a+} = (D^{+3})^a A^{--}, \qquad W^{a-} = \nabla^{--} W^{a+} \tag{4.9}$$

$$\nabla_{ab} W^{a\pm} = 0, \qquad \nabla_{ab} W^{a\pm} W^{b\pm} = 0 \tag{4.10}$$

Note that the last equation is not valid off-shell.

The Bäcklund transformations (BT) play an important role for integrable theories as transformations in the spaces of solutions. For the $SDYM$ and $SDSYM$ solutions these transformations have been considered in Refs[29, 30]. We shall discuss BT in the HS-formalism of SYM^1_6.

Let A^{--} and $\hat{A}^{--}$ be two different solutions of the SYM^1_6-system (3.4). Consider the corresponding bridges h_A and $h_{\hat{A}}$. Then the Bäcklund transformation between these solutions has the following form:

$$\hat{A}^{--} = D^+_a \hat{A}^{a(-3)} = B^{-1} A^{--} B + B^{-1} \partial^{--} B \tag{4.11}$$

where the B-matrix can be written in terms of two bridges

$$B(A, \hat{A}) = h_A h^{-1}_{\hat{A}} \tag{4.12}$$

It is easy to derive the equations for the matrix B in terms of the background solution A^{--}, h_A. Formally the new superfield variable $\hat{A}^{--}$ has an independent $\hat{\Lambda}$ transformation, and it is 'invariant' under the Λ-transformation of a background superfield. Eq(4.11) can be treated as a harmonic equation for B in terms of the background solution A^{--}, h_A and the second prepotential $\hat{A}^{a(-3)}$. The analyticity equations (3.4, 3.11) for the second solution $\hat{A}^{--}$ produce the following Λ-covariant equations

$$\nabla^{++}(A)\beta^+_a = 0, \qquad \nabla^{--}(A)\,\nabla^{--}(A)\beta^+_a = 0 \tag{4.13}$$

where $\beta^+_a = D^+_a B \; B^{-1}$. Validity of these equations is evident in the representation (4.12)

$$h^{-1}_A \beta^+_a h_A = h^{-1}_A D^+_a h_A - \hat{h}^{-1}_{\hat{A}} D^+_a \hat{h}_{\hat{A}} = u^+_i [A^i_a(z) - \hat{A}^i_a(z)] \tag{4.14}$$

This representation is equivalent to the following form of the Bäcklund transformation of the spinor CB gauge superfield:

$$\hat{A}^i_a(z) = A^i_a(z) + h^{-1}_A [u^i_- \beta^+_a - u^i_+ \nabla^{--}(A)\beta^+_a] h_A \tag{4.15}$$

The equations for B are simplified in the case of infinitesimal Bäcklund transformations $B = I + \delta B$

$$\delta B = D^+_{(a} \nabla^-_{b)} B^{ab} \tag{4.16}$$

The analyticity produces an additional restriction on B^{ab}.

5 V-frame analysis of $SU(2)$ solutions in the simplest harmonic gauge

The HS-integrability interpretation allows us to analyze the explicit constructions of the SYM^1_6-solutions by analogy with the harmonic formalism of $SDYM$ [3, 4] or SYM^6_3 [15]. Let us go back to the V-frame and consider the case of the gauge group $SU(2)$. We shall use a harmonic representation of the general $SU(2)$ prepotential [15, 24, 25]

$$V^{++} = (U^{+2})\, b^0(\zeta, u) + (U^0)\, b^{(+2)}(\zeta, u) + (U^{-2})\, b^{(+4)}(\zeta, u) \tag{5.1}$$

where b^0, $b^{(+2)}$, $b^{(+4)}$ are arbitrary real analytic superfields and (U^q) are matrix generators of the Lie algebra $SU(2)$ in a harmonic representation

$$(U^{\pm 2})^i_k = u^\pm_k u^{\pm i}, \qquad (U^0)^i_k = u^-_k u^{+i} + u^+_k u^{-i} \tag{5.2}$$

An analogous representation of the prepotential was used as a special Ansatz for instanton and monopole solutions in the harmonic formalism [26, 27].

Consider the infinitesimal gauge transformations of the harmonic components $b^{(q)}$

$$\delta b^0 = \partial^{++}_A \Lambda^{(-2)} + 2\Lambda^0 + 2b^0 \Lambda^0 - 2b^{(+2)} \Lambda^{(-2)} \tag{5.3}$$

$$\delta b^{(+2)} = \partial^{++}_A \Lambda^0 + \Lambda^{(+2)} + b^{(+4)} \Lambda^{(-2)} - b^0 \Lambda^{(+2)} \tag{5.4}$$

$$\delta b^{(+4)} = \partial^{++}_A \Lambda^{(+2)} + 2b^{(+2)} \Lambda^{(+2)} - 2b^{(+4)} \Lambda^0 \tag{5.5}$$

where $\Lambda^{(q)}$ are the real harmonic components of the analytic $SU(2)$-gauge matrix Λ. Remark that the $(\theta_+)^4$-component in (5.5) contains the term $\partial^{ab}\lambda_{ab}(x)$ with a total derivative of vector function from $\Lambda^{(+2)}$.

The simplest general gauge for $SU(2)$-prepotential is

$$V^{++}(b^0,\rho) = (U^{+2})\, b^0(\zeta,u) + (U^{-2})\,(\theta_+)^4\,\rho \tag{5.6}$$

where b^0 is an arbitrary analytic function and ρ is a constant part of the trace of the auxiliary scalar matrix field with $d=-2$ in $b^{(+4)}$ that can be written as $D^{ik}_{ik}(x) = \rho + \partial^{ab} f_{ab}(x_A)$. The $\rho \neq 0$ solutions characterize the phase of the SYM^1_6-theory with the spontaneous breaking of symmetry.

Stress that this (b^0,ρ)-gauge has the residual gauge invariance with $\Lambda^{(+2)}=0$, $\Lambda^0 = const$ and an arbitrary parameter $\Lambda^{(-2)}$. The additional condition $(\partial^{++})^5 b^0 = 0$ fix the Λ-gauge and results in the vanishing of harmonic components with isospin $T > 4$ in b^0 [25]

$$b^0(V_{iklm},u) = (D_4)^{iklm}\, V_{iklm} + 4(u^+u^-)_{ik}(D_4)^{lmn(i}\, V^{k)}_{lmn} + \tag{5.7}$$

$$(60/7)(u^{+2}u^{-2})_{ijkl}(D_4)^{mn(ij}V^{kl)}_{mn} + (100/9)(u^{+3}u^{-3})_{i_1\cdots i_6}(D_4)^{n(i_1\cdots}V^{\cdots i_6)}_n +$$

$$(50/9)(u^{+4}u^{-4})_{i_1\cdots i_8}(D_4)^{(i_1\cdots}\, V^{\cdots i_8)}$$

where an analogue of the Mezinchescu prepotential with $d=2$ [31] and the irreducible symmetrical combinations of harmonics $(u^{+q}u^{-q})_{i_1\cdots i_{2q}}$ are used. The analyticity of this representation follows from the identity (2.4).

The phase of SYM^1_6 and SYM^2_4 with $\rho = 0$ was considered in Refs[15, 24, 25]. The HZC-equation (2.23) has the following solution in the $(b^0,0)$-gauge

$$A^{--}(b^0,0) = (U^{+2})\, a_0^{(-4)} + (U^0)\, a_0^{(-2)} + (U^{-2})\, a_0^{(0)} \tag{5.8}$$

where $a_0^{(q)}$ are harmonic-quadrature functions of the prepotential b^0

$$a_0^{(0)} = \frac{b(z)}{1+b(z)}\,, \qquad b(z) = \int du b^0(z,u) \tag{5.9}$$

$$a_0^{(-2)}(z,u) = \int du_1 \frac{(u^-u_1^+)}{(u^+u_1^+)}\, \frac{b^0(z,u_1) - b(z)}{1+b(z)} \tag{5.10}$$

$$a_0^{(-4)}(z,u) = [1+b(z)]\,[\partial^{--}\, a_0^{(-2)} - a_0^{(-2)}\, a_0^{(-2)}] \tag{5.11}$$

Note that this solution has a singular point $b(z) = -1$.

The classical action of SYM^1_6 in the $(b^0,0)$-gauge has the following form [24, 25]:

$$S(b) = \int d^{14}z[\ln(1+b(z)) - b(z)] \tag{5.12}$$

where $b(z) = (D_4)^{iklm}\; V_{iklm}(z)$ is a constrained potential.

The SYM^1_6-equation of motion in the $(b^0,0)$-gauge has only one independent component

$$(D^+)^4\left[\frac{b(z)}{1+b(z)}\right] = 0 \tag{5.13}$$

A spinor part of the gauge CB-superfield can be written in terms of the single superfield $b(z)$ [25]

$$[A^l_a(z)]^k_i = \frac{1}{1+b(z)}\,[\delta^l_i D^k_a b(z) - (1/2)\delta^k_i D^l_a b(z)] \tag{5.14}$$

Note that the SYM^1_4-constraints (2.5) in this representation follow from the identity

$$(D_2)_{ab} b(z) = 0 \tag{5.15}$$

The harmonic equations (2.13) and (2.23) with the prepotential $V^{++}(b^0,\rho)$ (5.6) can be integrated in quadratures. The integration procedure uses a nilpotency of the term $\rho(\theta_+)^4$.

Eq(2.23) has the following harmonic components in the (b^0,ρ)-gauge:

$$\partial^{++}a_\rho^{(-4)} + 2(1+b^0)a_\rho^{(-2)} - \partial^{--}b^0 = 0 \tag{5.16}$$

$$\partial^{++}a_\rho^{(-2)} + (1+b^0)a_\rho^{(0)} - b^0 - \rho(\theta_+)^4 a_\rho^{(-4)} = 0 \tag{5.17}$$

$$\partial^{++}a^{(0)}\rho - 4\rho\theta^a_-(\theta^{+3})_a - 2\rho(\theta_+)^4 a_\rho^{(-2)} = 0 \tag{5.18}$$

Note that it is convenient to analyze harmonic equations in the central coordinates z, u.

Consider the harmonic equation for $a_\rho^{(0)}$ which follows from these equations

$$(\partial^{++})^2 a_\rho^{(0)} = 2\rho(\theta_+)^4[2 + b^0 - a_\rho^{(0)} - b^0 a_\rho^{(0)}] \tag{5.19}$$

Using (5.9) as a zero approximation one can obtain an exact solution for $a_\rho^{(0)}$ by two iterations and then the other harmonic components can be calculated.

The classical action in the (b^0,ρ)-gauge has the following form:

$$S(b^0,\rho)=\int d^{14}z\,du\,b^0\int_0^1 ds\,a^{(0)}(sb^0,\rho) \qquad (5.20)$$

where s is an auxiliary parameter.

6 The A-frame analysis of $SU(2)$-solutions

Now we shall discuss properties of the $SU(2)$-solution in the alternative A-frame. The first step of this approach is a solution of harmonic equations in the representation (3.2) and then the dynamical analyticity equation should be used.

Consider the harmonic (U^q)-components of the AZC-equation (3.3)

$$D^-_a a^{(0)}+\partial^{--}D^+_a a^{(0)}+2D^+_a a^{(-2)}+2a^{(-2)}D^+_a a^{(0)}-2a^{(0)}D^+_a a^{(-2)}=0 \qquad (6.1)$$

$$D^-_a a^{(-2)}+\partial^{--}D^+_a a^{(-2)}+D^+_a a^{(-4)}+a^{(-4)}D^+_a a^{(0)}-a^{(0)}D^+_a a^{(-4)}=0 \qquad (6.2)$$

$$D^-_a a^{(-4)}+\partial^{--}D^+_a a^{(-4)}+2a^{(-4)}D^+_a a^{(-2)}-2a^{(-2)}D^+_a a^{(-4)}=0 \qquad (6.3)$$

These equations are equivalent to the dynamical equations $D^+_a b^{(q)}(A)=0$ for the harmonic components of V^{++} (5.1) in the A-frame.

The analyticity equations imply the following condition

$$\nabla^{++}W^{+a}=\nabla^{++}(D^{+3})^a A^{--}=0 \qquad (6.4)$$

producing the relations between harmonic components of W^{+a}. Remark that the additional conditions $D^+_a a^{(0)}=0$ or $(D^{+3})^a a^{(-2)}=0$ correspond to pure gauge solutions of SYM^1_6.

By analogy with (6.4) one can obtain the general relations between the harmonic components of Eq(3.1):

$$(D^+)^4 a^{(-4)}=0 \Leftrightarrow (D^+)^4 a^{(0,-2)}=0 \qquad (6.5)$$

The on-shell dependence of the superfields $a^{(q)}$ allow us to simplify the SYM^1_6-equations.

Now the convenient 'hybrid' choice of the field variables will be considered. Let $a^{(0)}, a^{(-2)}, b^{(+2)}$ and $b^{(+4)}$ be the independent variables and b^0 and $a^{(-4)}$ be treated as the functions of these variables. We can use the gauge (5.6) and Eqs(5.16-5.18) in this frame, too.

Using Eq(5.17) one can obtain the relation for the dependent function of the hybrid frame

$$b^0(A)=\frac{1}{1-a^{(0)}_\rho}[\partial^{++}a^{(-2)}_\rho+a^{(0)}_\rho-\rho(\theta_+)^4 a^{(-4)}_\rho] \qquad (6.6)$$

The analyticity condition $D^+_a b^0(A)=0$ is a single dynamical equation in this approach. It should be stressed that this equation describe the general $SU(2)$ solution.

Consider a solution of the harmonic bridge equation (3.15) for the case $\rho=0$

$$h_A=\exp[(1/2)(U^0)\ln(1-a^{(0)}_0)][1-(U^{+2})a^{(-2)}_0] \qquad (6.7)$$

This solution has only one singular point $a^{(0)}_0=1$. More general solution can contain additional singularities. An arbitrariness in the bridge solution is connected with the gauge freedom of Eq(3.15). Eq(6.7) produce a relation for $a^{(-4)}_0$.

The polynomial form of the corresponding dynamical equation is

$$(1+\partial^{++}a^{(-2)}_0)\,D^+_a a^{(0)}_0+(1-a^{(0)}_0)\,D^+_a\partial^{++}a^{(-2)}_0=0 \qquad (6.8)$$

$$a^{(0)}_0(z)=(D_2)_{ab}A^{ab}(z),\quad a^{(-2)}_0(z,u)=D^+_a A^{a(-3)}(z,u) \qquad (6.9)$$

Remark that this one-component equation is covariant under the residual gauge transformations of the $(b^0,0)$-gauge . The consistency condition for this equation follows from the restriction (5.15)

$$(D_2)_{ab}\int du b^0(z,u)=(D_2)_{ab}\left[\frac{a^{(0)}_0}{1-a^{(0)}_0}\right]=0 \qquad (6.10)$$

One can try to solve these equations in superfields or in components and then use the b^0-solution for the construction of the bridge to the central basis.

Thus, the SYM^1_6-system reduces to Eqs(5.13) or (6.8) in the $(b^0,0)$-gauge. This reduction simplifies significantly the initial SYM^1_6-system and gives the hope to obtain the explicit solutions of this problem.

7 Conclusion

The harmonic-superspace integrability of SYM^1_6-theory guarantees the analogous property of its $N=2, D=4$ subsystem SYM^2_4. Consider the representation (3.3) in the Euclidean version of SYM_4^2

$$A^{--}(z,u) = D^+_\alpha A^{\alpha(-3)} + \bar{D}^+_{\dot{\alpha}} \bar{A}^{\dot{\alpha}(-3)} \qquad (7.1)$$

where two-component spinors are used.

The case $A^{\alpha(-3)} = 0$ corresponds to the general self-dual solution of SYM_4^2

$$W(A) = (\bar{D}^+)^2 A^{--} = 0 \qquad (7.2)$$

The self-dual prepotential $\bar{A}^{\dot{\alpha}(-3)}$ satisfies also the nonlinear AZC-equation (3.4).

Note that SYM_4^2-equations in HS are covariant under the discrete transformation

$$\theta^\alpha_i \leftrightarrow \bar{\theta}^{\dot{\alpha}}_i \qquad A^{\alpha(-3)} \leftrightarrow \bar{A}^{\dot{\alpha}(-3)} \qquad (7.3)$$

that is a residual form of the Lorentz transformation in $D = 6$. This discrete transformation corresponds to the duality transformation between self-dual and anti-self-dual solutions. Specific features of SYM_4^2-solutions will be discussed elsewhere.

It seems natural that the effective quantum action of SYM_4^2 [18] can be rewritten in terms of $N = 2$ superfields . Note that the simplest harmonic gauge for the gauge group $SU(3)$ contains analytic components b_3^0 and $b_8^{(+2)}$ corresponding to the Cartan generators of $SU(3)$ [24, 25]. Analogous harmonic gauges can be found for any gauge group.

The integrable theory SYM_4^3 can be described in the framework of SYM_4^2 with the special hypermultiplet interactions [2]. An analogous construction exists for the integrable SYM_6^2-theory in terms of HS_6^1-superfields. It seems natural to consider the A-frame HS-equations of more general interacting SYM-supergravity-matter systems. Any HS-integrable system can be reduced to the dynamical analyticity conditions and some solvable linear constraints. This formulation may help to build the explicit classical solutions and to study quantum solutions.

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