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ON POSSIBILITY OF CLASSICAL DESCRIPTIONS OF SPIN MOTION IN INHOMOGENEOUS ELECTROMAGNETIC FIELDS

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Nonrelativistic classical equations of motion of magnetic moment in inhomogeneous electromagnetic field are well known. A relativistic equation of motion for a polarization vector of a particle moving in a homogenous electromagnetic field was obtained by Bargmann, Michel and Telegdi in 1959 [1] on the basis of Bloch's statement [2] that the equation for the mean value of spin operator in such fields coincide with the classical one. A large number of papers (see for example [3], [4] and references therein) are aimed at constructing a classical Hamiltonian in the case of inhomogeneous electromagnetic and gravitational fields with the purpose of subsequent quantization.

A classical description of a particle is possible, if:

1. There is a classical trajectory, which is defined by the mean values of quantum operators, including the spin operator.

2. The polarization vector of a particle has a constant absolute value in the rest-frame associated with the motion along the classical trajectory.

In homogenous electromagnetic fields these assumptions are valid as long as the usual conditions for the quasiclassicity of the motion hold [1]; in inhomogeneous fields, however, their validity has further essential physical limits. These limits are associated, for instance, with the splitting of a beam in an inhomogeneous magnetic field (Stern-Gerlach effect [5]) and with the relaxation of a transverse (with reference to the field) polarization. In this case after the splitting only the trajectories for particles with a definite spin projection on the field direction have a physical meaning, while the averaged over the spin variables classical trajectory may appear to be lying between the real trajectories, where there are practically no particles. The relaxation of a transverse polarization changes the absolute value of the polarization vector, thus violating the assumption 2. The motion of a spin in an inhomogeneous magnetic field in the general case does not allow a classical description. The Bloch's statement is restricted to homogenous fields and is not true for this case.

We shall use a simple example allowing an exact analytical solution of a quantum problem, to demonstrate the mentioned peculiarities of the motion in an inhomogeneous field, and then the qualitative results of the analysis will be extended to the relativistic case.

1 Evolution of a Gaussian Wave Packet In a Linearly Inhomogeneous Magnetic Field

To obtain a quantitative evaluation of the applicability conditions for the classical description of a nonrelativistic motion of a neutral particle in a linearly inhomogeneous magnetic field in a geometry corresponding to Stern-Gerlach experiment we shall use the analytical solution of a quantum problem about the passage of a gaussian wave packet through such a field, obtained by Muller and Metz [6]. The paper [6] employs the Wigner-Weyl-Moyal (WWM) representation of quantum mechanics (see [7], [8], [9]), generalized by Varilly and Grasia-Bondia [10] to particles with an arbitrary spin. We shall present here the basic ideas and formulae of this representation for spin 1/2.

Consider a density matrix $\rho_{ab}(\vec{q_1}, \vec{q_2}, t)$ of a particle with spin 1/2 (a, b are spin variables). Let us construct a matrix distribution function by the formula

$$f_{ab}(\vec{q}, \vec{p}, t) = \frac{1}{(2\pi)^{3/2}} \int e^{i\vec{p}\vec{r}} \rho_{ab}\left(\vec{q} + \frac{\vec{r}}{2}, \vec{q} - \frac{\vec{r}}{2}, t\right) d^3r. \quad (1)$$

Varilli and Grasia-Bondia supplemented the classical phase space $\mathbf{R}^6$ with a unit sphere $\mathbf{S}^2$ - the set of "classical spin" values. They have formulated "the Stratonovich-Weyl rule" for the case being considered, i. e. the one-to-one correspondence between operators

A on Hilbert space and functions w_A on the phase space $\mathbf{R}^6 \times \mathbf{S}^2$ which reduces the calculation of $< A >$ to integrating over the phase space. This correspondence for a spin j is realized by

$$w_A = Sp\left(A \cdot \Delta^j(\theta, \phi)\right), \quad (2)$$

where the matrix elements $\Delta^j(\theta, \phi)$ may be expressed in terms of the spherical harmonics and Clebsch-Gordon coefficients. For a spin 1/2,

$$\Delta^{1/2}(\theta, \phi) = \frac{1}{2}\begin{pmatrix} 1 + \sqrt{3}\cos\theta & \sqrt{3}e^{i\phi}\sin\theta \\ \sqrt{3}e^{-i\phi}\sin\theta & 1 - \sqrt{3}\cos\theta \end{pmatrix},$$

We shall call *a distribution function* a function $f(\vec{q}, \vec{p}, \vec{n}, t)$ ($\vec{n}$ is a unit vector on $\mathbf{S}^2$), which corresponds to the matrix (1) according to the rule (2). Probabilities of finding a particle with a coordinate $\vec{q}$ and momentum $\vec{p}$ may be expressed as

$$P(\vec{q}) = \int f(\vec{p}, \vec{q}, \vec{n}) d^3p d\Omega, \quad (3)$$
$$P(\vec{p}) = \int f(\vec{p}, \vec{q}, \vec{n}) d^3q d\Omega,$$

Mean values of any operator A are given by

$$< A >= \int w_A(\vec{p}, \vec{q}, \vec{n}) f(\vec{p}, \vec{q}, \vec{n}) d^3q d^3p d\Omega.$$

Let us define *a twisted product* of two functions on the phase space by

$$(f \times g)(\gamma) = \int\int f(\gamma')g(\gamma'')L(\gamma, \gamma', \gamma'')d\gamma' d\gamma'',$$

where $\gamma = (\vec{q}, \vec{p}, \vec{n})$, the integrals are taken over the entire phase space, and the kernel L is given by

$$L(\gamma, \gamma', \gamma'') = \frac{1}{4\pi^2} Sp\left(\Delta^{1/2}(\vec{n})\Delta^{1/2}(\vec{n'})\Delta^{1/2}(\vec{n''})\right) \times$$

$$\exp\{2i[\vec{q}(\vec{p}'-\vec{p}'')+\vec{q}'(\vec{p}''-\vec{p})+\vec{q}''(\vec{p}-\vec{p}')]\}.$$

The time evolution of the distribution function is described by the equation

$$\frac{\partial f}{\partial t}=-i\,(w_H\times f-f\times w_H), \qquad (4)$$

where w_H is the function corresponding to Hamiltonian according to (2). The motion of a neutral particle in a magnetic field is described by the Pauli Hamiltonian

$$\hat{H}=\frac{1}{2m}\hat{\vec{p}}^2-\frac{\mu_0}{2}\vec{B}\hat{\vec{\sigma}}, \qquad (5)$$

where μ_0 is the magnetic moment of the particle. Consider the most simple linearly inhomogeneous magnetic field $\vec{B}=B'(-x,0,z)$ and a wave packet of a neutral particle with spin 1/2 moving along the y-axis is At $t=0$ the packet is described by the distribution function

$$f_0(\vec{q},\vec{p},\vec{n})=\frac{1}{2}\left(1+\sqrt{3}\vec{a}_0\vec{n}\right)f_0(x,0)\cdot f_0(y,p_y)\cdot f_0(z,0), \qquad (6)$$

where

$$f_0(q,p_q)=\frac{1}{\pi\sigma_p\sigma_q}\exp\left[\frac{-(q-q_0)^2}{\sigma_q^2}\right]\cdot\exp\left[\frac{-(p_q-p_{q0})^2}{\sigma_p^2}\right],$$

$\vec{a}_0$ is the polarization vector of the particle at $t=0$, and σ_p and σ_q are the packet widths in momentum and coordinate spaces. For a pure state they are connected by the uncertainty relationship $\sigma_p\sigma_q=1/2$. In this case there is a wave function, which is given at $t=0$ by

$$\Psi(x,y,z)=\frac{1}{\pi^{3/2}\sigma_x\sigma_y\sigma_z}\exp\left[-\frac{x^2}{\sigma_x^2}-\frac{y^2}{\sigma_y^2}-\frac{(z-z_0)^2}{\sigma_z^2}+ipy\right].$$

Its further evolution is described by the usual Schroedinger equation. This problem has been studied by Scully et al. [11]. The

consideration of the problem in the WWM formalism is somewhat more complete, for it allows to account not only for the quantum diffraction of the packet, but also for its spreading due to the initial dispersion of momenta, which usually plays the decisive role in actual experiments (see, for example, [5]). If $\sigma_x\ll z_0$, one may ignore the terms proportional to x in Hamiltonian (5). Then it becomes

$$\hat{H}=\frac{1}{2m}\hat{p_x}^2+\frac{1}{2m}\hat{p_y}^2+\left(\frac{1}{2m}\hat{p_z}^2-\frac{\mu_0}{2}B'z\hat{\sigma}_z\right).$$

For this Hamiltonian Muller and Metz have obtained the exact solution of the equation (4) with the initial distribution function (6):

$$f_0(\vec{q},\vec{p},\vec{n})=f_0\left(x-\frac{p_{x0}t}{m},p_x\right)\cdot f_0\left(y-\frac{p_{y0}t}{m},p_y\right)\cdot f^{(z)}(z,p_z,\vec{n},t), \qquad (7)$$

$$f^{(z)}(z,p_z,\vec{n},t)=\frac{1}{\pi}\frac{1+a_{0z}}{2}\exp\left[-\left(z-z_0-\frac{p_zt}{m}+\frac{\mu t^2}{4m}\right)^2\right]\times$$
$$\exp\left[-\left(p_z-\frac{\mu t}{2}\right)^2\right]\cdot\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\cos\theta\right)+$$
$$+\frac{1}{\pi}\frac{1-a_{0z}}{2}\times\exp\left[-\left(z-z_0-\frac{p_zt}{m}-\frac{\mu t^2}{4m}\right)^2\right]\exp\left[-\left(p_z+\frac{\mu t}{2}\right)^2\right]\times$$
$$\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\cos\theta\right)+$$
$$+\frac{\sqrt{3}\sin\theta}{2\pi}(a_{0x}\cos\varphi+a_{0y}\sin\varphi)\exp\left[-\left(z-z_0-\frac{p_zt}{m}\right)^2\right]\exp\left[-p_z^2\right]$$

where $\mu=\mu_0B'$, $\varphi=\phi-\mu t(z-pt/2m)$. (Here and below we use the units system in which the coordinate and momentum packet widths are equal to unity.)

2 Beam Splitting and Relaxation of Polarization In Inhomogeneous Magnetic Field

A probability to find a particle with a definite spin projection in a point z is defined according to (3) by

$$P_\uparrow(z) = \int w_\uparrow(\theta,\phi) f(z,p_z,\theta,\phi) \sin\theta d\theta d\phi dp_z, \tag{8}$$

$$P_\downarrow(z) = \int w_\downarrow(\theta,\phi) f(z,p_z,\theta,\phi) \sin\theta d\theta d\phi dp_z,$$

where $w_\uparrow = \frac{1}{2}\left(1+\sqrt{3}\cos\theta\right)$, $w_\downarrow = \frac{1}{2}\left(1-\sqrt{3}\cos\theta\right)$ are the functions corresponding by (2) to the projectors

$$U_\uparrow = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, U_\downarrow = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Upon integrating we obtain

$$P_\uparrow(z) = \frac{1+a_{0z}}{2} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{1+\left(\frac{t}{m}\right)^2}} \cdot \exp\left\{-\frac{\left(z-z_0-\frac{\mu t^2}{4m}\right)^2}{1+\left(\frac{t}{m}\right)^2}\right\}, \tag{9}$$

$$P_\downarrow(z) = \frac{1-a_{0z}}{2} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{1+\left(\frac{t}{m}\right)^2}} \cdot \exp\left\{-\frac{\left(z-z_0+\frac{\mu t^2}{4m}\right)^2}{1+\left(\frac{t}{m}\right)^2}\right\}. \tag{10}$$

The mean value of the z coordinate is

$$< z >= \int_{-\infty}^{+\infty} z\left(P_\uparrow(z)+P_\downarrow(z)\right) dz = \frac{a_{z0}\mu}{2m}\frac{t^2}{2} + z_0. \tag{11}$$

The beam splitting in momentum space is conveniently described by the possibility of finding a particle in the point $z = z_0$, which equals

$$P(z_0) = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{1+\left(\frac{t}{m}\right)^2}} \cdot \exp\left[-\frac{\left(\frac{\mu t^2}{4m}\right)^2}{1+\left(\frac{t}{m}\right)^2}\right]. \tag{12}$$

For the further analysis we shall need also the probabilities of finding a particle with a momentum p_z. They are given by

$$P_\uparrow(p_z) = \frac{1+a_{0z}}{2}\frac{1}{\sqrt{\pi}} \exp\left[-\left(p_z - \frac{\mu t}{2}\right)^2\right], \tag{13}$$

$$P_\downarrow(p_z) = \frac{1-a_{0z}}{2}\frac{1}{\sqrt{\pi}} \exp\left[-\left(p_z + \frac{\mu t}{2}\right)^2\right]. \tag{14}$$

The beam splitting in the momentum space is characterized by the probability $P(p_z = 0)$, which equals

$$P(p_z = 0) = \frac{1}{\sqrt{\pi}} \exp\left[-\left(\frac{\mu t}{2}\right)^2\right]. \tag{15}$$

At last, let us find the components of the polarization vector. By the rule (2), Pauli matrixes

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

correspond to the functions

$$w_{\sigma_x}(\theta,\phi) = \sqrt{3}\sin\theta\cos\phi,$$

$$w_{\sigma_y}(\theta,\phi) = -\sqrt{3}\sin\theta\sin\phi,$$

$$w_{\sigma_z}(\theta,\phi) = \sqrt{3}\cos\theta.$$

The mean values we are interested in are given by

$$a_i \equiv < \sigma_i >= \int\int Sp\left(\hat{\sigma}_i \hat{f}(\vec{q},\vec{p})\right) d^3q d^3p =$$

$$= \int\int\int\int w_{\sigma_i}(\theta,\phi) f(\vec{q},\vec{p},\theta,\phi) d^3p d^3q \sin\theta d\theta d\phi.$$

With the distribution function (7) we obtain

$$a_x = (a_{x0}\cos\mu z_0 t - a_{y0}\sin\mu z_0 t)\exp\left[-\left(\frac{\mu t}{2}\right)^2 - \left(\frac{\mu t^2}{4m}\right)^2\right], \quad (16)$$

$$a_y = (a_{x0}\sin\mu z_0 t + a_{y0}\cos\mu z_0 t)\exp\left[-\left(\frac{\mu t}{2}\right)^2 - \left(\frac{\mu t^2}{4m}\right)^2\right], \quad (17)$$

$$a_z = a_{z0}. \quad (18)$$

The first factors in (16), (17) correspond to the regular precession of the polarization vector with a frequency $\mu z_0 = \mu_0 H(z_0)$; the second factors describe the relaxation of the transverse polarization (here and later the words "longitudinal" and "transverse" refer to the direction in reference to the field, not to the particle velocity) It is convenient to introduce the absolute value of the transverse polarization vector, given by

$$a_t \equiv \sqrt{a_x^2 + a_y^2} = a_{t0}\exp\left[-\left(\frac{\mu t}{2}\right)^2 - \left(\frac{\mu t^2}{4m}\right)^2\right], \quad (19)$$

Formulae (12), (15) and (19) involve three characteristic times: $\tau_1 = (4m/\mu)^{1/2}$, $\tau_2 = 2/\mu$, $\tau_3 = m$. The physical meaning of these parameters is clear from (12) and (15): τ_1 is the time of the beam splitting in coordinate space in the case of quasiclassical motion ($\tau_3 \gg \tau_1$); τ_2 is the time of the beam splitting in the momentum space (it equals the time of the spatial splitting in the case of a great quantum diffraction $\tau_3 \ll \tau_1$); τ_3 is the quantum diffraction characteristic time. In dimensional units these times are: $\tau_1 = (2\sigma_q/a)^{1/2}$, $\tau_2 = \sigma_p/ma$, $\tau_3 = m\sigma_q/\sigma_p$, where $a = \mu_0/2m\,|dB_z/dz|$ is the classical acceleration due to inhomogeneous magnetic field.

If a point source is located at a distance d from a diaphragm of a radius σ_q, the value of σ_p is defined by the angular spreading of the beam $\sigma_q p/d$ and the quantum diffraction $\hbar/\sigma_q$. For a large angular

spreading $\sigma_q p/d \gg \hbar/\sigma_q$ we have $\sigma_p \simeq \sigma_q p/d$. In the opposite limit $\sigma_p \simeq \hbar/\sigma_q$.

The characteristic times are related by

$$\tau_1^2 = \tau_2\tau_3. \quad (20)$$

Using the introduced notation let us rewrite (19) as

$$a_t = a_{t0}\exp\left[-\left(\frac{t}{\tau_2}\right)^2 - \left(\frac{t}{\tau_1}\right)^4\right]. \quad (21)$$

Let us analyze this formula. Consider first the case of the quasiclassical motion, $\tau_3 \gg \tau_1$. Then (20) implies that $\tau_2 \ll \tau_1$. It is seen from (21) that the relaxation time of the transverse polarization in this case equals τ_2. The relaxation of the transverse polarization is governed by the beam splitting in momentum space and occurs much more quickly then the spatial separation of the beam.

In the case $\tau_3 \ll \tau_1$ (great quantum diffraction) (20) implies that $\tau_2 \gg \tau_1$. The times of the beam splitting in momentum and coordinate spaces are both equal to τ_2, and the relaxation time of the transverse polarization , as seen from (21), is equal to τ_1. Thus, the relaxation of the transverse polarization of a beam in an inhomogeneous magnetic field always occurs more quickly then the spatial splitting of the beam; in the quasiclassical case the characteristic time of the relaxation equals σ_p/ma, and in the case of great quantum diffraction – $(2\sigma_q/a)^{1/2}$.

The obtained results allow us to analyze the limits of applicability of the nonrelativistic classical equation for the motion of a polarization vector in the considered field, which was obtained by assuming the conservation of its absolute value. It has the form:

$$\dot{\vec{a}} = \mu_0\left[\vec{H}\times\vec{a}\right]. \quad (22)$$

The equations of motion for the quantum mean values of spin differ from (22). This fact is connected with the correlation between coordinate and spin variables, which is also responsible for the negative values of the distribution function [12]. It is readily seen from (16) and (17) that

$$\dot{a}_x = -H_z a_y - \left(\frac{2t}{\tau_2^2} + \frac{4t^3}{\tau_1^4}\right) a_x, \tag{23}$$

$$\dot{a}_y = H_z a_x - \left(\frac{2t}{\tau_2^2} + \frac{4t^3}{\tau_1^4}\right) a_y. \tag{24}$$

The additional terms in the right sides of (23) and (24) describe the relaxation of the transverse polarization. Their form depends on the model in use (gaussian beams). In general, the motion of the polarization vector in an inhomogeneous magnetic field does not allow a classical description. The equation (22) is an approximation; it describes the motion of the polarization vector rather well when the change of its absolute value over the precession period is negligible and time is small in comparison with the relaxation time of the transverse polarization, but it does not hold at greater t.

Qualitative results of the performed analysis of the transverse polarization relaxation remain valid for the relativistic motion of a particle in the conditions of Stern-Gerlach experiment. The characteristic times in this case differ from nonrelativistic ones because of the changed kinematics:

$$\tau_1 = \left(\frac{2\sigma_q \gamma}{a}\right)^{1/2}, \tau_2 = \frac{\sigma_p}{m_0 a}, \tau_3 = \frac{m_0 \sigma_q \gamma}{\sigma_p},$$

where m_0 is the rest mass, $a = \mu_0/2m_0\,|dB_z/dz|$, $\gamma = (1-v_y^2/c^2)^{-1/2}$ is the relativistic factor of the particle (we assume the transverse motion nonrelativistic) The relationship (20) holds in this case too.

So, the conditions of the classical description are violated because of the beam depolarization. The depolarization is due to the separation of the beams with different spin projections in momentum or coordinate space, i. e. with disappearing of interferention effects. This conclusion holds for a motion of particles, both neutral and charged, in arbitrary external fields.

Relativistic classical equations for a coordinate and a polarization vector in an arbitrary inhomogeneous electromagnetic field, according to [4], have the form (for a neutral particle):

$$\frac{du_\mu}{d\tau} = \frac{\mu_0}{4m}\epsilon_{abcd}F^{ab}_{;\mu}a^c u^d. \tag{25}$$

$$\frac{da^\mu}{d\tau} = -\mu_0 F^{\mu\nu}a_\nu + \mu_0 u^\mu F^{\lambda\nu}u_\lambda a_\nu - u^\mu \frac{du_\nu}{d\tau}a^\nu, \tag{26}$$

where u^μ is the four-velocity of the particle, a^μ is the polarization vector, $F^{\mu\nu}$ is the electromagnetic field tensor, and ϵ_{abcd} is an absolutely antisymmetric tensor.

In this case in addition to the limits of applicability of the classical equations discussed above one has to evaluate the validity of considering the terms in (26), connected with Thomas precession (third term in the right side of (26)). The relaxation of the transverse polarization over the period of Thomas precession must be negligible, for the condition $a_\nu a^\nu = const$ was used to deduce the equations (25 - 26).

To generalize the equation (25) on the motion in a gravitational field it is proposed [3] to introduce (in the first order of $\hbar$) a term

$$\frac{1}{2}R^{ab}_{\mu\nu}u^\nu \epsilon_{abcd}a^c u^d,$$

where $R^{ab}_{\mu\nu}$ is the Riemann tensor. This additional term, as well as the right side of (25), is linear in relation to the polarization vector and therefore must also cause the splitting of the quasiclassical trajectories and depolarization of the beam.

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