СООБЩЕНИЯ
ОБЪЕДИНЕННОГО
ИНСТИТУТА
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E2-95-410

A.P.Isaev*

$SL_q(N)$ DIFFERENTIAL CALCULUS FROM THE DIFFERENTIAL CALCULUS ON $GL_q(N)$

*E-mail address: isaevap@thsun1.jinr.dubna.su

1995

In the paper [1], the bicovariant differential algebra on $SL_q(N)$ (see below (10), (11)) with generators $\{T_{ij},\ L_{ij},\ \tilde{\Omega}_{ij}\}\ i,j=1,\dots N$ has been constructed. The elements T_{ij} are generators of the algebra $Fun(SL_q(N))$, while L_{ij} generate a matrix of the right-invariant Lie derivatives on $SL_q(N)$ and the elements $\tilde{\Omega}_{ij}$ define the basis of the right-invariant differential 1-forms on $SL_q(N)$. It has been shown [1] that this algebra is consistent with imposing the conditions:

$$det_q(T) = Det_q(L) = 1\ ,\ \ Tr_q(\tilde{\Omega}) = 0\ , \tag{1}$$

where we use

$$Det_q(L) = det_q(LT)\frac{1}{det_q(T)}\ ,\ \ Tr_q(\tilde{\Omega}) = \sum_{i=1}^{N} q^{-N-1+2i}\tilde{\Omega}_{ii}\ .$$

The last condition in (1) shows that the number of independent differential 1-forms on $SL_q(N)$, in the Faddeev-Pyatov approach, is equal to (N^2-1) just as in the classical case.

Another bicovariant differential algebra (which was considered as a differential algebra on $GL_q(N)$) has been proposed in [2] in the framework of the R- matrix approach [3] to the general theory of differential calculi on quantum groups [4]. The defining relations for this algebra have the form:

$$\begin{cases} \hat{R}_{12}T_1T_2 = T_1T_2\hat{R}_{12}\ ,\ \ T_1L_2 = \hat{R}_{12}L_1\hat{R}_{12}T_1\ ; \\ \\ T_1\Omega_2 = \hat{R}_{12}^{-1}\Omega_1\hat{R}_{12}^{-1}T_1\ ,\ \ \hat{R}_{12}^{-1}\Omega_1\hat{R}_{12}^{-1}\Omega_1 + \Omega_1\hat{R}_{12}^{-1}\Omega_1\hat{R}_{12} = 0\ , \\ \\ \hat{R}_{12}L_1\hat{R}_{12}L_1 = L_1\hat{R}_{12}L_1\hat{R}_{12}\ ,\ \ \hat{R}_{12}L_1\hat{R}_{12}\Omega_1 = \Omega_1\hat{R}_{12}L_1\hat{R}_{12}\ , \end{cases} \tag{2}$$

$$\begin{cases} T_1\Im_2 = \hat{R}_{12}\Im_1\hat{R}_{12}T_1\ ,\ \ \Im_1\hat{R}_{12}^{-1}\Omega_1\hat{R}_{12}^{-1} + \hat{R}_{12}^{-1}\Omega_1\hat{R}_{12}^{-1}\Im_1 = -\hat{R}_{12}^{-1}\ , \\ \\ \hat{R}_{12}L_1\hat{R}_{12}\Im_1 = \Im_1\hat{R}_{12}L_1\hat{R}_{12}\ ,\ \ \hat{R}_{12}\Im_1\hat{R}_{12}\Im_1 + \Im_1\hat{R}_{12}\Im_1\hat{R}_{12}^{-1} = 0\ . \end{cases} \tag{3}$$

Here we rewrite the defining relations for the differential algebra of [2] in terms of the right-invariant generators, use the notation L instead of Y, change $GL_q(N)$ R-matrix $\hat{R}_{12}$ to $\hat{R}_{12}^{-1}$ and denote right-invariant inner derivations by $\Im_{ij}$. For the algebra (2), (3) one can construct a 1- parametric family of differentials d_x satisfying the usual Leibnitz rule:

$$d_x A = [\xi_x, A]_\pm \,, \; \xi_x = \frac{q^N}{\lambda} Tr_q\,(\Omega(1 + x\,W\,L))\,, \; \xi_x^2 = 0\,, \qquad (4)$$

where $\lambda = q - q^{-1}$, $W = L(1 - \lambda\Omega\Im)L^{-1}$ (the operator $(1 - WL)/\lambda$ defines right-invariant vector fields on $GL_q(N)$), A is a function of $\{T, \Omega\}$ and x is a parameter. The last identity in (4) follows from the relation (for all x):

$$\hat{R}_{12}\Omega_x\hat{R}_{12}^{-1}\Omega_x = -\Omega_x\hat{R}_{12}^{-1}\Omega_x\hat{R}_{12}^{-1}\,, \; \Omega_x = \Omega\,(1 + xWL)\,.$$

In the paper [2], there was suggested a way to reduce the $GL_q(N)$ algebra (2), (3) to an algebra which was interpreted as a differential algebra on $SL_q(N)$. Unfortunately, this reduction to the $SL_q(N)$ case leads to N^2 independent differential 1-forms Ω_{ij}. Below we demonstrate that the $SL_q(N)$ differential algebra [1], with a correct number $(N^2 - 1)$ of independent differential 1-forms, can be obtained as a subalgebra of the $GL_q(N)$ differential algebra (2), (3).

Let us pass in eqs. (2), (3) to a new basis of differential forms, inner derivations and new generators of the quantum group $Fun(GL_q(N))$ (see [6]):

$$\Omega \to \Omega^L = L\Omega\,, \; \Im \to \Im^L = \Im L^{-1}\,, \; T \to (det_q T)^{-1/N}\,T\,. \qquad (5)$$

Then, relations (2), (3) take the following form in terms of the new generators:

$$\begin{cases} \hat{R}_{12}T_1T_2 = T_1T_2\hat{R}_{12}\,, \; q^{2/N}T_1\,L_2 = \hat{R}_{12}\,L_1\,\hat{R}_{12}T_1\,, \\ T_1\,\Omega_2^L = \hat{R}_{12}\Omega_1^L\,\hat{R}_{12}^{-1}T_1\,, \; \hat{R}_{12}\Omega_1^L\,\hat{R}_{12}\Omega_1^L + \Omega_1^L\,\hat{R}_{12}\,\Omega_1^L\,\hat{R}_{12}^{-1} = 0\,, \\ \hat{R}_{12}\,L_1\,\hat{R}_{12}\,L_1 = L_1\,\hat{R}_{12}\,L_1\,\hat{R}_{12}\,, \; \hat{R}_{12}\,L_1\,\hat{R}_{12}\,\Omega_1^L = \Omega_1^L\,\hat{R}_{12}\,L_1\,\hat{R}_{12}\,, \end{cases} \qquad (6)$$

$$\begin{cases} T_1\,\Im_2^L = \hat{R}_{12}\,\Im_1^L\,\hat{R}_{12}^{-1}T_1\,, \; \hat{R}_{12}\,\Im^L{}_1\,\hat{R}_{12}^{-1}\Im^L{}_1 + \Im^L{}_1\,\hat{R}_{12}^{-1}\,\Im^L{}_1\,\hat{R}_{12}^{-1} = 0\,, \\ \hat{R}_{12}\,L_1\,\hat{R}_{12}\,\Im^L{}_1 = \Im^L{}_1\,\hat{R}_{12}\,L_1\,\hat{R}_{12}\,, \end{cases} \qquad (7)$$

$$\Im^L{}_1\,\hat{R}_{12}^{-1}\,\Omega_1^L\,\hat{R}_{12} + \hat{R}_{12}\,\Omega_1^L\,\hat{R}_{12}^{-1}\,\Im^L{}_1 = -\hat{R}_{12}\,. \qquad (8)$$

Now we show that the subalgebra (6) contains the Faddeev-Pyatov $SL_q(N)$ subalgebra $\{T,\, L,\, \tilde{\Omega}\}$ where we have introduced the traceless generators $\tilde{\Omega}$ defined by

$$\tilde{\Omega} = \Omega^L - \frac{1}{N_q} Tr_q\Omega^L \cdot 1\,, \; N_q = \frac{q^N - q^{-N}}{q - q^{-1}}\,. \qquad (9)$$

Indeed, after substituting (9) into relations (6) and using formulas (which are a consequence of the fourth equation in (6))

$$(Tr_q\Omega^L)^2 = 0\,, \; [Tr_q\Omega^L,\, \tilde{\Omega}]_+ = \lambda q^N(\kappa_q - 1)\tilde{\Omega}^2\,,$$

where $\kappa_q = \lambda q^N(N_q + \lambda q^N)^{-1}$, we deduce the algebra [1]:

$$\begin{cases} \hat{R}_{12}T_1T_2 = T_1T_2\,\hat{R}_{12}\,, \; q^{2/N}T_1\,L_2 = \hat{R}_{12}\,L_1\,\hat{R}_{12}T_1\,, \\ T_1\,\tilde{\Omega}_2 = \hat{R}_{12}\,\tilde{\Omega}_1\,\hat{R}_{12}^{-1}\,T_1\,, \\ \hat{R}_{12}\,L_1\,\hat{R}_{12}\,L_1 = L_1\,\hat{R}_{12}\,L_1\,\hat{R}_{12}\,, \; \hat{R}_{12}\,L_1\,\hat{R}_{12}\,\tilde{\Omega}_1 = \tilde{\Omega}_1\,\hat{R}_{12}\,L_1\,\hat{R}_{12}\,, \end{cases} \qquad (10)$$

$$\hat{R}_{12}\tilde{\Omega}_1\hat{R}_{12}\tilde{\Omega}_1 + \tilde{\Omega}_1\hat{R}_{12}\tilde{\Omega}_1\hat{R}_{12}^{-1} = \kappa_q(\tilde{\Omega}^2 + \hat{R}_{12}\tilde{\Omega}^2\hat{R}_{12}) . \quad (11)$$

The algebra of differential 1-forms, with defining relations (11), has been proposed in [5]. It is clear that the whole algebra (6)-(8) includes the subalgebra $\{T, L, \tilde{\Omega}, \Im\}$ with defining relations (10), (11), (7) and, instead of the relation (8), we also have to take

$$\Im^L{}_1\hat{R}_{12}^{-1}\tilde{\Omega}_1\hat{R}_{12} + \hat{R}_{12}\tilde{\Omega}_1\hat{R}_{12}^{-1}\Im^L{}_1 = \frac{\kappa_q}{\lambda(1-\kappa_q)} - \hat{R}_{12} . \quad (12)$$

One can show that the algebra (6)-(8) (and, therefore, the subalgebra (10), (11), (7), (12)) possesses an additional central element [6]:

$$Z = Det_q\left((\overline{W}W)^{-1}\right) \equiv det_q\left((\overline{W}W)^{-1}T\right)\frac{1}{det_q(T)} , \quad (13)$$

$$Z = Det_q(W^{-1}) \cdot Det_q(\overline{W}^{-1}) ,$$

where the operators $\overline{W} = 1 - \lambda\Im^L\Omega^L$ and $W = 1 - \lambda\Omega^L\Im^L$ satisfy the commutation relations:

$$\hat{R}_{12}^{-1}\overline{W}_1\hat{R}_{12}W_1 = W_1\hat{R}_{12}^{-1}\overline{W}_1\hat{R}_{12} ,$$

$$\hat{R}_{12}^{-1}\overline{W}_1\hat{R}_{12}^{-1}\overline{W}_1 = \overline{W}_1\hat{R}_{12}^{-1}\overline{W}_1\hat{R}_{12}^{-1} , \quad (14)$$

$$\hat{R}_{12}^{-1}W_1\hat{R}_{12}^{-1}W_1 = W_1\hat{R}_{12}^{-1}W_1\hat{R}_{12}^{-1} .$$

Note that from these relations we immediately (see [7]) obtain:

$$\hat{R}_{12}^{-1}(\overline{W}W)_1\hat{R}_{12}^{-1}(\overline{W}W)_1 = (\overline{W}W)_1\hat{R}_{12}^{-1}(\overline{W}W)_1\hat{R}_{12}^{-1} . \quad (15)$$

We stress also that the combination $(\overline{W}W)$ is independent of the scalar generator $Tr_q\Omega^L$:

$$(\overline{W}W) = \frac{1}{1-\kappa_q} - \lambda[\tilde{\Omega}, \Im^L]_+ + \lambda^2(1-\kappa_q)\Im^L\tilde{\Omega}^2\Im^L .$$

It means that if we fix the central element (13), e.g. in the form $Z = 1$, we, therefore, impose a new additional relation on the generators $\Im^L_{ij}$. One can try to consider the relation $Z = 1$ as a constraint which finally reduces the subalgebra generated by $\{T, L, \tilde{\Omega}, \Im\}$ to the whole differential algebra on $SL_q(N)$. However, it is not quite clear: is the relation $Z = 1$ sufficient for removing the redundant generator $Tr_q\Im^L$ from this algebra to obtain the complete $SL_q(N)$ reduction?

Unfortunately, neither of the differential operators (4) (for the Schupp-Watts-Zumino algebra (2), (3)) yields an apropriate differential d (satisfying the usual Leibnitz rule) for the Faddeev-Pyatov differential subalgebra (10), (11). Thus, up to now, we have only one self-consistent construction of the exterior differential d (for the algebra (10), (11)) which satisfies the deformed Leibnitz rule (see [1]). Nevertheless, we hope that the method of obtaining the differential algebra on $SL_q(N)$ (10), (11) as a subalgebra of the differential algebra on $GL_q(N)$ (2), (3) will help us to clarify the Hopf and star structure of this subalgebra and also may help us to construct the explicit realization (e.g. via deformed commutator with nilpotent generators (4)) of an exterior differential d obeying the deformed Leibnitz rule [1].

To conclude this letter I would like to stress that the situation with bicovariant differential calculi on $SO_q(N)$ and $Sp_q(2n)$ groups is obscured up to now. Woronowicz's approach [4] leads, in this case, to the differential algebras which are not of the Poincaré- Birkhoff- Witt type and all attempts to improve this situation were failed (for the discussion see [8]).

Acknowledgement

I would like to thank P.N.Pyatov and A.A.Vladimirov for many valuable discussions.

This research was supported in part by the Russian Foundation of

Fundamental Research (grant No 95-02-05679-a), by the International Science Foundation (grant RFF 300) and by the INTAS grant No 93-127.

References

[1] Faddeev, L.D., Pyatov, P.N.: The differential calculus on quantum linear groups. hep-th/9402070, to appear in F.A.Berezin memorial volume.

[2] Schupp, P., Watts, P., Zumino, B.: Lett. Math. Phys. **25** (1992) 139-147; Zumino, B.: Differential Calculus on quantum spaces and Quantum Groups. Preprint LBL-33249 and UCB-PTH-92/41 (1992).

[3] Faddeev, L.D., Reshetikhin, N.Yu., Takhtajan, L.A.: Quantization of Lie groups and Lie algebras (in Russian). Algebra i Analiz **1** (1989) 178-206. English translation in: Leningrad Math. J. **1** (1990) 193.

[4] Woronowicz, S.L.: Commun. Math. Phys. **122** (1989) 125-170.

[5] Isaev, A.P., Pyatov, P.N.: J.Phys. A **28** (1995) 2227; hep-th/9311112.

[6] Isaev, A.P.: *R*-matrix aproach to the differential calculus on quantum groups, submitted to Algebra i Analiz.

[7] Majid Sh.: J.Math.Phys., **34** (1993)1176;
Isaev A.P., Vladimirov A.A.: Lett.Math.Phys. **33** (1995) 297.

[8] Arutyunov, G.E., Isaev, A.P., Popowicz, Z.: J.Phys. A **28** (1995) 4349.

Received by Publishing Department
on September 22, 1995.