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RELATIVISTIC SUSY QM AS DEFORMED SUSY QM

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Релятивистская SUSY квантовая механика
как деформированная SUSY квантовая механика

Построена релятивистская конечно-разностная SUSY квантовая механика. Показано, что она естественным образом связана с q-деформированной SUSY квантовой механикой. Даны простые примеры.

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Relativistic SUSY QM as Deformed SUSY QM

The relativistic finite-difference SUSY Quantum Mechanics (QM) is developed. We show that it is connected in a natural way with the q-deformed SUSY Quantum Mechanics. Simple examples are given.

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The relativistic QM is based on Snyder's idea of noncommutative space-time coordinates and the concept of relativistic configurational space. We refer the reader to the review papers [1-5], and the references therein. The natural connection of RQM with q-deformations is discussed in the articles [5, 6].

The one-dimensional relativistic Schrödinger equation has the form

$$(h-e)\,\psi(x) = (h_0 + V(x) - e)\,\psi(x) = 0, \tag{1}$$

where

$$\begin{gathered} h_0 = 2mc^2 \sinh^2 \tfrac{i\hbar}{2mc}\tfrac{d}{dx} = \tfrac{\hat{k}^2}{2m} \rightarrow -\tfrac{\hbar^2}{2m}\tfrac{d^2}{dx^2} \; in\ non-relativistic\ limit \\ \hat{k} = -2mc \sinh \tfrac{i\hbar}{2mc}\tfrac{d}{dx}, \qquad k = 2mc \sinh \tfrac{\chi}{2}, \qquad e = \tfrac{k^2}{2} \end{gathered} \tag{2}$$

We stress that the relativistic Schrödinger equation (1) is the finite-difference equation with a step equal to the Compton wave length $\frac{\hbar}{mc}$ of the particle. Usually, we shall use the unit system in which $\hbar = c = m = 1$. It is clear that the factorization method,that plays the key role in SUSY QM, must be modified in the finite-difference context [5]. Let us introduce a couple of ladder operators

$$A^{\pm} = \pm i\sqrt{2}\cdot\alpha(x)\cdot e^{\pm\rho(x)} \sinh\frac{i}{2}\frac{d}{dx}\cdot e^{\mp\rho(x)} \tag{3}$$

or

$$A^{\pm} = -i\sqrt{2}\cdot\alpha(x)\cdot e^{\pm\xi(x)}\left[\sinh\rho_{\frac{s}{2}}(x)\cosh\frac{i}{2}\frac{d}{dx} \mp \cosh\rho_{\frac{s}{2}}(x)\sinh\frac{i}{2}\frac{d}{dx}\right] \tag{4}$$

where

$$\rho_{\frac{s}{2}}(x) = \sinh\frac{i}{2}\frac{d}{dx}\rho(x) \qquad \rho_{\frac{c}{2}}(x) = \cosh\frac{i}{2}\frac{d}{dx}\rho(x) \tag{5}$$

$$\xi(x) = \rho(x) - \rho_{\frac{c}{2}}(x) \tag{6}$$

and $\rho(x)$ is the logarithm of the ground state wave function of eq. (1)

$$\psi_0(x) = e^{-\rho(x)} \tag{7}$$

In the nonrelativistic limit the operators $A^{\pm}$ turn into the usual ladder operators (cf. [5, 6]). In addition to the finite-difference character of the

operators (3, 4) there are two factors $\alpha(x)$ and $e^{\pm\xi(x)}$ that don't appear in the nonrelativistic case and whose presence enforces the difference with the nonrelativistic ladder operators. The factor $\alpha(x)$ is connected with a natural lattice variable and is also expressed in terms of $\rho(x)$ (see [5, 9]). Factors $e^{\pm\xi(x)}$ are connected with deformations. Some quantity (deformation)

$$q(x) = e^{a(x)} \tag{8}$$

must be introduced to cancel $e^{\pm\xi(x)}$. In the nonrelativistic case $\xi(x) \rightarrow 1$ and there is no deformation : $q(x) \rightarrow 1$. Let us consider the $q(x)$-mutator

$$[A^-, A^+]_{q(x)} = A^-\cdot q(x)\cdot A^+ - A^+\cdot q^{-1}(x)\cdot A^- =$$

$$= \frac{\alpha(x)}{2}\left\{\begin{array}{c} e^{\frac{i}{2}\frac{d}{dx}}\sinh Z(x)\,\alpha(x)\,e^{\frac{i}{2}\frac{d}{dx}} + e^{-\frac{i}{2}\frac{d}{dx}}\sinh Z(x)\,\alpha(x)\,e^{-\frac{i}{2}\frac{d}{dx}} - \\ -e^{\frac{i}{2}\frac{d}{dx}}\sinh\left(Z(x) + 2\rho_{\frac{s}{2}}(x)\right)\alpha(x)\,e^{-\frac{i}{2}\frac{d}{dx}} - \\ -e^{-\frac{i}{2}\frac{d}{dx}}\sinh\left(Z(x) - 2\rho_{\frac{s}{2}}(x)\right)\alpha(x)\,e^{\frac{i}{2}\frac{d}{dx}} \end{array}\right\} \tag{9}$$

where

$$Z(x) = 2\xi(x) + a(x) \tag{10}$$

Let us recall that the commutator of the nonrelativistic ladder operators $a^{\pm}$ does not contain the differentiation operators

$$[a^-, a^+] = \frac{d^2\rho(x)}{dx^2} \tag{11}$$

By analogy with (9) we shall require that there is no finite-difference derivatives $e^{\pm\frac{i}{2}\frac{d}{dx}}$ in the r.h.s. of (9). The simplest way to achieve this is to put

$$Z(x) = 0 \tag{12}$$

The last equation gives the relation connecting $\rho(x)$ and $a(x)$ (or $q(x)$):

$$a(x) = -2\xi(x) \qquad q(x) = e^{-2\xi(x)} \tag{13}$$

We have

$$\begin{gathered} [A^-, A^+]_{q(x)} = -2\alpha(x)\sinh\tfrac{i}{2}\tfrac{d}{dx}\left[\alpha(x)\sinh 2\rho_{\frac{s}{2}}(x)\right] \rightarrow \\ \rightarrow \tfrac{d^2\rho(x)}{dx^2} \end{gathered} \tag{14}$$

Now let us write down the basic relations of relativistic SUSY QM, i.e., the relativistic quantum mechanical system whose Hamiltonian is constructed of anticommuting charges Q [10]:

$$\hat{H} = \frac{1}{2} \cdot \{Q, Q^{\dagger}\}_{q(x)} = \frac{1}{2} \cdot \left(Q \cdot q(x)^{-1} \cdot Q^{\dagger} + Q^{\dagger} \cdot q(x) \cdot Q \right) \qquad (15)$$

$$\{Q, Q\} = \{Q^{\dagger}, Q^{\dagger}\} = 0 \qquad (16)$$

As in the nonrelativistic, case the supersymmetry property of Hamiltonian $\hat{H}$

$$\left[\hat{H}, Q\right] = \left[\hat{H}, Q^{\dagger}\right] = 0 \qquad (17)$$

is provided by nilpotency of the charge operators (16). The Hamiltonian $\hat{H}$ contains coordinates that are quantized by $q(x)$-mutators and anticommutators. They are mixed by deformed supersymmetry transformations. The explicit realization of Q and $Q^{\dagger}$ is

$$Q = i\sqrt{2} \cdot A^{+} \cdot \hat{\psi}^{\dagger}, \qquad Q^{\dagger} = -i\sqrt{2} \cdot A^{-} \cdot \hat{\psi} \qquad (18)$$

In the simplest case the bosonic degrees of freedom represented by the ladder operators $A^{\pm}$ are described by the momentum operator (2) and the position operator x with the commutation relation

$$\left[x, \hat{k}\right] = i \cosh \frac{i}{2} \frac{d}{dx} \qquad (19)$$

whereas $\hat{\psi}^{\dagger}$ and $\hat{\psi}$ are Fermi degrees of freedom with the corresponding anticommutation relations:

$$\left\{\hat{\psi}^{\dagger}, \hat{\psi}\right\} = 1, \quad \left\{\hat{\psi}, \hat{\psi}\right\} = \left\{\hat{\psi}^{\dagger}, \hat{\psi}^{\dagger}\right\} = 0 \qquad (20)$$

This yields (16) and

$$\hat{H} = H - \frac{1}{2} \cdot \left[\hat{\psi}^{\dagger}, \hat{\psi}\right] \cdot \Delta V(x). \qquad (21)$$

We introduce the operator

$$H = \tfrac{1}{2} \cdot \{A^{-}, A^{+}\}_{q(x)} = \tfrac{1}{2} \cdot \left(A^{-} \cdot q(x) \cdot A^{+} + A^{+} \cdot q^{-1}(x) \cdot A^{-}\right) =$$
$$= H_0 + \alpha(x)\, \alpha_{\frac{c}{2}}(x) - \alpha(x) \cdot \cosh \tfrac{i}{2} \tfrac{d}{dx} \left(\alpha(x) \cdot \cosh 2\rho_{\frac{s}{2}}(x)\right) \rightarrow$$
$$\rightarrow -\tfrac{1}{2} \tfrac{d^2}{dx^2} + \tfrac{1}{2} \left(\rho'(x)\right)^2, \qquad (22)$$

where the operator

$$H_0 = 2\left[\alpha(x) \cdot \sinh \frac{i}{2} \frac{d}{dx}\right]^2 = \frac{\hat{P}^2}{2} \rightarrow -\frac{1}{2} \frac{d^2}{dx^2} \qquad (23)$$

plays the role of free hamiltonian with the momentum operator

$$\hat{P} = -2\alpha(x) \cdot \sinh \frac{i}{2} \frac{d}{dx} \qquad (24)$$

modified by the interaction

$$H_{+} = A^{+} \cdot q^{-1}(x) \cdot A^{-} = H - \Delta V(x)$$
$$H_{-} = A^{-} \cdot q(x) \cdot A^{+} = H + \Delta V(x) \qquad (25)$$

$$\Delta V(x) = -\alpha(x) \sinh \frac{i}{2} \frac{d}{dx} \left[\alpha(x) \sinh 2\rho_{\frac{s}{2}}(x)\right] \qquad (26)$$

In the (2×2)-representation

$$\hat{\psi}^{\dagger} = \sigma_{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\psi} = \sigma_{+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad (27)$$

$$\left[\hat{\psi}^{\dagger}, \hat{\psi}\right] = -\sigma_3 \qquad (28)$$

we find from (21)

$$\hat{H} = H + \tfrac{1}{2} \cdot \Delta V(x) \cdot \sigma_3 =$$
$$= \begin{pmatrix} H_{-} & 0 \\ 0 & H_{+} \end{pmatrix} = \begin{pmatrix} A^{-} \cdot q(x) \cdot A^{+} & 0 \\ 0 & A^{+} \cdot q^{-1}(x) \cdot A^{-} \end{pmatrix} \qquad (29)$$

EXAMPLES

1) RELATIVISTIC OSCILLATOR (q-oscillator) [5]- [8].

In this case, we have [5, 6]:

$$\rho(x) = \frac{m\omega x^2}{2\hbar} \tag{30}$$

The deformation parameter is a constant and we come to the q-oscillator with

$$q(x) = const = q = e^{-\frac{\omega\hbar}{4mc^2}} \tag{31}$$

and

$$\alpha(x) = \frac{1}{\cos\frac{\omega x}{2c}}. \tag{32}$$

The finite-difference ladder operators have the form

$$A^{\pm} = \pm i\sqrt{2}\cdot e^{\pm\frac{\omega}{8}}\cdot\left(\sinh\frac{i}{2}\frac{d}{dx} \mp i\tan\frac{\omega x}{2}\cdot\cosh\frac{i}{2}\frac{d}{dx}\right). \tag{33}$$

SUSY Hamiltonian (29) becomes

$$\hat{H} = \begin{pmatrix} e^{-\frac{\omega}{4}}A^-A^+ & 0 \\ 0 & e^{\frac{\omega}{4}}A^+A^- \end{pmatrix} = \begin{pmatrix} h+e_0 & 0 \\ 0 & h-e_0 \end{pmatrix} \to$$
$$\to \begin{pmatrix} -\frac{1}{2}\frac{d^2}{dx^2}+\frac{\omega^2x^2}{2}+\frac{\omega}{2} & 0 \\ 0 & -\frac{1}{2}\frac{d^2}{dx^2}+\frac{\omega^2x^2}{2}-\frac{\omega}{2} \end{pmatrix} \tag{34}$$

where

$$e_0 = 2\sinh\frac{\omega}{4} \to \frac{\omega}{2} \tag{35}$$

and

$$h = \{A^-,A^+\}_q = 2\left\{\left(\frac{1}{\cos\frac{\omega x}{2}}\cdot\cosh\frac{i}{2}\frac{d}{dx}\right)^2 - \cosh\frac{\omega}{4}\right\} = \tag{36}$$
$$= \frac{\hat{P}}{2} + V(x)$$

$$\hat{P} = -\frac{2}{\cos\frac{\omega x}{2}}\cdot\sinh\frac{i}{2}\frac{d}{dx} \to -i\frac{d}{dx} \tag{37}$$

The relativistic oscillator potential is

$$V(x) = \frac{\cosh\frac{\omega}{4}\cdot\left[\sin^2\frac{\omega x}{2} - \sinh^2\frac{\omega}{4}\right]}{\left[\cos^2\frac{\omega x}{2} + \sinh^2\frac{\omega}{4}\right]} \to \frac{\omega^2x^2}{2} \tag{38}$$

The spectrum is

$$\begin{pmatrix} e_n^- = 2\left(e^{\frac{2n+1}{4}\omega} - e^{-\frac{\omega}{4}}\right) & 0 \\ 0 & e_n^+ = 2\left(e^{\frac{2n+1}{4}\omega} - e^{\frac{\omega}{4}}\right) \end{pmatrix} \to$$
$$\to \begin{pmatrix} \left(n+\frac{1}{2}\right)\omega + \frac{\omega}{2} & 0 \\ 0 & \left(n+\frac{1}{2}\right)\omega - \frac{\omega}{2} \end{pmatrix} \tag{39}$$

Thus, we have two q-oscillators with zero point of energy shifted by $\pm e_0$. In other words, the energies of q-supersymmetric partners are connected by

$$e_{n+1}^- = q^{-2}\cdot e_n^+ \tag{40}$$

2) THE RADIAL PART OF THREE - DIMENSIONAL RELATIVISTIC SCHRÖDINGER EQUATION [2, 3, 4, 5]:

This equation

$$H_l s_l(r,\chi) = \left\{2\sinh^2\frac{i}{2}\frac{d}{dr} + \frac{l(l+1)}{2r(r+i)}e^{i\frac{d}{dr}}\right\} s_l(r,\chi) = \tag{41}$$
$$= (\cosh\chi - 1)\, s_l(r,\chi)$$

can be considered as one-dimensional with the potential $\frac{l(l+1)}{2r(r+i)}e^{i\frac{d}{dr}}$. Solutions of this equation, i.e., the free relativistic radial waves have the form

$$s_l(r,\chi) = \sqrt{\frac{\pi\sinh\chi}{2}}\cdot(-i)^{l+1}\cdot\frac{\Gamma(ir+l+1)}{\Gamma(ir)}\cdot P_{ir-\frac{1}{2}}^{-(l+\frac{1}{2})}(\cosh\chi) \tag{42}$$

In the nonrelativistic limit, these functions turn into free solutions of the Schrödinger equation

$$s_l(r,\chi) \to s_l(pr) = \sqrt{\frac{\pi r p}{2}}\cdot J_{l+\frac{1}{2}}(pr) \tag{43}$$

In this case the relativistic finite-difference ladder operators have the form

$$\lambda^{\pm} = \pm\frac{i}{\sqrt{2}}\left[\frac{ir\mp(l+1)}{ir} - e^{i\frac{d}{dr}}\right] \to a^{\pm} = \mp\left[\frac{d}{dr} \pm \frac{l+1}{r}\right] \tag{44}$$

Hence,

$$H_+ = H_l = \lambda^+\cdot e^{i\frac{d}{dr}}\cdot\lambda^- \qquad H_- = H_{l+1} = \lambda^-\cdot e^{i\frac{d}{dr}}\cdot\lambda^+ \tag{45}$$

In contrast with the nonrelativistic case, the rising and lowering operators $\Lambda^{\pm}$, which shift the value of the angular momentum

$$\Lambda^+ s_{l+1}(r,\chi) = s_l(r,\chi) \qquad \Lambda^- s_l(r,\chi) = s_{l+1}(r,\chi) \tag{46}$$

and the ladder operators (44) factorizing the Hamiltonian are different:

$$\begin{aligned} \Lambda^+ &= \tfrac{i}{\sinh\chi}\cdot\left[\cosh\chi - \tfrac{ir-l-2}{ir-1}\cdot e^{i\frac{d}{dr}}\right] \longrightarrow a^+ \\ \Lambda^- &= -\tfrac{i}{\sinh\chi}\cdot\left[\cosh\chi - \tfrac{ir+l}{ir-1}\cdot e^{i\frac{d}{dr}}\right] \rightarrow a^- \end{aligned} \tag{47}$$

Let us consider the identity

$$[-H_{l+1} + H_l]\cdot H_l - [H_{l+1} - H_l]\cdot H_l \equiv 0 \tag{48}$$

Using the relation

$$\lambda^- e^{i\frac{d}{dr}} - e^{i\frac{d}{dr}}\lambda^- = -\frac{i}{\sqrt{2}}\cdot[H_{l+1} - H_l] \tag{49}$$

we have

$$[-H_{l+1} + H_l]\cdot H_l - i\sqrt{2}\cdot\left[\lambda^- e^{i\frac{d}{dr}} - e^{i\frac{d}{dr}}\lambda^-\right]\cdot H_l = 0 \tag{50}$$

After acting on $s_l(r,\chi)$ and taking into account (41) and the relation

$$\Lambda^- = \frac{i}{\sinh\chi}\cdot\left[-(\cosh\chi - 1) + i\sqrt{2}\cdot e^{i\frac{d}{dr}}\cdot\lambda^-\right], \tag{51}$$

we come to the formula

$$H_{l+1}\cdot\Lambda^- s_l(r,\chi) = \Lambda^-\cdot H_l s_l(r,\chi) \tag{52}$$

which allows us to consider relativistic l and $l+1$ states as deformed supersymmetric partner states. If $s_l(r,\chi)$ is the eigenstate of H_l, then $(\Lambda^- s_l(r,\chi))$ is the eigenstate of H_{l+1} with the same eigenvalue

$$\begin{aligned} &H_l s_l(r,\chi) = (\cosh\chi - 1)\cdot s_l(r,\chi) \rightarrow \\ &\rightarrow H_{l+1}\cdot(\Lambda^- s_l(r,\chi)) = (\cosh\chi - 1)\cdot(\Lambda^- s_l(r,\chi)) \end{aligned} \tag{53}$$

The nonrelativistic (undeformed) analog of (52) is the relation

$$H_{l+1}\cdot a^- s_l(pr) = a^-\cdot H_l s_l(pr) \tag{54}$$

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