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QUANTUM DEFORMATIONS OF THE SELF-DUALITY EQUATION AND CONFORMAL TWISTORS

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Квантовые деформации уравнения самодуальности
и конформные твисторы

Рассматривается некоммутативная алгебра комплексных твисторов и их дифференциалов, основанная на квантовой группе $GL_q(4) \times SL_q(2)$. Обсуждаются также вещественные и псевдовещественные q-твисторы. Мы рассматриваем уравнение самодуальности для квантовых групп в рамках локальной алгебры дифференциальных форм на q-твисторных пространствах. Построены квантовые деформации общих многоинстантонных решений и определены соответствующие некоммутативные алгебры модулей. Общая q-инстантонная связность представляет собой функцию q-твисторов и генераторов алгебры q-модулей.

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Quantum Deformations of the Self-Duality Equation
and Conformal Twistors

A noncommutative algebra of the complex q-twistors and their differentials is considered on the basis of the quantum $GL_q(4) \times SL_q(2)$ group. Real and pseudoreal q-twistors are discussed too. We consider the quantum-group self-duality equation in the framework of the local gauge algebra of differential forms on q-twistor spaces. Quantum deformations of the general multi-instanton solutions are constructed. The corresponding noncommutative algebras of moduli are introduced. The general q-instanton connection is a function of the q-twistors and the q-moduli.

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1 Introduction

Noncommutative generalizations of the local gauge theories have been considered in the framework of different approaches [1-7]. The mathematically strict approach of Refs[3,4] is based on the noncommutative global generalizations of the classical fibre bundles. We prefer to study the local structure of the quantum-group gauge theory in terms of the deformed connection and curvature differential forms A and F. The basic algebra of these gauge forms should be covariant under the action of the quantum gauge group [5,6]

$$A \rightarrow TAT^{-1} + dTT^{-1} \tag{1.1}$$

where T and dT are elements of the differential complex on the quantum group.

Consider some classical or quantum space with the coordinates z and let $T(z)$ and $dT(z, dz)$ be noncommutative 'functions' on this space. We shall treat these functions as generators of the local gauge differential complex if the map

$$T \rightarrow T(z), \quad dT \rightarrow dT(z, dz) \tag{1.2}$$

conserves all relations between T and dT.

The general function $T(z)$ is some formal expansion with noncommutative coefficients. Thus, the localization of the quantum group is equivalent to the definition of an infinite-dimensional noncommutative Hopf algebra.

The q-deformations of the Grassmann and twistor spaces were studied in Refs[8-10]. In section 2 we consider the differential calculus on the 4-dimensional deformed complex twistor space $T_q(4, C)$. The real forms of the q-twistor space are also discussed .

Section 3 is devoted to the description of the quantum-group gauge fields on the q-twistor space. We consider the algebraic relation for the connection form in the $GL_q(N)$ gauge theory [5,6] that defines the algebraic properties of the 'off-shell' gauge fields. The generalizations of the Yang-Mills and self-duality equations are discussed. Note that one can use the 'pure gauge' $U(1)$ field and q-traceless curvature 2-forms in the $U_q(N)$ gauge theory [7]. The noncommutative analogue of the BPST one-instanton solution [11] was constructed in the deformed 4-dimensional Euclidean space [7].

The deformed analogue of the t'Hooft multi-instanton twistor solution for the gauge group $GL_q(2)$ (or $U_q(2)$) is considered in section 4. We use a multidimensional extension of the q-twistor algebra by the set of noncommutative $6D$-vector generators b. The potential of our solution is a sum of the central (z,b)-functions obeying the q-twistor Laplace equation.

The deformed generalization of the Atiyah-Drinfel'd-Hitchin-Manin solution [12] for the gauge group $GL_q(N)$ contains q-twistor functions u and $\tilde{u}$. We generalize the classical conformal constructions of Ref[13]. One can consider the linear twistor functions v and $\tilde{v}$ that depend on the noncommutative moduli b and $\tilde{b}$. The functions u and v are submatrices of the quantum $GL_q(N+2p)$ matrix for the instanton number p. The consistency relations for the deformed ADHM-construction can be proved in the framework of a differential calculus on $GL_q(N+2p)$. The self-duality condition is equivalent to the bilinear constraint on the moduli b and $\tilde{b}$.

A preliminary version of this work was published in Ref[14]. Note that we use here the modified notation and definitions of some basic quantities.

2 Differential calculus on the deformed twistor space

The conformal covariant description of the classical ADHM solution was considered in Ref[13]. This approach uses real forms of the complex $GL(4,C)\times SL(2,C)$ twistors where $GL(4,C)$ is the complex conformal group. It is convenient to discuss firstly the deformed complex twistors.

Let R^{ab}_{cd} be a solution of the Yang-Baxter equation satisfying also the Hecke relation

$$R\,R'\,R = R'\,R\,R' \tag{2.1}$$

$$R^2 = I + (q-q^{-1})R \tag{2.2}$$

where q is a complex parameter. Note that the standard notation for these R-matrices is $R=\hat{R}_{12}$, $R'=\hat{R}_{23}$ [15]. We use the symbols $a,b\ldots h = 1\ldots 4$ for the 4D-spinor indices.

The multiparameter 4D R-matrix [16-18] and corresponding inverse matrix R^{-1} can be written in the following simple form:

$$(R^{\pm 1})^{ab}_{cd} = \delta^a_c\delta^b_d[q^{\pm 1} - q^{\epsilon(a-b)}] + r(ab)\delta^a_d\delta^b_c \tag{2.3}$$

where $\epsilon(b-a)=0,\ \pm 1$ is a sign function and $r(ab)$ are complex parameters satisfying the relations $r(ab)r(ba)=1$. It is evident that this formula is valid for an arbitrary number N .

The standard $GL_q(4)$ solution corresponds to the case $r(ab)=1$ [15]. The choice $q=1$ leads to the unitary R-matrix [19]

$$R^2 = I \tag{2.4}$$

Consider also the $SL_q(2,C)$ R-matrix

$$R^{\alpha\beta}_{\mu\nu} = q\delta^\alpha_\mu\delta^\beta_\nu + \varepsilon^{\alpha\beta}(q)\varepsilon_{\mu\nu}(q) \tag{2.5}$$

where $\varepsilon(q)$ is the deformed antisymmetric symbol

$$\varepsilon^{12}(q) = -\varepsilon_{12}(q) = \frac{1}{\sqrt{q}},\quad \varepsilon^{21}(q) = -\varepsilon_{21}(q) = -\sqrt{q} \tag{2.6}$$

The q-deformed flag spaces and twistors were considered in Refs[8-10]. We shall treat the complex q-twistors z^α_a as generators of the noncommutative algebra with the basic relation

$$R^{\alpha\beta}_{\mu\nu}\, z^\mu_a\, z^\nu_b = z^\alpha_c\, z^\beta_d\, R^{dc}_{ba} \tag{2.7}$$

This relation for the (4×2) rectangular matrix z is analogous to the RTT-relations for the square quantum matrices. The consistency conditions for (2.7) are pairs of the Yang-Baxter and Hecke relations (2.1, 2.2) for the independent 4D and 2D R-matrices with the unique common parameter q.

The differential calculus on the complex q-twistor space $T_q(4,C)$ can be constructed by the analogy with the bicovariant differential complex on the quantum linear group [20-23]. Consider the relations between z^α_a and their differentials dz^α_a

$$z^\alpha_a\, dz^\beta_b = R^{\alpha\beta}_{\mu\nu}\, dz^\mu_c\, z^\nu_d\, R^{dc}_{ba} \tag{2.8}$$

$$dz^\alpha_a\, dz^\beta_b = -R^{\alpha\beta}_{\mu\nu}\, dz^\mu_c\, dz^\nu_d\, R^{dc}_{ba} \tag{2.9}$$

The elements z and dz are generators of the external algebra $\Lambda T_q(4,C)$ on the q-twistor space. The operator of external derivative d on $\Lambda T_q(4,C)$ is nilpotent and satisfies the ordinary Leibniz rule.

The symmetry properties of dz can be obtained from Eq(2.9)

$$P^{(+)}_2 dz\, dz' P^{(+)}_4 = 0 = P^{(-)}_2 dz\, dz' P^{(-)}_4 \tag{2.10}$$

where $P_2^{(\pm)}$ and $P_4^{(\pm)}$ are the projection operators for $SL_q(2)$ and $GL_q(4)$, respectively [15]

$$P^{(+)} + P^{(-)} = I, \quad R = qP^{(+)} - q^{-1}P^{(-)} \tag{2.11}$$

One can define the algebra of partial derivatives ∂_α^a on $T_q(4, C)$

$$R^{ab}_{cd}\, \partial^c_\alpha\, \partial^d_\beta = \partial^a_\mu\, \partial^b_\nu\, R^{\nu\mu}_{\beta\alpha} \tag{2.12}$$

$$\partial^a_\alpha\, z^\beta_b = \delta^a_b\, \delta^\beta_\alpha + R^{\beta\mu}_{\alpha\nu}\, R^{da}_{cb}\, z^\nu_d\, \partial^c_\mu \tag{2.13}$$

$$\partial^a_\alpha\, dz^\beta_b = R^{\beta\mu}_{\alpha\nu}\, R^{da}_{cb}\, dz^\nu_d\, \partial^c_\mu \tag{2.14}$$

Consider a definition of the deformed ε_q-symbol for $GL_q(4)$

$$\varepsilon^{abcd}_q = -qR^{ba}_{fe}\, \varepsilon^{efcd}_q = [P_4^{(-)}]^{ba}_{fe}\, \varepsilon^{efcd}_q = -q^{\epsilon(b-a)} r(ba)\, \varepsilon^{bacd}_q \tag{2.15}$$

Analogous relations are valid for other neighboring pairs of indices.

The q-twistors obey the following identity:

$$\varepsilon^{abcd}_q\, z^\beta_b\, z^\mu_c\, z^\nu_d = 0 \tag{2.16}$$

Introduce the $SL_q(2)$-invariant bilinear function of q-twistors

$$y_{ab} = \frac{q^2}{1+q^2}\, \varepsilon_{\alpha\beta}(q)\, z^\alpha_a\, z^\beta_b = [P_4^{(-)}]^{dc}_{ba}\, y_{cd} \tag{2.17}$$

This vector satisfies the following commutation relation

$$y_{ab}\, z^\alpha_c = q^{-1} R^{ed}_{ha} R^{fh}_{cb}\, z^\alpha_d\, y_{ef} \tag{2.18}$$

In consequence of Eq(2.16) the coordinate y is an isotropic vector in the deformed 6D space

$$(y, y) = \varepsilon^{abcd}_q\, y_{ab}\, y_{cd} = 0 \tag{2.19}$$

This $GL_q(4, C)$ covariant equation determines the 4D deformed subspace of the complex 6D quantum plane. The classical analogue of this subspace is a complex sphere S^C_4.

Consider a duality transformation $*$ of the basic q-twistor 2-forms by analogy with Ref[7]

$$*\, dz\, dz' = dz\, dz'\, P_4^{(+)} - dz\, dz'\, P_4^{(-)} \tag{2.20}$$

where $P_4^{(\pm)}$ are the $GL_q(4)$ projectors (2.11). Note that a self-dual part $dz\, dz'\, P_4^{(+)}$ is proportional to the $SL_q(2)$-invariant conformal tensor

$$\varepsilon_{\alpha\beta}(q)\, dz^\alpha_a\, dz^\beta_b \tag{2.21}$$

Real q-twistors can be treated as a representation of the real quantum group $SL_q(2, R) \times GL_q(4, R)$. The classical analogue of these twistors are connected with the real pseudo-Euclidean (2,2)-space [13]. Consider the R-matrices (2.3,2.5) and the conditions $|q| = 1$ and $|r(ab)| = 1$, then under the complex conjugation

$$\overline{R^{ab}_{cd}} = (R^{-1})^{ba}_{dc} \tag{2.22}$$

$$\overline{R^{\alpha\beta}_{\mu\nu}} = (R^{-1})^{\beta\alpha}_{\nu\mu} \tag{2.23}$$

These formulas correspond to anti-involution of the real $T_q(2, 2)$ twistors

$$\overline{z} = z, \quad \overline{dz} = dz \tag{2.24}$$

$$\overline{z\, z'} = z'\, z, \quad \overline{z\, dz'} = dz'\, z \tag{2.25}$$

The pseudoreal Euclidean q-twistors have a more complicated anti-involution

$$\overline{z^\alpha_a} = \varepsilon_{\alpha\beta}(q) z^\beta_b\, C^b_a(q) \tag{2.26}$$

where $C(q)$ is the charge-conjugation matrix for the Euclidean conformal quantum group $U^*_q(4) = D \times SU^*_q(4)$ and D is a real one-parameter dilatation.

It is convenient to use the simple representation

$$C(q) = \begin{pmatrix} \varepsilon^{\alpha\beta}(q) & 0 \\ 0 & \varepsilon^{\dot\alpha\dot\beta}(q) \end{pmatrix} \tag{2.27}$$

For the real q we have the following properties:

$$\overline{\varepsilon^{\alpha\beta}(q)} = \varepsilon^{\alpha\beta}(q) = -\varepsilon_{\alpha\beta}(q) \tag{2.28}$$

$$\overline{C(q)} = C(q), \quad C^2(q) = -I \tag{2.29}$$

$$\overline{\overline{z^\alpha_a}} = -\varepsilon^{\alpha\beta}(q)\varepsilon_{\beta\gamma}(q) z^\gamma_b\, (C^2)^b_a(q) = z^\alpha_a \tag{2.30}$$

$$\overline{R^{dc}_{ba}} = C^d_e(q)\, C^c_f(q)\, R^{fe}_{gh}\, C^g_a(q)\, C^h_b(q) \tag{2.31}$$

We do not here consider the conformal quantum group $SU_q(2, 2)$ [24,25] and corresponding q-twistors.

3. Quantum-group gauge theory on the q-twistor space

The classical gauge field on some domain $\{x^m\}$ of the basic space corresponds to the connection 1-form

$$A(x,dx)=dx^m A_m(x) \tag{3.1}$$

which can be decomposed in terms of the gauge-group generators. For the domain with the coordinates $\tilde{x}(x)$ one should define the transformed connection

$$\tilde{A}(\tilde{x},d\tilde{x})=T(x)AT^{-1}(x)+dT(x)T^{-1}(x) \tag{3.2}$$

where $T(x)$ is a matrix of the local gauge transformation.

The components of the matrix $A(x,dx)$ satisfy the anticommutativity conditions

$$\{A^i_k, A^l_m\}=0 \tag{3.3}$$

The classical gauge group formally has an infinite number of generators. A constructive example of the classical gauge algebra is the affine (Kac-Moody) algebra. The quantum affine algebras can be considered as a basis of the quantum gauge theory on the classical two-dimensional space.

The formal quantization of the gauge groups on the multi-dimensional classical or quantum spaces is a difficult problem. Let R_N be the constant R-matrix for the quantum group $GL_q(N)$ and x^M are the coordinates of some basic space. Consider the simplest possible relations for the components of the quantum gauge matrix

$$R_N T(x)T'(x)=T(x)T'(x)R_N \tag{3.4}$$

$$T^i_k(x)=\sum_{n=0}^{\infty}\frac{1}{n!}(T^i_k)_{M_1\cdots M_n}x^{M_1}\ldots x^{M_n} \tag{3.5}$$

where $i,k\ldots=1\ldots N$.

The quantum-group gauge matrix is well defined if the relation (3.4) generates the consistent set of relations between the coefficients $(T^i_k)_{M_1\cdots M_n}$.

Quantum deformations of the $GL(N)$ gauge connection can be treated in terms of the noncommutative gauge algebra for the components of the deformed connection 1-form [5,6]

$$(AR_NA+R_NAR_NAR_N)^{ik}_{mn}=$$

$$A^i_l(R_N)^{lk}_{rn}A^r_m+(R_N)^{ik}_{jl}A^j_r(R_N)^{rl}_{st}A^t_p(R_N)^{pt}_{mn}=0 \tag{3.6}$$

These relations generalize the classical anticommutativity conditions (3.3). The gauge algebra is an analogue of the relations between components of the right-invariant 1-forms $\omega=dTT^{-1}$ in the framework of the bicovariant differential calculus on $GL_q(N)$ [20-23]

$$\omega R_N\omega+R_N\omega R_N\omega R_N=0,\quad d\omega-\omega^2=0 \tag{3.7}$$

Thus, the form ω can be considered as a pure gauge $GL_q(N)$-field. The general $GL_q(N)$ connection A has the nontrivial curvature 2-form

$$F=dA-A^2 \tag{3.8}$$

Explicit constructions of the deformed gauge fields on the q-twistor space contain also the noncommutative elements (moduli) which generate some algebra B

$$A^i_k(z,dz,B)=dz^\alpha_a(A^a_\alpha)^i_k(z,B) \tag{3.9}$$

The appearance of additional noncommutative elements is necessary for the consistency of the algebra (3.6) and the relations of the q-twistor algebra (2.8,2.9).

The (anti)self-duality equation for the gauge field (3.9) can be defined with the help of the relations (2.20)

$$*F=\frac{1}{2}(*dz^\alpha_a dz^\beta_b)F^{ab}_{\alpha\beta}(z,B)=\pm F \tag{3.10}$$

or in terms of the deformed field-strength

$$[P^{(\pm)}_4]^{ab}_{cd}F^{cd}_{\alpha\beta}(z,B)=0 \tag{3.11}$$

The solutions of this equation and the explicit construction of the algebra B will be considered in sections 4 and 5.

A quantum deformation of the Yang-Mills equation has the standard form in the framework of the external algebra $\Lambda T_q(4,C)$

$$\nabla*F=d*F+[A,*F]=0 \tag{3.12}$$

The bicovariant differential calculus with the ordinary Leibniz rule for the operator d and the gauge-connection algebra (3.6) are consistent only for the case of the nonsemisimple quantum group $GL_q(N)$. The gauge algebra produce the restriction

$$\alpha = \mathrm{Tr}_q A \neq 0 \tag{3.13}$$

Nevertheless, one should use the gauge-covariant conditions [7]

$$d\alpha = 0, \quad \mathrm{Tr}_q A^2 = 0, \quad \mathrm{Tr}_q F = 0 \tag{3.14}$$

These restrictions generate the effective reduction of the Abelian gauge field α in the framework of the gauge group $GL_q(N)$.

4 Quantum deformations of the t'Hooft multi-instanton solution

A simple form of the manifest multi-instanton solution in the Euclidean space was discussed in Refs[26-28]. This solution can be written in terms of the potential Φ satisfying the Laplace equation. The classical twistor version of the t'Hooft solution was considered in Ref[13].

Consider firstly the deformed Laplace equation in the complex q-twistor space $T_q(4,C)$. Using Eq(2.13) one can obtain the action of the partial q-twistor derivative on the isotropic $6D$-vector

$$\partial^c_\alpha\, y_{ab} = \varepsilon_{\alpha\beta}(q)\,[P_4^{(-)}]^{dc}_{ba}\, z^\beta_d \tag{4.1}$$

Introduce the formal differential operator that acts only on the $6D$ vector variables

$$\partial^{dc} \triangleright y_{ab} = [P_4^{(-)}]^{dc}_{ba} \tag{4.2}$$

Now we can write the following relations:

$$dy_{ab} = \varepsilon_{\alpha\beta}(q)\,[P_4^{(-)}]^{dc}_{ba}\, dz^\alpha_c\, z^\beta_d \tag{4.3}$$

$$d\Phi(y) = dz^\alpha_c\, \partial^c_\alpha\, \Phi(y) = dy_{ab}\, \partial^{ba}\, \Phi(y) \tag{4.4}$$

$$\partial^c_\alpha\, \Phi(y) = \varepsilon_{\alpha\beta}(q)\, z^\beta_b\, \partial^{bc}\, \Phi(y) \tag{4.5}$$

The $SL_q(2,C)$-invariant analogue of the Laplace operator has the following form:

$$\Delta^{ba} = -\frac{q}{1+q^2}\varepsilon^{\alpha\beta}(q)\,\partial^b_\beta\,\partial^a_\alpha \tag{4.6}$$

By definition, we have

$$\Delta^{ba} \triangleright y_{cd} = [P_4^{(-)}]^{ba}_{dc} = \partial^{ba} \triangleright y_{cd} \tag{4.7}$$

In this section we shall use the standard $GL_q(4,C)$ R-matrix corresponding to Eq(2.3) with $r(ab) = 1$. This R-matrix satisfies the following identity:

$$\varepsilon_q^{abcd}\, R^{a'h}_{ea}\, R^{b'e}_{fb}\, R^{c'f}_{gc}\, R^{d'g}_{h'd} = q\, \delta^h_{h'}\, \varepsilon_q^{a'b'c'd'} \tag{4.8}$$

Introduce the additional noncommutative moduli b^p_{ab} where p is an arbitrary number

$$(b^p, b^p) = \varepsilon_q^{abcd} b^p_{ab}\, b^p_{cd} = 0 \tag{4.9}$$

$$y_{ab}\, b^p_{cd} = R^{ea'}_{ga}\, R^{fg}_{cb}\, R^{c'b'}_{he}\, R^{d'h}_{df}\, b^p_{a'b'}\, y_{c'd'} \tag{4.10}$$

$$b^p_{ab}\, b^{\hat p}_{cd} = q^{-2} R^{ea'}_{ga}\, R^{fg}_{cb}\, R^{c'b'}_{he}\, R^{d'h}_{df}\, b^{\hat p}_{a'b'}\, b^p_{c'd'} \tag{4.11}$$

and $p \leq \hat p$ in the last equation.

This (B,y)-algebra has the following central elements

$$X_p = (y, b^p) = \varepsilon_q^{abcd}\, y_{ab}\, b^p_{cd} \tag{4.12}$$

The commutativity of X_p with y and $b^{\hat p}$ can be proved with the help of Eq(4.8).

Let us introduce the commutation relations between b^p, and z

$$b^p_{ab}\, z^\gamma_c = R^{eh}_{ga}\, R^{fg}_{cb}\, z^\gamma_h\, b^p_{ef} \tag{4.13}$$

An analogous relation for b^p, and dz can be obtained as the external derivative d of this formula by using $[d, b^p] = 0$.

Equation(2.9) generates the relation for y and dz

$$y_{ab}\, dz^\gamma_c = q\, R^{eh}_{ga}\, R^{fg}_{cb}\, dz^\gamma_h\, y_{ef} \tag{4.14}$$

Write the corresponding relation for the elements X_p

$$X_p\, dz^\alpha_a = q^2\, dz^\alpha_a\, X_p \tag{4.15}$$

Now one can determine the derivative of the central functions

$$\partial^a_\alpha \frac{1}{X_p} = -\frac{1}{q^2 X_p^2}\,\partial^a_\alpha X_p \tag{4.16}$$

$$\partial^a_\alpha \frac{1}{X^2_{p}} = -\frac{1+q^2}{q^4}\frac{1}{X^3_p}\partial^a_\alpha X_p \tag{4.17}$$

It is easy to check the following identity for the isotropic vectors b^p :

$$\varepsilon^{\alpha\beta}(q)\,\partial^b_\beta X_p \partial^a_\alpha X_p = -q^{-1}\varepsilon^{abcd}_q\, b^p_{cd}\, X_p \tag{4.18}$$

We can obtain the solutions of the deformed Laplace equation (q-harmonic functions)

$$\Delta^{ba}\frac{1}{X_p} = \frac{1}{q^5(1+q^2)}\frac{1}{X^2_p}\varepsilon^{\alpha\beta}(q)\,[\partial^b_\beta\,\partial^a_\alpha X_p - (1+q^2)\frac{1}{X_p}\partial^b_\beta X_p \partial^a_\alpha X_p] = 0 \tag{4.19}$$

By analogy with Ref[13] one can consider the deformed t'Hooft Ansatz for the $GL_q(2)$ self-dual gauge field

$$A^\alpha_\beta = q^{-3}\,dz^\alpha_a\,(\partial^a_\mu\Phi)\,\Phi^{-1}\varepsilon^{\sigma\mu}(q)\varepsilon_{\sigma\beta}(q) \tag{4.20}$$

$$\mathrm{Tr}_q A = -q^3 d\Phi\,\Phi^{-1}, \quad \mathrm{Tr}_q dA = 0 \tag{4.21}$$

where the potential function Φ for the instanton number P is a sum of q-harmonic functions with different elements b^p

$$\Phi = \sum_{p=1}^{P}\frac{1}{X_p} = \sum_{p=1}^{P}(y, b^p)^{-1} \tag{4.22}$$

The anti-self-dual part of the corresponding curvature form vanishes in consequence of Eq(4.19)

$$(F - *F)^\alpha_\beta \sim dz^\alpha_e\, dz^\gamma_f\, [P_4^{(-)}]^{fe}_{ac}\,\Delta^{ac}\Phi\,\Phi^{-1}\,\varepsilon_{\beta\gamma}(q) = 0 \tag{4.23}$$

Note that the isotropic vector b^p has 5 independent elements so (4.20) is the $5P$-parameter solution.

5 Quantum deformations of the ADHM-solution

The covariant formulation of the ADHM multi-instanton solution in the classical twistor space was considered in Ref[13]. We shall discuss the quantum deformations of this formalism.

Let us consider the gauge group $GL_q(N,C)$. The ADHM-solution for the instanton number p can be connected with some $GL_q(N+2p,C)$ matrix q-twistor function. Introduce the notation for indices of different types : $A,B\ldots = 1\ldots p$; $I,K,L,M\ldots = 1\ldots N+2p$ and $i,k,l\ldots = 1\ldots N$. The ADHM Ansatz for the general self-dual $GL_q(N,C)$ field contains the deformed twistor functions $u^i_I(z)$ and $\tilde u^I_i(z)$

$$A^i_k = du^i_I\,\tilde u^I_k, \quad u^i_I\,\tilde u^I_k = \delta^i_k \tag{5.1}$$

The commutation relations for the u and $\tilde u$ twistors are

$$(R_N)^{ik}_{lm}\,u^l_I\,u^m_K = u^i_L\,u^k_M\,\mathbf{R}^{LM}_{IK} \tag{5.2}$$

$$\mathbf{R}^{KI}_{ML}\,\tilde u^L_i\,\tilde u^M_k = \tilde u^I_l\,\tilde u^K_m\,(R_N)^{ml}_{ki} \tag{5.3}$$

$$\tilde u^I_l\,(R_N)^{li}_{mk}\,u^m_K = u^i_L\,\mathbf{R}^{IL}_{KM}\,\tilde u^M_k \tag{5.4}$$

where the R-matrices for $GL_q(N,C)$ and $GL_q(N+2p,C)$ are used.

Introduce also the relation for the differentials du

$$\tilde u^I_i\,(R_N)^{ik}_{lm}\,du^l_K = du^k_L\,(\mathbf{R}^{-1})^{IL}_{KM}\,\tilde u^M_m \tag{5.5}$$

$$du^i_L\,du^k_M\,(\mathbf{R}^{-1})^{LM}_{IK} = -(R^{-1}_N)^{ik}_{lm}\,du^l_I\,du^m_K \tag{5.6}$$

These relations are necessary for proving a validity of the gauge algebra (3.6) in the framework of the ADHM-Ansatz

$$(AR_NA)^{ik}_{mn} = du^i_I\,du^k_L\,(\mathbf{R}^{-1})^{IL}_{KM}\,\tilde u^M_n\,\tilde u^K_m = -(R_NAR_NAR_N)^{ik}_{mn} \tag{5.7}$$

Consider also the linear twistor functions v and $\tilde v$

$$v^{A\alpha}_I = z^\alpha_a\,b^{aA}_I \tag{5.8}$$

$$\tilde v^{IA\alpha} = z^\alpha_a\,\tilde b^{aIA} \tag{5.9}$$

where b and $\tilde b$ are the noncommutative q-instanton moduli

$$b^{aA}_I\,z^\alpha_b = R^{da}_{cb}\,z^\alpha_d\,b^{cA}_I \tag{5.10}$$

$$\tilde b^{aIA}\,z^\alpha_b = R^{da}_{cb}\,z^\alpha_d\,\tilde b^{cIA} \tag{5.11}$$

The relations between b and $\tilde b$ will be defined below.

Introduce the following condition for the functions v and $\tilde v$:

$$v^{A\alpha}_I\,\tilde v^{IB\beta} = g^{AB}(z)\,\varepsilon^{\alpha\beta}(q) \tag{5.12}$$

where $g(z)$ is the nondegenerate $(p \times p)$ matrix with the central elements

$$g^{AB}(z) = q^{-2}\, \tilde{y}_{cd}\, b_I^{cA}\, \tilde{b}^{dIB} \qquad (5.13)$$

The condition (5.12) is equivalent to the restriction on the elements of the B-algebra

$$[P^{(+)}]^{ab}_{cd}\, b_I^{cA}\, \tilde{b}^{dIB} = 0 \qquad (5.14)$$

Write the basic commutation relations of the B-algebra

$$R^{ab}_{cd}\, b_I^{cA}\, b_K^{dB} = b_L^{aB}\, b_M^{bA}\, \mathbf{R}^{ML}_{KI} \qquad (5.15)$$
$$\mathbf{R}^{IK}_{LM}\, \tilde{b}^{aLA}\, \tilde{b}^{bMB} = R^{ab}_{cd}\, \tilde{b}^{cIB}\, \tilde{b}^{dKA} \qquad (5.16)$$
$$R^{ab}_{cd}\, b_I^{cA}\, \tilde{b}^{dKB} = \mathbf{R}^{KL}_{IM}\, \tilde{b}^{aMB}\, b_L^{bB} \qquad (5.17)$$

Remark that a formal permutation of the indices A and B is commutative.

Consider the new functions

$$\tilde{v}^I_{A\alpha} = \tilde{v}^{IB\beta}\, g_{BA}(z)\, \varepsilon_{\beta\alpha}(q) \qquad (5.18)$$

where we use the matrix g_{BA} inverse of the matrix (5.13)

$$g_{BA}(z)\, g^{AC}(z) = \delta^C_B \qquad (5.19)$$

Now one can construct the full quantum $GL_q(N+2p, C)$ matrices

$$\mathbf{U} = \begin{pmatrix} u^i_I \\ v_I^{A\alpha} \end{pmatrix}, \qquad \mathbf{S}(\mathbf{U}) = \mathbf{U}^{-1} = \begin{pmatrix} \tilde{u}^I_i \\ \tilde{v}^I_{A\alpha} \end{pmatrix} \qquad (5.20)$$

The standard $GL_q(N+2p, C)$ commutation relations for these matrices contain Eqs(5.2-5.4) and the relations for v and $\tilde{v}$ functions

$$\widetilde{\mathbf{R}}\, \mathbf{U}\, \mathbf{U}' = \mathbf{U}\, \mathbf{U}'\, \mathbf{R} \qquad (5.21)$$
$$\mathbf{R}\, \mathbf{S}'\, \mathbf{S} = \mathbf{S}'\, \mathbf{S}\, \widetilde{\mathbf{R}} \qquad (5.22)$$
$$\mathbf{S}\, \widetilde{\mathbf{R}}\, \mathbf{U} = \mathbf{U}'\, \mathbf{R}\, \mathbf{S}' \qquad (5.23)$$

where the R-matrix for $GL_q(N+2p, C)$ can be written in the following form

$$\widetilde{\mathbf{R}} = \begin{pmatrix} (R_N)^{ik}_{mn} & 0 & 0 & 0 \\ 0 & \lambda \delta^A_C \delta^\alpha_\mu \delta^k_n & \delta^A_D \delta^\alpha_\nu \delta^k_m & 0 \\ 0 & \delta^i_n \delta^B_C \delta^\beta_\mu & 0 & 0 \\ 0 & 0 & 0 & \delta^A_D \delta^B_C R^{\alpha\beta}_{\mu\nu} \end{pmatrix}$$

where $\lambda = q - q^{-1}$.

The equations (5.5-5.6) follow from are the relations

$$\mathbf{S}\, \widetilde{\mathbf{R}}\, d\mathbf{U} = d\mathbf{U}'\, \mathbf{R}^{-1}\, \mathbf{S}' \qquad (5.24)$$

It should be stressed that the bicovariant differential calculus on $GL_q(N+2p, C)$ is the basis of the deformed ADHM-construction for the group $GL_q(N, C)$.

Write explicitly the orthogonality and completeness conditions for the deformed ADHM-twistors:

$$u^i_I\, \tilde{v}^{IA\alpha} = 0 \qquad (5.25)$$
$$v_I^{A\alpha}\, \tilde{u}^I_i = 0 \qquad (5.26)$$
$$\delta^I_K = \tilde{u}^I_i\, u^i_K + \tilde{v}^{IA\alpha}\, g_{AB}(z) \varepsilon_{\alpha\beta}(q)\, v_K^{B\beta} \qquad (5.27)$$

Now we are in a position to verify the self-duality of the connection (5.1)

$$dA^i_k - A^i_l\, A^l_k = du^i_I\, (\tilde{u}^I_l\, u^l_M - \delta^I_M) d\tilde{u}^M_k = \qquad (5.28)$$
$$= -q^{-4} u^i_I\, \tilde{b}^{cIA} D^a_c\, g_{AB}(z) \varepsilon_{\alpha\beta}(q)\, dz^\alpha_a\, dz^\beta_b\, b^{bB}_M\, \tilde{u}^M_k$$

where D^a_c is the $GL_q(4)$ metric. This curvature contains only the self-dual q-twistor 2-form (2.21).

The real forms of the deformed ADHM-construction are based on the quantum groups $U_q(N)$ and $GL_q(N, R)$.

It should be stressed that all R-matrices of our deformation scheme satisfy the Hecke relation with the common parameter q. The other possible parameters of different R-matrices are independent. The case $q = 1$ corresponds to the unitary deformations ($R^2 = I$) of the twistor space and the gauge groups. It is evident that the trivial deformation of the z-twistors is consistent with the nontrivial unitary deformation of the gauge sector and vice versa.

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References

[1] A.Connes, M.Rieffel, Contemp.Math., **62** (1987) 237

[2] I.Ya.Aref'eva, I.V.Volovich, Mod. Phys. Lett., **A6** (1991) 893

[3] T.Brzezinski, Sh.Majid , Commun. Math. Phys., **157** (1993) 591

[4] M.J.Pflaum, Comm.Math.Phys., **166** (1994) 279

[5] A.P.Isaev, Z.Popowicz , Phys.Lett., **B281** (1992) 271 ; Phys.Lett., **B307** (1993) 353

[6] A.P.Isaev, J.Math.Phys., **35** (1995) 6784

[7] B.M.Zupnik, JETP Lett., **61** (1995) 443; Preprints JINR E2-94-449, hep-th/9411186; E2-94-487, q-alg/9412010

[8] V.Lakshmibai , N.Yu.Reshetikhin, C.R. Acad.Sci. Paris, **313** (1991) 121; Contemp. Math., **134** (1992) 145

[9] S. Levendorski, Ya.Soibelman, Comm.Math.Phys., **139** (1991) 141

[10] S.A.Merkulov, Z.Phys., **C52** (1991) 583

[11] A.A.Belavin, A.M.Polyakov, A.S.Schwartz and Yu.S.Tyupkin, Phys.Lett., **B59** (1975) 85

[12] M.F.Atiyah, V.G.Drinfel'd, N.J.Hitchin and Yu.I.Manin, Phys. Lett., **A65** (1978) 185
Yu.I.Manin, Gauge Fields and Complex Geometry, Nauka Publ., Moscow, 1984

[13] W.Siegel, Preprint ITP-SB-94-66, Stony Brook, 1994; hep-th/9412011

[14] B.M.Zupnik, Pis'ma ZhETF, **62** (1995) 365; q-alg/9507024

[15] N.Yu.Reshetikhin, L.A.Takhtadjan, L.D.Faddeev, Algeb. Anal., **1** (1989) 178

[16] D.D.Demidov, Yu.I.Manin, E.E.Mukhin and D.V.Zhdanovich, Prog. Theor. Phys. Suppl., **102** (1990) 203

[17] A.Sudbery, J.Phys., **A23** (1990) L697

[18] A.Schirrmacher, Z.Phys., **C50** (1991) 321

[19] B.M.Zupnik, Teor.Mat.Phys., **95** (1993) 403, hep-th/9211065; Symmetry methods in physics, v.2, p.599, E2-94-347, (Dubna, 1994), hep-th/9307149

[20] S.L.Woronowicz, Comm. Math. Phys., **122** (1989) 125

[21] A.Sudbery, Phys. Lett., **B284** (1992) 61

[22] P.Schupp, P.Watts, B.Zumino, Comm. Math. Phys., **157** (1993) 305

[23] A.Schirrmacher, in: Groups and related topics, ed. R.Gielerak et al. (Kluwer Acad. Publ. 1992) p.55

[24] M.Chaichian, A.P.Demichev, Phys. Lett., **304B** (1994) 220

[25] V.K.Dobrev, Phys. Lett., **341B** (1994) 133

[26] G. t'Hooft, unpublished

[27] R.Jackiw, C.Hohl, C.Rebbi , Phys. Rev., **D15** (1977) 1672

[28] E.Corrigan, D.Fairlie, Phys.Lett., **67B** (1977) 69

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