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GENERALIZED HELMHOLTZ COILS AND ALL THAT

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Обобщенные катушки Гельмгольца и всё такое прочее

Изучаются магнитные поля, связанные с токами, протекающими по поверхности одной или двух сфер. Найдены распределения токов, являющиеся обобщениями катушек Гельмгольца и позволяющие устранить из мультипольного разложения любую наперёд заданную систему мультиполей.

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Generalized Helmholtz Coils and All That

We study magnetic fields which are due to currents flowing on the surfaces of one or two spheres. Current distributions generalizing Helmholtz coils are found which cancel any set of multipoles occurring in a multipole expansion.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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1.Introduction

This investigation has been initiated by two nice papers [1,2] which are appeared recently in the American Journal of Physics and which are devoted to the study of electromagnetic properties of the so-called Helmholtz coils.However, when reading these papers we felt ourselves like spectators observing the tricks in a circus. It was not clear to us whether the cancellation of a particular multipole is a curious yet isolated fact or manifestation of something more general. The purpose of this paper is to clarify this point. The plan of our exposition is as folows. In sect.2, we present the known facts concerning magnetic fields originating from currents confined to the surface of the sphere. These facts are needed for the subsequent consideration. In sect.3, comparing different representations of the vector potential (VP) we find sums and integrals involving Legendre functions. They are absent in the available mathematical handbooks, treatises and original publications. In sect.4, we study magnetic fields originating from various currents flowing on the surfaces of one or two spheres. Current distributions that generalize so-called Helmholtz coils and which cancel any set of multipoles occurring in multipole expansion are found in the same section.

2.Preliminairies

The major part of this section deals with the known facts (see, e.g., Chapter YII of ref.[3]) which will be needed later. Consider the sphere S of the radius R on the surface of which the current flows in the lateral direction:

$$\vec{j} = \vec{n_\phi} \cdot f(\theta) \cdot \delta(r - R) \tag{1}$$

The vector potential (VP) corresponding to this density is:

$$\vec{A} = \int \frac{\vec{j}(\vec{r'})}{|\vec{r} - \vec{r'}|} \cdot dV' \tag{2}$$

Using the expansion

$$\frac{1}{|\vec{r} - \vec{r'}|} = 4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r_<^{\,l}}{r_>^{\,l+1}} \cdot Y_{lm}(\theta, \phi) \cdot Y_{lm}{}^*(\theta', \phi') ,$$

we obtain $\vec{A} = A \cdot \vec{n_\phi}$, where

$$A = \frac{4\pi}{c} \sum_l \frac{1}{2l+1} \frac{R^{l+2}}{r^{l+1}} \cdot P_l^1(\cos\theta) \cdot f_l$$

outside S and

$$A = \frac{4\pi}{c} \sum_l \frac{1}{2l+1} \frac{r^l}{R^{l-1}} \cdot P_l^1(\cos\theta) \cdot f_l$$

inside it. Here

$$f_l = \int_0^\pi P_l^1(\cos\theta) \cdot f(\theta) \cdot \sin\theta \cdot d\theta \tag{3}$$

and P_l^m is the adjoint Legendre polynomial normalized to unity ($\int_{-1}^{1} P_l^m(x) \cdot P_k^m(x) \cdot dx = \delta_{k,l}$. The following their properties will be used later [4]:

1) $P_l^{-m}(x) = (-1)^m \cdot P_l^m(x)$

2) $P_l^m(-x) = (-1)^{l-m} \cdot P_l^m(x)$

3) $P_{2n}^1(0) = 0, \quad P_{2n+1}^1(0) = (-1)^{n+1} \dfrac{(2n+1)!!}{2^{n+1} \cdot n!} \cdot \sqrt{\dfrac{4n+3}{(n+1)(2n+1)}}$

4) $P_l^m(x)$ has $l-m$ zeroes in the interval $(-1 < x < 1)$.

5) $P_l^m(x)$ with m fixed form a complete basis. In the interval $(-1, 1)$ any function of x which vanishes at the endpoints of this interval can be represented in the form: $f(x) = \sum_l f_l \cdot P_l^m(x)$.

6) We write out explicit expressions for the simplest Legendre functions which will be needed later:

$$P_1^1(x) = -\frac{\sqrt{3}}{2} \cdot \sqrt{1-x^2}, \quad P_2^1(x) = -\frac{\sqrt{15}}{2} \cdot x \cdot \sqrt{1-x^2}$$

$$P_3^1(x) = -\sqrt{21/32} \cdot \sqrt{1-x^2} \cdot (5x^2-1)$$

$$P_4^1(x) = -\frac{3\sqrt{5}}{8\sqrt{2}} \cdot \sqrt{1-x^2} \cdot x \cdot (7x^2-3)$$

$$P_5^1(x) = -\frac{\sqrt{165}}{16} \cdot \sqrt{1-x^2} \cdot (21x^4 - 14x^2 + 1)$$

In what follows we shall not indicate the argument of P_l^m if it equals $\cos\theta$. The magnetic field ($\vec{H} = curl \vec{A}$) is equal to

$$H_r = -\frac{4\pi}{c} \sum \frac{\sqrt{l(l+1)}}{2l+1} \frac{R^{l+2}}{r^{l+2}} \cdot P_l \cdot f_l$$

$$H_\theta = \frac{4\pi}{c} \sum \frac{l}{2l+1} \frac{R^{l+2}}{r^{l+2}} \cdot P_l^1 \cdot f_l \tag{4}$$

outside S and

$$H_r = -\frac{4\pi}{c} \sum \frac{\sqrt{l(l+1)}}{2l+1} \frac{r^{l-1}}{R^{l-1}} \cdot P_l \cdot f_l$$

$$H_\theta = -\frac{4\pi}{c} \sum \frac{l+1}{2l+1} \frac{r^{l-1}}{R^{l-1}} \cdot P_l^1 \cdot f_l \tag{5}$$

inside it. Let

$$f(\theta) = C_0 \cdot P_k^1, \qquad C_0 = const \tag{6}$$

Then, $f_l = C_0 \cdot \delta_{l,k}$ and

$$A = \frac{4\pi}{c} \frac{C_0}{2k+1} \frac{R^{k+2}}{r^{k+1}} \cdot P_k^1 \quad (r > R)$$

$$A = \frac{4\pi}{c} \frac{C_0}{2k+1} \frac{r^k}{R^{k-1}} \cdot P_k^1 \quad (r < R) \tag{7}$$

This means that the current density (1) with f given by (6) generates magnetic moment with multipolarity k. Further, putting $f = \sum_l C_l \cdot P_l^1$ we obtain VP

$$A = \frac{4\pi}{c} \cdot \sum_l \frac{C_l}{2l+1} \cdot \frac{R^{l+2}}{r^{l+1}} \cdot P_l^1 \quad (r > R)$$

$$A = \frac{4\pi}{c} \cdot \sum_l \frac{C_l}{2l+1} \cdot \frac{r^l}{R^{l-1}} \cdot P_l^1 \quad (r < R) \tag{8}$$

We see that it is possible to reproduce any multipole combination via the suitable continuous distibution of currents on the surface of S. Consider the current lying in the equatorial plane of the sphere S. For this we put $f = I_0 \cdot \delta(\theta - \frac{\pi}{2})$. Then, $f_l = I_0 \cdot P_l^1(0)$. It follows from the property 3) that only odd values of l survive in the expansion of VP:

$$A = \frac{4\pi I_0}{c} \sum_n \frac{1}{4n+3} \cdot \frac{R^{2n+3}}{r^{2n+2}} \cdot P_{2n+1}^1 \cdot P_{2n+1}^1(0)$$

$$H_r = -\frac{4\pi}{c} \sum \frac{\sqrt{2(2n+1)(n+1)}}{4n+3} \frac{R^{2n+3}}{r^{2n+3}} \cdot P_{2n+1} \cdot P_{2n+1}^1(0)$$

$$H_\theta = \frac{4\pi}{c} \sum \frac{2n+1}{4n+3} \frac{R^{2n+3}}{r^{2n+3}} \cdot P_{2n+1}^1 \cdot P_{2n+1}^1(0) \tag{9}$$

outside S and

$$A = \frac{4\pi I_0}{c} \sum_n \frac{1}{4n+3} \cdot \frac{r^{2n+1}}{R^{2n}} \cdot P_{2n+1}^1 \cdot P_{2n+1}^1(0)$$

$$H_r = -\frac{4\pi}{c} \sum \frac{\sqrt{2(2n+1)(n+1)}}{4n+3} \frac{r^{2n}}{R^{2n}} \cdot P_{2n+1} \cdot P_{2n+1}^1(0)$$

$$H_\theta = -\frac{4\pi}{c} \sum \frac{2n+2}{4n+3} \frac{r^{2n}}{R^{2n}} \cdot P_{2n+1}^1 \cdot P_{2n+1}^1(0) \tag{10}$$

inside S. Let the current configuration be symmetric with respect to the equatorial plane ($\theta = \frac{\pi}{2}$) of S. Then $f(\theta) = f(\pi - \theta)$ and

$$A = \frac{4\pi}{c} \cdot \sum \frac{f_{2n+1}}{4n+3} \cdot \frac{R^{2n+3}}{r^{2n+2}} \cdot P_{2n+1}^1 \quad (r > R)$$

$$A = \frac{4\pi}{c} \cdot \sum \frac{f_{2n+1}}{4n+3} \cdot \frac{r^{2n+1}}{R^{2n}} \cdot P_{2n+1}^1 \quad (r < R) \tag{11}$$

i.e., magnetic field contains only odd multipoles. On the other hand, if $f(\theta) = -f(\pi - \theta)$, then

$$A = \frac{4\pi}{c} \sum \frac{f_{2n}}{4n+1} \frac{R^{2n+2}}{r^{2n+1}} \cdot P_{2n}^1 \quad (r > R)$$

$$A = \frac{4\pi}{c} \sum \frac{f_{2n}}{4n+1} \frac{r^{2n}}{R^{2n-1}} \cdot P_{2n}^1 \quad (r < R) \tag{12}$$

i.e., the magnetic field contains only even multipoles for the antisymmetric current distribution.

3.On sums and integrals involving Legendre functions

On the other hand, one may find closed expressions for VP without performing the multipole expansion. For this we insert (1) into (2) and integrate over ϕ'. The result is

$$A = \frac{2R^{3/2}}{c\sqrt{r\sin\theta}} \cdot \int d\theta' \cdot \sqrt{sin\theta'} \cdot f(\theta') \cdot Q_{1/2}(\cosh\mu) \qquad (13)$$

Here $Q_{1/2}$ is the Legendre function of the 2-nd kind and

$$\cosh\mu = \frac{R^2 + r^2 - 2Rr\cos\theta\cos\theta'}{2Rr\sin\theta\sin\theta'}$$

Since A in (2) and (13) are the same one obtains the following equality:

$$\frac{R^{3/2}}{\sqrt{r\sin\theta}} \cdot \int d\theta' \cdot \sqrt{sin\theta'} \cdot f(\theta') \cdot Q_{1/2}(\cosh\mu) =$$

$$= 2\pi \sum_l \frac{1}{2l+1} \frac{R^{l+2}}{r^{l+1}} \cdot P_l^1(\cos\theta) \cdot f_l \quad (r > R)$$

and

$$= 2\pi \sum_l \frac{1}{2l+1} \frac{r^l}{R^{l-1}} \cdot P_l^1(\cos\theta) \cdot f_l \quad (r < R) \qquad (14)$$

Consider particular cases of this equality. Let $f(\theta') = P_l^1(\cos\theta')$.Then,

$$R^{3/2}\sqrt{r\sin\theta} \cdot \int d\theta' \cdot \sqrt{sin\theta'} \cdot P_l^1(\cos\theta') \cdot Q_{1/2}(\cosh\mu) =$$

$$= 2\pi \frac{1}{2l+1} \frac{R^{l+2}}{r^{l+1}} \cdot P_l^1(\cos\theta) \quad (r > R)$$

and

$$= 2\pi \frac{1}{2l+1} \frac{r^l}{R^{l-1}} \cdot P_l^1(\cos\theta) \quad (r < R) \qquad (15)$$

Let $f(\theta') = \delta(\theta' - \theta_0)$. Then,

$$\frac{R^{3/2}}{\sqrt{r\sin\theta\sin\theta_0}} \cdot Q_{1/2}\left(\frac{R^2 + r^2 - 2rR\cos\theta\cos\theta_0}{2Rr\sin\theta\sin\theta_0}\right) =$$

$$= 2\pi \sum_l \frac{1}{2l+1} \frac{R^{l+2}}{r^{l+1}} \cdot P_l^1 \cdot P_l^1(\cos\theta_0) \quad (r > R) \quad and$$

$$= 2\pi \sum_l \frac{1}{2l+1} \frac{r^l}{R^{l-1}} \cdot P_l^1 \cdot P_l^1(\cos\theta_0) \quad (r < R) \qquad (16)$$

If, in addition, $r = R$, then

$$\frac{1}{\sqrt{\sin\theta \cdot \sin\theta_0}} Q_{1/2}\left(\frac{1 - \cos\theta\cos\theta_0}{\sin\theta\sin\theta_0}\right) = 2\pi \sum \frac{1}{2l+1} P_l^1(\cos\theta) P_l^1(\cos\theta_0)$$

For $\theta_0 \to 0$ this equation transforms into the known one [4]:

$$\frac{1}{\sqrt{1 - \cos\theta}} = 2 \sum P_l(\cos\theta)/\sqrt{2l+1}.$$

For $\theta_0 = \pi/2$ Eq.(16) reduces to

$$\frac{R^{3/2}}{\sqrt{r\sin\theta}} \cdot Q_{1/2}\left(\frac{r^2 + R^2}{2Rr\sin\theta}\right) = 2\pi \sum \frac{1}{4n+3} \frac{R^{2n+3}}{r^{2n+2}} \cdot P_{2n+1}^1 \cdot P_{2n+1}^1(0) \quad (r > R)$$

$$and \quad = 2\pi \sum \frac{1}{4n+3} \frac{r^{2n+1}}{R^{2n}} \cdot P_{2n+1}^1 \cdot P_{2n+1}^1(0) \quad (r < R)$$

For $r = R$ this equation goes into

$$Q_{1/2}(1/\sin\theta)/\sqrt{\sin\theta} = 2\pi \sum P_{2n+1}^1(\cos\theta) P_{2n+1}^1(0)/(4n+3)$$

These equations are easily generalized if in Eq.(1) we take the fictitious vector $\vec{n}_\phi^m = \cos m\phi\, \vec{n}_y - \sin m\phi\, \vec{n}_x$ instead of $\vec{n}_\phi = \cos\phi\, \vec{n}_y - \sin\phi\, \vec{n}_x$. Substituting $\vec{n}_\phi^m$ in (2) and using the same procedure as above we get

$$\frac{1}{\sqrt{\sin\theta \cdot \sin\theta_0}} Q_{m-1/2}\left(\frac{1 - \cos\theta\cos\theta_0}{\sin\theta\sin\theta_0}\right) = 2\pi \sum \frac{1}{2l+1} P_l^m(\cos\theta) P_l^m(\cos\theta_0)$$

For $\theta_0 = \pi/2$ and $\theta_0 \to 0$ this equation takes the forms

$$Q_{m-1/2}(1/\sin\theta)/\sqrt{\sin\theta} = 2\pi \sum P_{2n+1}^m(\cos\theta) P_{2n+1}^m(0)/(4n+3) \quad and$$

$$\frac{\sin^m\theta}{(1 - \cos\theta)^{m+1/2}} = \frac{(-1)^m 2^{m+1/2}}{\Gamma(m+1/2)} \sqrt{\pi} \sum \sqrt{\frac{2(l+m)!}{(2l+1)(l-m)!}} P_l^m(\cos\theta)$$

The last equation simplifies if we use Legendre polynomials $\tilde{P}_l^m$ which are not normalized to unity ($\int [\tilde{P}_l^m(x)]^2 dx = 2(l+m)!/(2l+1)(l-m)!$):

$$\frac{\sin^m\theta}{(1 - \cos\theta)^{m+1/2}} = \frac{(-1)^m 2^{m+1/2}}{\Gamma(m+1/2)} \sqrt{\pi} \sum \tilde{P}_l^m(\cos\theta)$$

These sums and integrals are absent in available treatises, mathematical handbooks and original papers. This trick, i.e., obtaining of new mathematical identities from the comparison of different representation of the same function is not, in fact, new. Many other useful sums and integrals involving Legendre functions may be found in refs. [5].

4. On the Helmloltz coils and their generalizations

4.1. Currents on the single sphere.

Although continuous current distributions considered in sect.2 permit one to obtain prescribed magnetic field both inside and outside S, they, in fact, are not very convenient for practical applications. Suppose we carry out an experiment inside S. For this we should have access to the interior of S. However, the continuous distribution of currents on the surface of S prevents us from this. The question arises: Could the discrete currents flowing on the surface of S be used for producing the magnetic field with a more or less acceptable properties inside S ? The simplest discrete current configurations producing more or less uniform magnetic field inside S were found by James Clerk Maxwell [6]. The nice exposition of his ideas with impressive illustrations may be found in quoted refs.[1,2]. We shall move step by step from the simplest cases to more complicated.

1) Let

$$f(\theta) = I_0 \cdot (\delta(\theta - \theta_1) + \delta(\theta - \pi + \theta_1)) \tag{17}$$

(two currents symmetric relative to the equator of S and having the same direction). Then,

$$A = \frac{8\pi I_0}{c} \sin\theta_1 \cdot \sum \frac{1}{4n+3} \cdot \frac{R^{2n+3}}{r^{2n+2}} \cdot P^1_{2n+1} \cdot P^1_{2n+1}(x_1) \quad (r > R)$$

$$A = \frac{8\pi I_0}{c} \sin\theta_1 \cdot \sum \frac{1}{4n+3} \cdot \frac{r^{2n+1}}{R^{2n}} \cdot P^1_{2n+1} \cdot P^1_{2n+1}(x_1) \quad (r < R) \tag{18}$$

Here $x_k = \cos\theta_k$. Let θ_1 be a particular zero of $P_{2k+1}{}^1$. Then, the term with $n = k$ is missing in the development (10). In particular ([1,2] and [6] (Art. 713)), if $k = 1$, then $\sin\theta_1 = 2/\sqrt{5}$ and the first correction term ($n = 2$) to the main dipole term ($n = 0$) falls like r^{-6} for $r \to \infty$. As $P_1{}^1(x_1) = 0$ only if $x_1 = \pm 1$ (this corresponds to the poles of the sphere),so current density (17) can not kill the dipole term in (18).

2) Let

$$f(\theta) = I_0 \cdot (\delta(\theta - \theta_1) - \delta(\theta - \pi + \theta_1)) \tag{19}$$

(two currents symmetric relative to the equator of S and flowing in the opposite direction).

Then,

$$A_\phi = \frac{8\pi I_0}{c} \cdot \sin\theta_1 \cdot \sum \frac{1}{4n+1} \frac{R^{2n+2}}{r^{2n+1}} \cdot P^1_{2n} \cdot P^1_{2n}(x_1) \tag{20}$$

Let θ_1 be a particular zero of $P_{2n}{}^1$. Then term with $n = k$ is missing in (12). In particular, if $k = 2$, then $\sin\theta_1 = 2/\sqrt{7}$ and the first correction term ($n = 3$) to the main quadrupole one ($n = 1$) falls like r^{-7} for $r \to \infty$. Obviously, the current distribution (19) can not be used to remove the quadrupole term in (19). Although the r.h.s. of (20) disappears for $\theta_1 = \pi/2$, but for this θ_1 the current density (19) disappears as well.

3) Let

$$f(\theta) = I_0 \cdot \delta(\theta - \pi/2) + I_1 \cdot (\delta(\theta - \theta_1) + \delta(\theta - \pi + \theta_1)) \tag{21}$$

Then,

$$f_{2n+1} = I_0 \cdot P^1_{2n+1}(0) + 2 \cdot I_1 \sin\theta_1 \cdot P^1_{2n+1}(x_1) \tag{22}$$

We require now the vanishing of f_{2k+1} and f_{2m+1}. This gives:

$$I_0 \cdot P^1_{2k+1}(0) + 2I_1 \sin\theta_1 \cdot P^1_{2k+1}(x_1) = 0$$

,

$$I_0 \cdot P^1_{2m+1}(0) + 2I_1 \sin\theta_1 \cdot P^1_{2m+1}(x_1) = 0 \tag{23}$$

Excluding I_0 and I_1 one finds the following equation defining θ_1

$$\frac{P^1_{2k+1}(x_1)}{P^1_{2m+1}(x_1)} = \frac{P^1_{2k+1}(0)}{P^1_{2m+1}(0)} \tag{24}$$

After the root of this equation is found, one of Eqs.(23) may be used to determine I_1 as a function of I_0 and θ_1

$$I_1 = -\frac{I_0}{2\sin\theta_1} \cdot \frac{P^1_{2k+1}(0)}{P^1_{2k+1}(x_1)} \tag{25}$$

Consider the simplest cases. i) The case $k = 1, m = 2$ was considered in [1] and [6] (Art.715) Using the explicit expressions for $P_3{}^1$ and $P_5{}^1$ one easily obtains $\sin\theta_1 = 2/\sqrt{7}$, $I_1 = 49I_0/64$. This current configuration consists of the current lying in the equatorial plane and two currents at $\theta = \theta_1$ and $\theta = \pi - \theta_1$. All currents flow in the same direction. The first correction term to the main dipole one falls as r^{-8} for $r \to \infty$. ii)The question arises: Is it possible to fit the parameters entering into (21) in such a way as to cancel the dipole term in Eq.(11)? Substituting $k = 0$ into Eq.(24) we get

$$P^1_{2m+1}(x_1) = \sin\theta_1 P^1_{2m+1}(0) \tag{26}$$

The lowest value of m for which this equation has a nontrivial solution is $m = 2$. It turns out that $x_1 = \pm\sqrt{2/3}$, $I_1 = -3I_0/2$. The treated current configuration consists of the current lying in the equatorial plane and two currents at $\theta = \theta_1$ and $\theta = \pi - \theta_1$ flowing in the direction opposite to the equatorial one. The first correction term to the main octupole one falls as r^{-8} for $r \to \infty$.

4) Let

$$f(\theta) = I_1 \cdot (\delta(\theta - \theta_1) + \delta(\theta - \pi + \theta_1)) + I_2 \cdot (\delta(\theta - \theta_2) + \delta(\theta - \pi + \theta_2)) \tag{27}$$

Then, the conditions for the disappearance of f_{2k+1} and f_{2m+1} are:

$$I_1 \sin\theta_1 \cdot P^1_{2k+1}(x_1) + I_2 \sin\theta_2 \cdot P^1_{2k+1}(x_2) = 0,$$

$$I_1 \sin\theta_1 \cdot P^1_{2m+1}(x_1) + I_2 \sin\theta_2 \cdot P^1_{2m+1}(x_2) = 0. \tag{28}$$

For θ_1 and I_1 fixed we get the equations defining θ_2 and I_2:

$$\frac{P^1_{2k+1}(x_2)}{P^1_{2m+1}(x_2)} = \frac{P^1_{2k+1}(x_1)}{P^1_{2m+1}(x_1)}, \quad I_2 = -I_1 \cdot \frac{\sin\theta_1 P^1_{2k+1}(x_1)}{\sin\theta_2 P^1_{2k+1}(x_2)} \tag{29}$$

Again, consider the simplest cases. i) The case $k = 1$, $m = 2$ was considered in [6] (Art.714). Then, one has:

$$\sin^2\theta_2 = \frac{\frac{6}{7} - \sin^2\theta_1}{1 - \frac{5}{4}\sin^2\theta_1}, \qquad I_2 = 14 I_1 \sin^2\theta_1 \frac{(1 - \frac{5}{4}\sin^2\theta_1)^3}{\frac{6}{7} - \sin^2\theta_1} \qquad (30)$$

These equations have solutions if $\sin\theta_1$ lies within the intervals $0 < \sin\theta_1 < 2/\sqrt{7}$ and $\sqrt{6/7} < \sin\theta_1 < 1$. The treated current configuration consists of four currents which lie at $\theta = \theta_1, \pi - \theta_1, \theta_2, \pi - \theta_2$ and flow in the same direction. As in the case 3i) the first correction term to the main dipole one falls as r^{-8} for $r \to \infty$. The position and intensity of the first current pair are arbitrary while that of the second current are given by (30). ii) The choice $k = 0, m = 2$ is the simplest one for which the dipole term disappears. In this case the position and intensity of the first current pair are arbitrary (with the reservation that $\sin\theta_1 \geq 1/\sqrt{3}$), while the position and intensity of the second pair are given by:

$$\sin\theta_2 = \sqrt{\frac{4}{3} - \sin^2\theta_1} \quad I_2 = -I_1 \frac{\sin^2\theta_1}{\frac{4}{3} - \sin^2\theta_1}$$

It turns out that directions of currents flowing in these pairs are opposite.

5) Let $f(\theta) = I_1\delta(\theta - \theta_1) + I_2\delta(\theta - \theta_2)$. Then,

$$A = \frac{4\pi}{c}\sum \frac{1}{2l+1}\frac{R^{l+2}}{r^{l+1}} \cdot P_l^1 \cdot [I_1 \sin\theta_1 \cdot P_l^1(x_1) + I_2 \sin\theta_2 \cdot P_l^1(x_2)] \quad (r > R)$$

$$A = \frac{4\pi}{c}\sum \frac{1}{2l+1}\frac{r^l}{R^{l-1}} \cdot P_l^1 \cdot [I_1 \sin\theta_1 \cdot P_l^1(x_1) + I_2 \sin\theta_2 \cdot P_l^1(x_2)] \quad (r < R) \qquad (31)$$

Let $I_1 \sin\theta_1 \cdot P_k{}^1(x_1) + I_2 \sin\theta_2 \cdot P_k{}^1(x_2) = 0$. Then, the term with $l = k$ is absent in the development (31). In particular, for $k = 1$ one gets $I_2 = -I_1 \cdot \sin^2\theta_1/\sin^2\theta_2$. In this case, the dipole term is absent both outside and inside the sphere S. Treated current configuration consists of two currents whose positions on the sphere are arbitrary. The intensity of one of the currents is also arbitrary, while the intensity of the second current is a function of the parameters just mentioned.

6) Let $f(\theta) = I_1\delta(\theta - \theta_1) + I_2\delta(\theta - \theta_2) + I_3\delta(\theta - \theta_3)$. We require the disappearance of the k and m terms in the development (3):

$$I_1 \sin\theta_1 \cdot P_k{}^1(x_1) + I_2 \sin\theta_2 \cdot P_k{}^1(x_2) + I_3 \sin\theta_3 \cdot P_k{}^1(x_3) = 0$$

$$I_1 \sin\theta_1 \cdot P_m{}^1(x_1) + I_2 \sin\theta_2 \cdot P_m{}^1(x_2) + I_3 \sin\theta_3 \cdot P_m{}^1(x_3) = 0 \qquad (32)$$

Resolve these Eqs. relative to I_2, I_3

$$I_2 = -I_1 \frac{\sin\theta_1}{\sin\theta_2} \cdot \frac{\Delta(1,3)}{\Delta(2,3)}, \quad I_3 = -I_1 \frac{\sin\theta_1}{\sin\theta_3} \cdot \frac{\Delta(2,1)}{\Delta(2,3)}$$

Here $\Delta(i,j) = P_k{}^1(x_i)P_m{}^1(x_j) - P_k{}^1(x_j)P_m{}^1(x_i)$. These Eqs. define the current forces I_2, I_3 as function of $I_1, \theta_1, \theta_2, \theta_3$. Consider the particular choices of k, m. Take $k = 1, m = 2$. Then,

$$I_2 = -I_1 \frac{\sin^2\theta_1}{\sin^2\theta_2} \cdot \frac{x_3 - x_1}{x_3 - x_2}, \quad I_3 = -I_1 \frac{\sin^2\theta_1}{\sin^2\theta_3} \cdot \frac{x_1 - x_2}{x_3 - x_2}$$

Since dipole and quadrupole terms dissappear, the expansion begins with a octupole term. Further, putting $k = 2, m = 3$ we suppress quadrupole and octupole terms. In this case

$$I_2 = -I_1 \frac{\sin^2\theta_1}{\sin^2\theta_2} \cdot \frac{x_3 - x_1}{x_3 - x_2} \cdot \frac{5x_1x_3 + 1}{5x_2x_3 + 1},$$

$$I_3 = -I_1 \frac{\sin^2\theta_1}{\sin^2\theta_3} \cdot \frac{x_1 - x_2}{x_3 - x_2} \cdot \frac{5x_1x_2 + 1}{5x_2x_3 + 1}$$

Treated current configuration consists of three currents whose positions on the sphere are arbitrary. The intensities of two currents are also arbitrary, while the intensity of the third current is a function of the parameters just mentioned.

7) Let $f(\theta)$ be of the form

$$f(\theta) = \sum_{s=1}^{k} I_s \delta(\theta - \theta_s) \qquad (33)$$

Substitute this into (3) and require the disappearance of the first $(k-1)$ multipoles. This gives

$$\sum_{s=1}^{k} I_s \sin\theta_s \cdot P_m^1(\cos\theta_s) = 0 \qquad m = 1, \ldots, k-1$$

or, in a slightly different form:

$$\sum_{s=2}^{k} I_s \sin\theta_s \cdot P_m^1(\cos\theta_s) = -I_1 \sin\theta_1 \cdot P_m^1(\cos\theta_1) \quad m = 1, \ldots, k-1 \qquad (34)$$

This system of Eqs. may be viewed as one defining $I_2, \ldots, I_k$ as a function of I_1, $\sin\theta_s$ $(s = 1, \ldots, k)$. It has solutions if $\det P_m{}^1(\cos\theta_s)$ $(m = 1, \ldots, k-1,\ s = 2, \ldots, k)$ differs from zero. It vanishes only for very special values of θ_s which are of no interest for us. This means that for the given I_1 and θ_s $(s = 1, \ldots, k)$ one can reach the disappearance of the first $k-1$ multipoles both inside and outside S. It follows from this that the magnetic field is concentrated near the surface of S as k grows. This configuration consists of k currents whose positions are arbitrary. The intensities of $k-1$ currents are arbitrary, while the intensity of the last current depends on the parameters just mentioned.

8) Sometimes, one needs the uniform magnetic field confined to the closed volume V. Let this volume be the interior of the sphere S. From Eq.(5) we observe that uniform magnetic field directed along the Z axis corresponds to the $l = 1$ term in Eq.(5). Thus, to obtain uniform magnetic field inside S one should suppress higher harmonics in (3) as much as possible. Again, seek $f(\theta)$ in the form

$$f(\theta) = \sum_{s=1}^{k} I_s \delta(\theta - \theta_s) \tag{35}$$

We substitute this into Eq.(3), collect terms at $P_m{}^1(\cos\theta_s)$ and require the disappearance of $k-1$ terms begining from $m = 2$. This gives

$$\sum_{s=2}^{k} I_s \sin\theta_s \cdot P_m^1(\cos\theta_s) = -I_1 \sin\theta_1 \cdot P_m^1(\cos\theta_1), \quad m = 2, \ldots, k$$

We may express $I_2, \ldots, I_k$ through $I_1, \theta_1, \ldots, \theta_k$. This is possible if

$$\det P_m^1(\cos\theta_s) \quad (m = 2, \ldots, k \quad s = 2, \ldots, k)$$

differs from zero. This does not take place except for the very special combinations of θ_s. So, using k coils it is possible to kill $k-1$ multipoles ($k \geq 2$). For k sufficiently large the magnetic field will be uniform inside S everywhere except for the surface of S. Outside S the magnetic field approximately reduces to that of magnetic dipole. Turning to Eqs.(35) we feel that they contain too much degrees of freedom ($I_1, \theta_1, \ldots, \theta_k$). Is it possible to dispose them in such a way as to kill more multipoles (for the same value of k)? We try to cancel k multipoles by k currents:

$$f(\theta) = \sum_{s=1}^{k} I_s \delta(\theta - \theta_s) \tag{36}$$

This leads to the equation:

$$\sum_{s=1}^{k} I_s \sin\theta_s \cdot P_m^1(\cos\theta_s) = 0, \qquad m = 2, \ldots, k+1 \tag{37}$$

This system of Eqs. has solutions if

$$\det P_m^1(\cos\theta_s) = 0 \quad (s = 1, \ldots, k, \quad m = 2, \ldots, k+1) \tag{38}$$

We consider a particular case ($k = 2$) to demonstrate that Eqs.(36)-(38) have nontrivial solutions. Eq.(38) reduces to

$$P_2^1(\cos\theta_1) \cdot P_3^1(\cos\theta_2) - P_3^1(\cos\theta_1) \cdot P_2^1(\cos\theta_2) = 0$$

Disregarding the nonphysical solution $\theta_1 = \theta_2$ we get

$$\cos\theta_2 = -\frac{1}{5\cos\theta_1}, \quad I_2 = 5 \cdot I_1 \cdot \frac{\sin^2\theta_1 \cdot \cos^4\theta_1}{\cos^2\theta_1 - 1/25}$$

Thus, the specific choice of θ_s permits us to cancel more multipoles for the same k.

4.2. Currents on two spheres.

So far we have considered the current densities distributed over the surface of the sphere S. The drawback of such distributions is that cancellation of a particular multipole inside S inevitably leads to the disappearance of the same multipole outside S. Much more possibilities arise if the currents are ditributed over the surfaces of two concentric spheres S_1 and S_2. Let the radii of these spheres be R_1 and R_2 and let currents over them flow in the lateral direction:

$$\vec{j}_1 = \vec{n}_\phi f_1(\theta)\delta(r - R_1), \quad \vec{j}_2 = \vec{n}_\phi f_1(\theta)\delta(r - R_2) \tag{39}$$

The VP corresponding to this current distribution has only the ϕ component that is equal to

$$A = \frac{4\pi}{c} \sum \frac{1}{2l+1} r^l P_l^1 [\frac{f_l^{(1)}}{R_1{}^{l-1}} + \frac{f_l^{(2)}}{R_2{}^{(l-1)}}]$$

-inside S_1,

$$A = \frac{4\pi}{c} \sum \frac{1}{2l+1} \frac{1}{r^{l+1}} P_l^1 [f_l^{(1)} R_1{}^{l+2} + f_l^{(2)} R_2{}^{(l+2)}]$$

-outside S_2 and

$$A = \frac{4\pi}{c} \sum \frac{1}{2l+1} P_l^1 [\frac{R_1{}^{l+2}}{r^{l+1}} f_l^{(1)} + \frac{r^l}{R_2{}^{l-1}} f_l^{(2)}]$$

-between S_1 and S_2. Here, as before,

$$f_l^{(1,2)} = \int P_l^1 f_{(1,2)}(\theta) \sin\theta \cdot d\theta$$

As P_l^1 form a complete system we may develop f_1 and f_2 over them:

$$f_1(\theta) = \sum C_l{}^1 \cdot P_l^1(\cos\theta), \quad f_2(\theta) = \sum C_l{}^2 \cdot P_l^1(\cos\theta) \tag{40}$$

After substitution in A one gets:

$$A = \frac{4\pi}{c} \sum \frac{1}{2l+1} \cdot r^l P_l^1 (\frac{C_l{}^1}{R_1{}^{l-1}} + \frac{C_l{}^2}{R_2{}^{l-1}}) \tag{41}$$

-inside S_1,

$$A = \frac{4\pi}{c} \sum \frac{1}{2l+1} \cdot \frac{1}{r^{l+1}} P_l^1 [C_l{}^1 R_1{}^{l+2} + C_l{}^2 R_2{}^{l+2}] \tag{42}$$

-outside S_2 and

$$A = \frac{4\pi}{c} \sum \frac{1}{2l+1} P_l^1 [C_l{}^1 \frac{R_1{}^{l+2}}{r^{l+1}} + C_l{}^2 \frac{r^l}{R_2{}^{l-1}}] \tag{43}$$

-between S_1 and S_2.

Consider the particular choices of C_l^1 and C_l^2.

1) Let $C_l^1 = C_l/R_1^{l+2}$, $C_l^2 = -C_l/R_2^{l+2}$. Then, A=0 outside S_2,

$$A = \frac{4\pi}{c}\sum\frac{1}{2l+1}\cdot r^l P_l^1 C_l(\frac{1}{R_1^{2l+1}} - \frac{1}{R_2^{2l+1}})$$

inside S_1 and (44)

$$A = \frac{4\pi}{c}\sum\frac{1}{2l+1}\cdot P_l^1 C_l(\frac{1}{r^{l+1}} - \frac{r^l}{R_2^{2l+1}})$$

-between S_1 and S_2. In this case the magnetic field is completely inside S_2 and we have a kind of magnetic capacitor.

2) Let $C_l^1 = C_l R_1^{l-1}$, $C_l^2 = -C_l R_2^{l-1}$. Then, A=0 inside S,

$$A = \frac{4\pi}{c}\sum\frac{1}{2l+1}\cdot r^{-l-1} P_l^1 C_l(R_1^{2l+1} - R_2^{2l+1})$$

outside S_2 and (45)

$$A = \frac{4\pi}{c}\sum\frac{1}{2l+1}\cdot P_l^1 C_l(\frac{R_1^{2l+1}}{r^{l+1}} - r^l)$$

-between S_1 and S_2. In this case the magnetic field equals zero inside S_1. If only one of C_l differs from zero, then only one of the multipoles gives a contribution either outside S_2 or inside S_1.

3) Let the δ-type current

$$\vec{j_1} = \vec{n_\phi}\cdot f_1, \quad f_1(\theta) = \delta(\theta - \theta_1)$$

flows on the surface of the inner sphere S_1. Developing over $P_l^1(\cos\theta)$ we get

$$\delta(\theta - \theta_1) = \sin\theta_1\cdot\sum P_l^1 P_l^1(x_1) \tag{46}$$

This is a particular case of a more general well-known formula:

$$\delta(\theta - \theta_1) = \sin\theta_1\cdot\sum P_l^m P_l^m(x_1)$$

(no sum over m here). Comparing (40) with (46) one gets: $C_l^1 = \sin\theta_1\cdot P_l^1(x_1)$. The question arises: Is it possible to distribute currents on the surface of the external sphere S_2 in such a way as to cancel magnetic field produced by the δ-type current (45) in the space exterior to S_2? It follows from (42) that magnetic field disappears outside S_2 if $C_l^2 = -\sin\theta_1\cdot P_l^1(x_1)\cdot R_1^{l+2}/R_2^{l+2}$. This gives the following current density on S_2

$$f_2(\theta) = -\sum\frac{R_1^{l+2}}{R_2^{l+2}}P_l^1(x_1)P_l^1 \tag{47}$$

and VP inside S_1

$$A = \frac{4\pi}{c}\sin\theta_1\sum\frac{1}{2l+1}P_l^1 P_l^1(x_1)\frac{r^l}{R_1^{l-1}}(1 - \frac{R_1^{2l+1}}{R_2^{2l+1}}) \tag{48}$$

Let us slightly change the problem. Let the δ-type current ($f_2 = \delta(\theta - \theta_2)$) flows on the surface of the outer sphere S_2. Is it possible to find a current distribution on the inner sphere S_1 that exactly cancells magnetic field outside S_2? The reasoning similar to the previous one gives:

$$f_1(\theta) = -\sin\theta_2\sum\frac{R_2^{l+2}}{R_1^{l+2}}P_l^1(x_2)P_l^1 \tag{49}$$

It turns out that this function is highly singular one (more singular than δ- function). This is due to the factor R_2^{l+2}/R_1^{l+2} which is greater than 1. Note that if this factor were equal to 1, we had (up to a sign) the δ function in the r.h.s. of (49). On the other hand, the factor R_1^{l+2}/R_2^{l+2} in Eq.(47)is smaller than 1. This ensures us that r.h.s. of (47) is not a singular function. The easiest way to see this is to multiply Eq.(47) by itself and integrate over θ:

$$\int[f_2(\theta)]^2\sin\theta d\theta = \sum[\frac{R_1^{l+2}}{R_2^{l+2}}P_l^1(x_1)]^2 \tag{50}$$

It follows [4] from the asymptotical ($l\to\infty$) behavior of P_l^1 that the sum in the r.h.s. of (50) is finite. Thus, $f_2(\theta)$ is not singular function (or, at least, quadratically integrable). In the same way, the magnetic field of the δ-type current flowing on the surface of the outer sphere S_2 may be compensated inside S_1 by the nonsingular current distribution on the surface of the inner sphere S_1. This situation strongly resembles the electrostatical one where the point charge is screened by the oppositely charged sphere surrounding this charge.

4) Now we install Helmholtz coils considered in sect.4.1 on each of the spheres S_1 and S_2 (under the same angles). It turns out that the number of multipoles which could be cancelled is doubled if the proper choice of the current forces is made. We illustrate this using the case 1) from 4.1 as an example. The charge density distributed over S_1 and S_2 is

$$\vec{j} = (I_1\delta(r - R_1) + I_2\delta(r - R_2))\cdot(\delta(\theta - \theta_1) + \delta(\theta - \theta_1)) \tag{51}$$

The corresponding VP is

$$A = \frac{8\pi}{c}\sin\theta_1\cdot\sum\frac{1}{4n+3}\frac{1}{r^{2n+2}}(I_1 R_1^{2n+3} + I_2 R_2^{2n+3})P_{2n+1}^1\cdot P_{2n+1}^1(x_1) \tag{52}$$

outside S_2 and

$$A = \frac{8\pi}{c}\sin\theta_1\cdot\sum\frac{1}{4n+3}r^{2n+1}(\frac{I_1}{R_1^{2n}} + \frac{I_2}{R_2^{2n}})P_{2n+1}^1\cdot P_{2n+1}^1(x_1) \tag{53}$$

inside S_1. We have now two degrees of freedom: connected with an angle θ and with one of the linear combinations($(I_1 R_1{}^{2n+3} + I_2 R_2{}^{2n+3})$, $(I_1/R_1{}^{2n} + I_2/R_2{}^{2n})$). We choose θ_1 to be equal to the root of P^1_{2m+1} and adjust the current forces in one of the following two ways: either

$$I_1 R_1{}^{2k+3} + I_2 R_2{}^{2k+3} = 0 \qquad (54)$$

or

$$I_1/R_1{}^{2n} + I_2/R_2{}^{2n} = 0 \qquad (55)$$

For this choice the m multipole drops out in the whole space, while k multipole drops out either outside the outer sphere S_2 or inside the inner sphere S_1 depending on which of the two choices (54) or (55) is made. The particular case $k = 0, m = 1$ was considered in ref.[2]

5.Conclusion.

We have shown how to construct current distributions generalizing the idea of Helmholtz coils and permitting one to cancel any set of multipoles in the multipole expansion of an arbitrary magnetic field. This may be useful in electrical engineering, for the construction of current configurations that are insentitive to the external electromagnetic disturbances, (details may be found in ref.[2]), for the optimal choice of magnetic field configurations which are needed for the plasma confinement in thermo-nuclear reactors, atomic traps, etc. On the other hand, this consideration may be viewed as an exercise demonstrating the power of special functions when they are applied to the problems of electrodynamics.

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