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ON THE FOUR-PHOTON DECAY OF THE NEUTRAL PION

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Although a C-violating decay $\pi^0 \to 3\gamma$ is expected to have an extremely small branching ratio [1], beyond the reach of any present or future experimental facilities, its experimental study has attracted considerable attention [2] because any observed anomaly in this process would be a clear signal of a new physics.

Any $\pi^0 \to 3\gamma$ searching experiment has as a by-product, information about the allowed decay $\pi^0 \to 4\gamma$, that is a potential background for $\pi^0 \to 3\gamma$. The experimental upper limit on the branching ratio $Br(\pi^0 \to 4\gamma)$ was gradually improved [3, 4, 5] and lowered up to $2 \cdot 10^{-8}$ in [2]. Some theoretical estimates can be found in the literature for $Br(\pi^0 \to 4\gamma)$ [6, 7, 8], with rather broad ranges from 10^{-9} to 10^{-16}. In our opinion, the results of [8] are more reliable, the authors giving the most thorough investigation of the subject.

As argued in [8], the dominant contribution to the $\pi_0 \to 4\gamma$ branching ratio is expected to come from the purely electromagnetic photon-splitting graph of Fig. 1, contributions from any hadronic intermediate states being less significant, especially for a PCAC-satisfying models. The calculation of this contribution is still absent to our best knowledge, and will be performed in the present note.

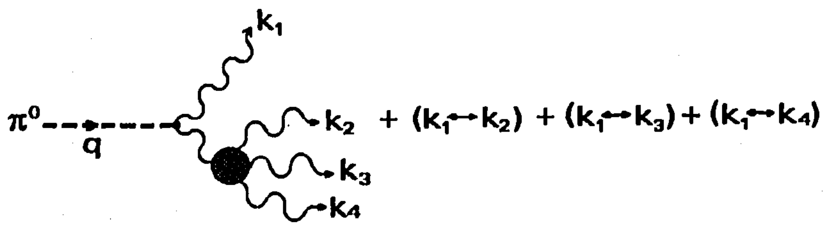

Fig. 1: The electromagnetic photon-splitting graph.

Using the standard covariant phase-space calculation technique [9] and factoring out some numerical constants from the decay amplitude, we can write

$$Br(\pi^0 \to 4\gamma) \simeq \frac{\Gamma(\pi^0 \to 4\gamma)}{\Gamma(\pi^0 \to 2\gamma)} = \frac{1}{6\pi}\left(\frac{\alpha}{8\pi}\right)^4 R, \tag{1}$$

where

$$R = \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_{s_2/s_1}^{1-s_1+s_2} \frac{du_1}{\sqrt{\lambda(1, s_2, s_2')}} \int_{u_2^-}^{u_2^+} du_2 \int_{-1}^{1} \frac{d\zeta}{\sqrt{1-\zeta^2}} F(s_1, s_2, u_1, u_2, t_2(\zeta)). \tag{2}$$

In (2) we have introduced a demensionless version of Kumar's invariant variables [9].

$$s_1 = \frac{1}{m^2}(q-k_1)^2, \quad s_2 = \frac{1}{m^2}(q-k_1-k_2)^2, \quad u_1 = \frac{1}{m^2}(q-k_2)^2, \quad u_2 = \frac{1}{m^2}(q-k_3)^2, \tag{3}$$

m being the pion mass.

One more invariant variable $t_2 = \frac{1}{m^2}(q-k_2-k_3)^2$ is a linear function of the integration variable ζ

$$t_2 = u_1 - \frac{1}{2}(1+u_1)(1-u_2) - \frac{1}{2}(1-u_1)(1-u_2)\left[\xi\eta - \sqrt{(1-\xi^2)(1-\eta^2)}\zeta\right], \tag{4}$$

where

$$\xi = \frac{\lambda(1,s_2,s_2') - (1-s_1)^2 + (1-u_1)^2}{2(1-u_1)\sqrt{\lambda(1,s_2,s_2')}}, \quad \eta = \frac{(1-s_3')^2 - (1-u_2)^2 - \lambda(1,s_2,s_2')}{2(1-u_2)\sqrt{\lambda(1,s_2,s_2')}}, \tag{5}$$

$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2(xy + xz + yz)$ is a conventional triangle function, and

$$s_2' = 1 + s_2 - u_1 - s_1, \quad s_3' = 2 - s_1 - u_1 - u_2. \tag{6}$$

At last, the limits of integration for the u_2-variable in (2) are

$$u_2^{\pm} = 1 - \frac{1}{2}(u_1 + s_1) \pm \frac{1}{2}\sqrt{\lambda(1,s_2,s_2')}. \tag{7}$$

Function F stands for a half sum of the squared helicity amplitudes

$$F = \sum_{\{\lambda\}} |M_{+\lambda_2\lambda_3\lambda_4}|^2. \tag{8}$$

To evaluate these helicity amplitudes, it is convenient to use the light-light scattering tensor from [10]. In fact, as virtuality of the intermediate photon is $\sim m$ and much bigger than the electron mass, we have used the asymptotic form of the light-light scattering amplitudes for the massless electron in the loop [11, 12].

Let us note, however, that we can not use the polarization vectors from [10] because of an additional photon and the need for photon permutations. Instead we have taken polarization vectors in the form which appeared useful in various QED calculations [13]:

$$\varepsilon_\mu^{(\lambda_m)}(k) = N_m\left[q^{(m)}\cdot k\ p_\mu^{(m)} - p^{(m)}\cdot k\ q_\mu^{(m)} + i\lambda_m\varepsilon_{\mu\nu\lambda\sigma}p^{(m)\nu}\ k^\lambda\ q^{(m)\sigma}\right]$$
$$N_m^{-1} = 2\sqrt{p^{(m)}\cdot q^{(m)}\ p^{(m)}\cdot k\ q^{(m)}\cdot k}, \quad m = 1 \div 4, \quad {p^{(m)}}^2 = {q^{(m)}}^2 = 0. \tag{9}$$

Where for various photons we take

$$p^{(1)} = k_2, \quad p^{(2)} = k_3, \quad p^{(3)} = k_4, \quad p^{(4)} = k_1,$$
$$q^{(1)} = k_4, \quad q^{(2)} = k_1, \quad q^{(3)} = k_2, \quad q^{(4)} = k_3. \tag{10}$$

The polarization vectors from [10] can also be expressed in this form ($8\Delta = k_2\cdot k_3\ k_3\cdot k_4\ k_2\cdot k_4$):

$$u_\mu'^{(-\lambda_2)} = \frac{1}{4\sqrt{2\Delta}}[k_4\cdot k_2\ k_{3\mu} - k_3\cdot k_2\ k_{4\mu} + i\lambda_2\varepsilon_{\mu\nu\lambda\sigma}k_3^\nu k_2^\lambda k_4^\sigma],$$
$$u_\mu^{(\lambda_3)} = \frac{1}{4\sqrt{2\Delta}}[k_2\cdot k_3\ k_{4\mu} - k_4\cdot k_3\ k_{2\mu} + i\lambda_3\varepsilon_{\mu\nu\lambda\sigma}k_4^\nu k_3^\lambda k_2^\sigma] \equiv \varepsilon_\mu^{(\lambda_3)}, \tag{11}$$
$$u_\mu^{(-\lambda_4)} = \frac{1}{4\sqrt{2\Delta}}[k_3\cdot k_4\ k_{2\mu} - k_2\cdot k_4\ k_{3\mu} + i\lambda_4\varepsilon_{\mu\nu\lambda\sigma}k_2^\nu k_4^\lambda k_3^\sigma].$$

But $u_\mu'^{(-\lambda_2)}$ and $u_\mu^{(-\lambda_4)}$ differ by the phase factors from $\varepsilon_\mu^{(\lambda_2)}$ and $\varepsilon_\mu^{(\lambda_4)}$ (note that, in contrast with [10], $u_\mu'^{(-\lambda_2)}$ corresponds to the $+\lambda_2$ circular polarization for the second photon because now it is also outgoing). Therefore, while using the expressions from [10], we should not forget the relevant phase factors. For example,

$$u'^{(\lambda_2)} \cdot \varepsilon^{(\lambda_2)} = \frac{N_2\, k_2 \cdot k_3}{2\sqrt{2\Delta}} \Phi(\lambda_2; 1234),$$

where

$$\Phi(\lambda; 1234) = k_1 \cdot k_2\; k_3 \cdot k_4 + k_1 \cdot k_3\; k_2 \cdot k_4 - k_1 \cdot k_4\; k_2 \cdot k_3 + i\lambda\, [k_1, k_2, k_3, k_4],$$
$$[k_1, k_2, k_3, k_4] = \varepsilon_{\mu\nu\lambda\sigma} k_1^\mu k_2^\nu k_3^\lambda k_4^\sigma. \tag{12}$$

Owing to the remarkable cyclic symmetry in the definition (9), (10) of the $\varepsilon_\mu^{(\lambda_i)}$ polarization vectors, for the $\pi^0 \to 4\gamma$ helicity amplitudes we get (remind that some numerical factors have already been taken out in (1)):

$$M_{\lambda_1\lambda_2\lambda_3\lambda_4} = \frac{1}{k_1 \cdot k_2\; k_1 \cdot k_3\; k_1 \cdot k_4\; k_2 \cdot k_3\; k_2 \cdot k_4\; k_3 \cdot k_4} \sum_{cyclic} \frac{\Phi(\lambda_2; 1234)\; \Phi(\lambda_4; 1432)}{k_2 \cdot k_4} \tag{13}$$

$$\times \left\{ A(\lambda_1; 1234)\varepsilon^{(1)}_{-\lambda_2,\lambda_3,\lambda_4}(234) + B(\lambda_1; 1234)\varepsilon^{(1)}_{-\lambda_2,\lambda_4,\lambda_3}(243) + 2C(\lambda_1; 1234)\varepsilon^{(2)}_{-\lambda_2,\lambda_3,\lambda_4}(234) \right\}.$$

Here the summation extends over simultaneous cyclic permutations of $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and (k_1, k_2, k_3, k_4):

$$\sum_{cyclic} F(1234) = F(1234) + F(2341) + F(3412) + F(4123).$$

The $\varepsilon^{(1)}_{\{\lambda\}}$ and $\varepsilon^{(2)}_{\{\lambda\}}$ amplitudes have been defined in [11] (see also [12]) and we reproduce them in the appendix. For the A, B and C functions we have

$$A(\lambda_1; 1234) = \varepsilon^{\mu\nu\lambda\sigma}\; k^1_\mu\; q_\nu\; \varepsilon^{(\lambda_1)}_\lambda \left[k_3 - \frac{(q-k_1) \cdot k_3}{(q-k_1) \cdot k_2}\, k_2 \right]_\sigma,$$

$$B(\lambda_1; 1234) = -\varepsilon^{\mu\nu\lambda\sigma}\; k^1_\mu\; q_\nu\; \varepsilon^{(\lambda_1)}_\lambda \left[k_4 - \frac{(q-k_1) \cdot k_4}{(q-k_1) \cdot k_2}\, k_2 \right]_\sigma, \tag{14}$$

$$C(\lambda_1; 1234) = i\, \varepsilon^{\mu\nu\lambda\sigma}\; k^1_\mu\; q_\nu\; \varepsilon^{(\lambda_1)}_\lambda\; \varepsilon_{\sigma\mu'\nu'\lambda'}\, (k_1 - q)^{\mu'}\, k_2^{\nu'}\, k_3^{\lambda'}.$$

Their explicit expressions are rather cumbersome and are given in the appendix.

The polarization vectors (9) and (11) become ill-defined for collinear photons. Fortunately, this kinematical region gives a negligible contribution to the decay width. In fact the corresponding fictitious kinematical singularities don't cause any considerable trouble in numerical calculations because the phase factors also vanish for collinear photons and only the singularities corresponding to the three simultaneously collinear photons remain.

There are no infrared divergencies in our problem (when the energy of any photon goes to zero), as is easily seen from the explicit expressions of the $\varepsilon^{(1)}, \varepsilon^{(2)}$ amplitudes. The contribution of the muon in the fermion loop can be neglected at least by a factor $(\omega/m_\mu)^3$, $\omega \sim m/4$, ω is the mean photon energy, due to the known low energy behavior of the light-light scattering amplitude.

The numerical calculations give the result

$$Br(\pi^0 \to 4\gamma) \simeq (2.6 \pm 0.1) \cdot 10^{-11}. \tag{15}$$

This is about three orders of magnitude below the present experimental limit.

A Appendix

The $\varepsilon^{(1)}_{\{\lambda\}}$ and $\varepsilon^{(2)}_{\{\lambda\}}$ amplitudes are defined in our case as

$$\varepsilon^{(1)}_{\lambda_2\lambda_3\lambda_4}(234) = \frac{1}{2(q-k_1)^2} E^{(1)}_{\lambda_2\lambda_3\lambda_4}(234)$$
$$\varepsilon^{(2)}_{\lambda_2\lambda_3\lambda_4}(234) = \frac{1}{4} E^{(2)}_{\lambda_2\lambda_3\lambda_4}(234) \qquad \text{(A.1)}$$

and $E^{(1)}$ and $E^{(2)}$ are defined in [10]. The expressions for $\varepsilon^{(1)}_{\{\lambda\}}$ and $\varepsilon^{(2)}_{\{\lambda\}}$ can be found in [11], and here we just reproduce them.

$$\varepsilon^{(1)}_{+++}(234) = \frac{2(1-\nu_3)(1-\nu_4)}{\nu_3} + \left[\frac{2(1-\nu_3)^2(1-\nu_4)}{\nu_3(1-\nu_2)} + \frac{2(1-\nu_3)(1-\nu_4)}{\nu_3^2}\right.$$
$$\left. - \frac{(1-\nu_3)^2}{\nu_3}\right] \ln(1-\nu_3) + \left[\frac{2(1-\nu_3)(1-\nu_4)}{1-\nu_2} + \frac{(1-\nu_3)(1-\nu_4)}{\nu_4}\right] \ln(1-\nu_4)$$
$$+ \left[\frac{(1-\nu_3)(\nu_4-\nu_3)}{1-\nu_2} - \frac{2(1-\nu_3)^2(1-\nu_4)}{(1-\nu_2)^2}\right] \left(\frac{\pi^2}{6} - \mathrm{Li}(\nu_3) - \mathrm{Li}(\nu_4) - \ln(1-\nu_3)\ln(1-\nu_4)\right),$$

$$\varepsilon^{(1)}_{-++} = 0, \quad \varepsilon^{(1)}_{++-}(234) = -\frac{\nu_2}{\nu_3}\varepsilon^{(1)}_{+++}(324) + \frac{\nu_4}{\nu_3}\varepsilon^{(1)}_{+++}(342), \qquad \text{(A.2)}$$

$$\varepsilon^{(1)}_{+-+}(234) = \varepsilon^{(1)}_{+++}(432), \quad \varepsilon^{(1)}_{-\lambda_2,-\lambda_3,-\lambda_4}(234) = -\varepsilon^{(1)}_{\lambda_2,\lambda_3,\lambda_4}(234).$$

$$\varepsilon^{(2)}_{+++}(234) = \left[\frac{2(1-\nu_3)}{1-\nu_2} - \frac{1-\nu_3}{\nu_3}\right] \ln(1-\nu_3) + \left[\frac{2(1-\nu_4)}{1-\nu_2} - \frac{1-\nu_4}{\nu_4}\right] \ln(1-\nu_4)$$
$$- \left[\frac{2(1-\nu_3)(1-\nu_4)}{(1-\nu_2)^2} + \frac{\nu_2}{(1-\nu_2)}\right] \left(\frac{\pi^2}{6} - \mathrm{Li}(\nu_3) - \mathrm{Li}(\nu_4) - \ln(1-\nu_3)\ln(1-\nu_4)\right),$$

$$\varepsilon^{(2)}_{-++} = -2, \quad \varepsilon^{(2)}_{++-}(234) = \varepsilon^{(2)}_{+++}(324) \qquad \text{(A.3)}$$

$$\varepsilon^{(2)}_{+-+}(234) = \varepsilon^{(2)}_{+++}(432), \quad \varepsilon^{(2)}_{-\lambda_2,-\lambda_3,-\lambda_4}(234) = \varepsilon^{(2)}_{\lambda_2,\lambda_3,\lambda_4}(234),$$

where

$$\nu_2 = 1 - \frac{2\, k_3 \cdot k_4}{(q-k_1)^2}, \quad \nu_3 = 1 - \frac{2\, k_2 \cdot k_4}{(q-k_1)^2}, \quad \nu_4 = 1 - \frac{2\, k_2 \cdot k_3}{(q-k_1)^2}, \quad \nu_2 + \nu_3 + \nu_4 = 2, \qquad \text{(A.4)}$$

$$\mathrm{Li}(\nu) = -\int_0^\nu \frac{dx}{x} \ln(1-x). \qquad \text{(A.5)}$$

The expressions for the A, B and C functions look like

$$A(\lambda_1; 1234) = \left[k_1 \cdot k_4 + \left(1 + \frac{(q-k_1)\cdot k_3}{(q-k_1)\cdot k_2}\right) k_1 \cdot k_2\right] [k_1, k_2, k_3, k_4]$$

$$-i\Big[k_1\cdot k_4\; q\cdot k_2\left(k_1\cdot k_3-\frac{(q-k_1)\cdot k_3}{(q-k_1)\cdot k_2}k_1\cdot k_2\right)+k_1\cdot k_2\; q\cdot k_1\left(k_3\cdot k_4-\frac{(q-k_1)\cdot k_3}{(q-k_1)\cdot k_2}k_2\cdot k_4\right)$$
$$-k_1\cdot k_2\; q\cdot k_4\left(k_1\cdot k_3-\frac{(q-k_1)\cdot k_3}{(q-k_1)\cdot k_2}k_1\cdot k_2\right)-k_2\cdot k_3\; k_1\cdot k_4\; q\cdot k_1\Big]\;\lambda_1, \qquad (A.6)$$

$$B(\lambda_1;1234)=\left[k_1\cdot k_4-\frac{(q-k_1)\cdot k_4}{(q-k_1)\cdot k_2}k_1\cdot k_2\right]\;[k_1,k_2,k_3,k_4]$$
$$+i\Big[k_1\cdot k_4\; q\cdot k_2\left(k_1\cdot k_4-\frac{(q-k_1)\cdot k_4}{(q-k_1)\cdot k_2}k_1\cdot k_2\right)-k_1\cdot k_2\; q\cdot k_4\left(k_1\cdot k_4-\frac{(q-k_1)\cdot k_4}{(q-k_1)\cdot k_2}k_1\cdot k_2\right)$$
$$-k_1\cdot k_2\; k_2\cdot k_4\; q\cdot k_1\,\frac{(q-k_1)\cdot k_4}{(q-k_1)\cdot k_2}-k_1\cdot k_4\; k_2\cdot k_4\; q\cdot k_1\Big]\;\lambda_1, \qquad (A.7)$$

$$C(\lambda_1;1234)=\{(k_1\cdot k_2\; q\cdot k_4-k_1\cdot k_4\; q\cdot k_2)\,[k_1,k_2,k_3,k_4]\,\lambda_1$$
$$+i\big[(k_1\cdot k_4\; k_2\cdot k_3-k_1\cdot k_2\; k_3\cdot k_4)\,(q^2\; k_1\cdot k_2-q\cdot k_1\; k_1\cdot k_2-q\cdot k_1\; q\cdot k_2)$$
$$-\;k_1\cdot k_2\; k_3\cdot k_4(k_1\cdot k_3\; q\cdot k_1+q\cdot k_1\; q\cdot k_3-q^2\; k_1\cdot k_3)$$
$$+(k_1\cdot k_2\; q\cdot k_4-k_1\cdot k_4\; k_2\cdot q)\,(k_1\cdot k_2\; q\cdot k_3-k_1\cdot k_3\; q\cdot k_2)\big]\}\;\frac{1}{(q-k_1)^2}. \qquad (A.8)$$

Note that in the squared helicity amplitudes only $[k_1,k_2,k_3,k_4]^2$ appears, and it can be expressed in terms of the scalar products between photon momenta:

$$[k_1,k_2,k_3,k_4]^2\equiv\left(\varepsilon_{\mu\nu\lambda\sigma}\; k_1^\mu k_2^\nu k_3^\lambda k_4^\sigma\right)^2=-\lambda(k_1\cdot k_2\; k_3\cdot k_4,\;\; k_1\cdot k_3\; k_2\cdot k_4,\;\; k_1\cdot k_4\; k_2\cdot k_3), \qquad (A.9)$$

where λ is the triangle function.

At last we list the expressions for various scalar products in terms of the invariant variables used in the phase space integral (2):

$$\begin{aligned}
k_1\cdot k_2&=\frac{m^2}{2}(1+s_2-s_1-u_1), & k_2\cdot k_3&=\frac{m^2}{2}(1+t_2-u_1-u_2),\\
k_1\cdot k_3&=\frac{m^2}{2}(u_1-t_2-s_2), & k_2\cdot k_4&=\frac{m^2}{2}(s_1+u_1+u_2-s_2-t_2-1),\\
k_1\cdot k_4&=\frac{m^2}{2}t_2, & k_3\cdot k_4&=\frac{m^2}{2}s_2,\\
q\cdot k_1&=\frac{m^2}{2}(1-s_1), & q\cdot k_2&=\frac{m}{2}(1-u_1),\\
q\cdot k_3&=\frac{m^2}{2}(1-u_2), & q\cdot k_4&=\frac{m^2}{2}(-1+s_1+u_1+u_2).
\end{aligned} \qquad (A.10)$$

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