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CALCULATION OF HEAT-KERNEL COEFFICIENTS
AND USAGE OF COMPUTER ALGEBRA

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Introduction

The calculation of the quark determinant modulus can be reduced to the calculation of the Schwinger proper-time integral [1]:

$$\log(\det \mathbf{A}) = \int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau} \text{Tr} K(\tau), \quad (1)$$

where $K(\tau) = e^{-\mathbf{A}\tau}$ is the so-called “heat kernel” for operator $\mathbf{A} = \widehat{\mathbf{D}}^\dagger \widehat{\mathbf{D}}$, with $\widehat{\mathbf{D}} \equiv \gamma^\mu \mathbf{D}_\mu$ being Dirac differential operator in the presence of the external background fields. The coefficients h_k of the expansion of the interaction part of the heat kernel in powers of the proper time τ are known as the heat-kernel coefficients. They determine the low-energy expansion of the effective one-loop action [2]. Therefore, the calculation of these coefficients is an alternative method to the direct calculation of Feynman diagrams [3], thus having a fundamental character.

This paper is an extension of our previous paper [4]. We discuss the implementation of the classical DeWitt algorithm [5] to calculate the higher-order heat-kernel coefficients by means of computer algebra. After presenting the results obtained with this method we briefly compare them with those of other groups using different techniques. This method is demonstrated to obtain also the nonlocal corrections taking into account the finite sizes of mesons, that leads to the modification of the heat-kernel equation. We also discuss technical questions connected with the work with large expressions in Lisp-based computer algebra systems such as Reduce. Our techniques allow us to push the calculations of the heat-kernel coefficients to higher orders.

1 General outline of the heat-kernel techniques

The logarithm of the determinant of a positive definite operator \mathbf{A} is defined in proper-time regularization with the integral relation Eq. (1). The trace Tr is to be understood as a space-time integration and a “normal” trace over Dirac, color and

flavor indices. The operator \mathbf{A} is an elliptic one in usual cases:

$$\mathbf{A} \equiv \widehat{\mathbf{D}}^\dagger \widehat{\mathbf{D}} = d_\mu d^\mu + a(x) + \mu^2,$$

where $d_\mu = \partial_\mu + \Gamma_\mu$, operator Γ_μ describes vector gauge fields, $a(x)$ is a local operator without free derivatives. The explicit form of these operators depends on the particular model. The effective parameter μ fixes the regularization in the region of low momenta.

The heat kernel $K(\tau) = e^{-\mathbf{A}\tau}$ satisfies the equation

$$\frac{\partial}{\partial \tau} K(\tau) + \mathbf{A}K(\tau) = 0$$

with the boundary condition

$$K(\tau = 0) = 1.$$

The asymptotic behavior of \mathbf{A} at short distances is defined by the “free” part

$$\mathbf{A}_0 = \partial_\mu \partial^\mu + \mu^2.$$

Using the ansatz

$$K = K_0 H$$

it is convenient to separate from the heat-kernel its “free” part K_0 ,

$$\langle x | K_0 | y \rangle = \langle x | \exp(-(\partial_\mu \partial^\mu + \mu^2)\tau) | y \rangle = \frac{1}{(4\pi\tau)^2} \exp \left\{ -\mu^2\tau + \frac{(x-y)^2}{4\tau} \right\},$$

which satisfies equation

$$\frac{\partial K_0}{\partial \tau} + \mathbf{A}_0 K_0 = 0,$$

with the boundary condition

$$K_0(\tau = 0) = 1.$$

The “interaction” part H of the heat kernel satisfies the equation

$$\left(\frac{\partial}{\partial \tau} + \frac{1}{\tau} z_\mu d^\mu + d^\mu d_\mu + a \right) H(x, y; \tau) = 0,$$

$$H(x, y = x; \tau = 0) = 1, \tag{2}$$

with $z_\mu = x_\mu - y_\mu$. The differential operator d_μ acts on x only. Using now an expansion for $H(\tau)$ in powers of τ

$$H(x, y; \tau) = \sum_{k=0}^{\infty} h_k(x, y) \cdot \tau^k$$

a recursive relation is obtained from Eq. (2):

$$(n + z_\mu d^\mu) h_n(x, y) = -(a + d_\mu d^\mu) h_{n-1}(x, y), \quad (3)$$

with the boundary condition

$$z_\mu d^\mu h_0(x, y) = 0. \quad (4)$$

After the integration over τ in Eq. (1) the following expression for the quark determinant is obtained

$$\log |\det i\widehat{\mathbf{D}}| = -\frac{1}{2} \log(\det \widehat{\mathbf{D}}^\dagger \widehat{\mathbf{D}}) = -\frac{1}{2} \frac{\mu^4}{(4\pi)^2} \sum_k \frac{\Gamma(k-2, \mu^2/\Lambda^2)}{\mu^{2k}} \text{Tr} h_k,$$

where $\Gamma(\alpha, x) = \int_x^\infty dt e^{-t} t^{\alpha-1}$ is the incomplete gamma function and $h_k \equiv h_k(x, y = x)$.

2 Recursive algorithm

In order to construct a recursive algorithm let us use the basic property of the heat-kernel coefficients providing the recursive relation (3) between $h_n(x, y)$ and $h_{n-1}(x, y)$. The boundary condition (4) for $h_0(x, y)$ is determined from Eq. (3) with $n = 0$ and $h_{-1}(x, y) = 0$. The task is to find the coincidence limit $h_n \equiv h_n(x, y)|_{z=0}$. We cannot set $z = 0$ in Eq. (3) directly, since when acting by differential operator d_α on this equation we get a nonvanishing contribution: $d_\alpha(z_\mu d^\mu h_n)|_{z=0} = g_{\alpha\mu} \cdot d^\mu h_n|_{z=0} = d_\alpha h_n|_{z=0}$.

One can see that the usage of Eq. (3) for the calculation of h_n will produce terms like $d_\alpha d_\beta \dots h_n(x, y)|_{z=0}$. In order to get the recursive relation for them we apply the product of m differential operators $d_\alpha d_\beta \dots d_\omega$ on Eq. (3) and take limit $z = 0$:

$$\underbrace{d_\alpha d_\beta \dots d_\omega}_m h_n(x, y)|_{z=0} = -\frac{1}{n+m} \left\{ d_\alpha d_\beta \dots d_\omega (a + d_\mu d^\mu) h_{n-1}(x, y) + P_{\alpha\beta\dots\omega} h_n(x, y) \right\} \Big|_{z=0} \quad (5)$$

where $(n+m) > 0$ and

$$P_{\alpha\beta\dots\omega} = \underbrace{d_\alpha d_\beta \dots d_\omega}_m \cdot z_\mu d^\mu \Big|_{z=0} - m d_\alpha d_\beta \dots d_\omega.$$

For $P_{\alpha\beta\dots\omega}$ we have the recursive relation

$$P_{\alpha\beta\dots\omega} = d_\alpha P_{\beta\dots\omega} + R_{\beta\dots\omega; \alpha} \quad (6)$$

with the boundary condition $P = 0$, where $R_{\beta\dots\omega;\alpha} = [d_\beta \dots d_\omega, d_\alpha]$. Performing a commutation of the differential operator d_α successively through the others d_β, \dots, d_ω , one can move it to the left side. Then the two products with m differential operators cancel each other, and only terms with $(m - 2)$ differential operators are left over. Finally we will get the recursive relation

$$R_{\beta\gamma\dots\omega;\alpha} = \Gamma_{\beta\alpha} \cdot d_\gamma \dots d_\omega + d_\beta \cdot R_{\gamma\dots\omega;\alpha} \quad (7)$$

with the boundary condition $R_{,\alpha} = 0$. Thus, one has to use the recursive relation (5) starting from $m = 0$ to calculate $h_n(x, y)|_{z=0}$. After each iteration it is necessary to commute all differential operators arising from $d_\mu d^\mu$ or $P_{\alpha\beta\dots\omega}$ to the right side up to $h_k(x, y)$ introducing commutators of the type

$$S_\mu = [d_\mu, a], \quad S_{\mu\nu} = [d_\mu, S_\nu], \quad S_{\alpha\mu\nu} = [d_\alpha, S_{\mu\nu}], \quad \text{etc.}$$

$$\Gamma_{\mu\nu} = [d_\mu, d_\nu], \quad K_{\alpha\mu\nu} = [d_\alpha, \Gamma_{\mu\nu}], \quad K_{\beta\alpha\mu\nu} = [d_\beta, K_{\alpha\mu\nu}], \quad \text{etc.} \quad (8)$$

n and m change under these iterations in the following way: either $(n \rightarrow n - 1)$, or $(m \rightarrow m - 2)$, or $(n \rightarrow n - 1; m \rightarrow m + 2)$. It is easy to show that after $2n$ iterations only $h_0(x, y)$ remains without differential operators. At the end one gets the desired result by setting the limit $z = 0$ where $h_0(x, y)|_{z=0} = 1$.

Following the strategy outlined above, the calculation of the heat-kernel coefficients is straightforward but cumbersome. The lengthy calculations can be performed only with computer support. The calculation of the heat-coefficients is a recursive process which can be done by computer algebra very conveniently using the recursive relations (5-7) until substitutions cannot be made any further.

3 Heat-kernel coefficients

The expressions for the heat-kernel coefficients contain a large number of terms which can be related to each other with the following transformations: cyclic properties of trace, commutator identities (8), removing physically redundant total derivatives, renaming of the dummy indices and Jacobi identities for symbols $K_{\alpha\mu\nu}$.

It is necessary to reduce the expressions to some minimal basis of linearly independent terms for the final presentation of the results and their comparison with the results of other papers. As soon as we know, it is still not possible for nontrivial cases to bring all expressions to a canonical form using some identities. The solution of this problem is very important since sometimes it is easier to perform a calculation in a

noncanonical basis than to bring it to a canonical form or test its equivalence with the similar expression in another presentation. For example, we had to perform by hand the reduction of the effective chiral lagrangians of the order p^6 in the momentum expansion to a minimal basis [6], after most of the initial “trivial” simplifications had been done automatically with computer algebra.

So far we do not know an efficient general algorithm which allows to transform the total heat-kernel coefficients into some unique minimal basis. For the reduction of the heat-kernel coefficients to the minimal basis let us use what is always possible to do — test whether two expressions are equivalent or not. Here it is sufficient to express all $S_{\mu\dots}$ and $\Gamma_{\mu\nu\dots}$ according to their definitions in terms of operators a and d_μ . The comparison of such expressions is easier since the only equivalent transformations here are cyclic permutations and renaming of dummy indices. If we can test the equivalence of any expressions with some unknown coefficients, then we can expand any expression in the minimal basis solving the system of linear equations for the coefficients of this expansion. The minimal basis itself can also be constructed by rejecting the linearly dependent terms and including the linearly independent ones. These operations were fully automated by means of the computer algebra and they have allowed us to obtain the heat-kernel coefficients h_1, \dots, h_5 in the minimal-basis form:

$$\begin{aligned}
h_0 &= 1, \\
h_1 &= -a, \\
\text{Tr } h_2 &= \frac{1}{2} \text{Tr} \left\{ a^2 + \frac{1}{6} \Gamma_{\mu\nu}^2 \right\}, \\
\text{Tr } h_3 &= \frac{1}{6} \text{Tr} \left\{ -a^3 + \frac{1}{2} S_\mu^2 - \frac{1}{2} a \Gamma_{\mu\nu}^2 + \frac{1}{10} K_{\nu\nu\mu}^2 - \frac{1}{15} \Gamma_{\mu\nu} \Gamma_{\nu\alpha} \Gamma_{\alpha\mu} \right\}, \\
\text{Tr } h_4 &= \frac{1}{24} \text{Tr} \left\{ a^4 + a^2 S_{\mu\mu} + \left(\frac{4}{5} a^2 \Gamma_{\mu\nu}^2 + \frac{1}{5} (a \Gamma_{\mu\nu})^2 \right) - \frac{2}{5} a S_\mu K_{\nu\nu\mu} + \frac{1}{5} S_{\mu\mu}^2 \right. \\
&\quad + \frac{4}{15} a \Gamma_{\mu\nu} \Gamma_{\nu\rho} \Gamma_{\rho\mu} + \left(-\frac{2}{5} a K_{\nu\nu\mu}^2 + \frac{2}{15} S_{\alpha\alpha} \Gamma_{\mu\nu}^2 - \frac{8}{15} S_{\beta\gamma} \Gamma_{\gamma\alpha} \Gamma_{\alpha\beta} \right) \\
&\quad + \left(\frac{17}{210} \Gamma_{\mu\nu}^2 \Gamma_{\alpha\beta}^2 + \frac{2}{35} \Gamma_{\mu\nu} \Gamma_{\nu\rho} \Gamma_{\mu\sigma} \Gamma_{\sigma\rho} + \frac{1}{105} \Gamma_{\mu\nu} \Gamma_{\nu\rho} \Gamma_{\rho\sigma} \Gamma_{\sigma\mu} \right. \\
&\quad \left. + \frac{1}{420} \Gamma_{\mu\nu} \Gamma_{\rho\sigma} \Gamma_{\mu\nu} \Gamma_{\rho\sigma} \right) + \frac{16}{105} (K_{\mu\alpha\alpha\nu} \Gamma_{\nu\rho} \Gamma_{\rho\mu} + K_{\alpha\alpha\mu} K_{\beta\beta\nu} \Gamma_{\mu\nu}) + \frac{1}{35} K_{\mu\alpha\alpha\nu}^2 \left. \right\}, \\
\text{Tr } h_5 &= \frac{1}{240} \text{Tr} \left\{ -a^5 + (-2a^3 S_{\mu\mu} - a^2 S_\mu^2) + \left(-a^3 \Gamma_{\mu\nu}^2 - \frac{2}{3} a^2 \Gamma_{\mu\nu} a \Gamma_{\mu\nu} \right) \right. \\
&\quad \left. + \left(\frac{2}{3} a^2 S_\mu K_{\nu\nu\mu} - \frac{2}{3} S_\mu S_\nu a \Gamma_{\mu\nu} \right) + \left(-a S_{\mu\mu} S_{\nu\nu} - \frac{2}{3} S_{\mu\mu} S_\nu^2 \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left(-\frac{2}{7}a^2\Gamma_{\mu\nu}\Gamma_{\nu\alpha}\Gamma_{\alpha\mu} - \frac{8}{21}a\Gamma_{\mu\nu}a\Gamma_{\nu\alpha}\Gamma_{\alpha\mu} \right) \\
& + \left(\frac{4}{7}a^2K_{\mu\nu}K_{\alpha\sigma\nu} + \frac{3}{7}aK_{\mu\mu\nu}aK_{\alpha\sigma\nu} - \frac{8}{7}aS_{\mu\mu}\Gamma_{\nu\alpha}^2 + \frac{4}{7}aS_{\mu}K_{\nu\nu\alpha}\Gamma_{\mu\alpha} \right. \\
& \quad + \frac{8}{7}aS_{\mu}\Gamma_{\mu\nu}K_{\alpha\sigma\nu} - \frac{4}{21}a\Gamma_{\mu\nu}S_{\alpha\alpha}\Gamma_{\mu\nu} - \frac{11}{21}S_{\mu}^2\Gamma_{\nu\alpha}^2 + \frac{20}{21}S_{\mu}K_{\nu\nu\alpha}a\Gamma_{\mu\alpha} \\
& \quad \left. + \frac{2}{21}S_{\mu}S_{\nu}\Gamma_{\mu\alpha}\Gamma_{\alpha\nu} - \frac{10}{21}S_{\mu}S_{\nu}\Gamma_{\nu\alpha}\Gamma_{\alpha\mu} + \frac{2}{7}S_{\mu}\Gamma_{\mu\nu}S_{\alpha}\Gamma_{\alpha\nu} + \frac{1}{42}S_{\mu}\Gamma_{\nu\alpha}S_{\mu}\Gamma_{\nu\alpha} \right) \\
& + \left(\frac{8}{21}S_{\mu\nu}S_{\nu}K_{\alpha\sigma\nu} - \frac{4}{21}S_{\mu}S_{\nu}K_{\mu\alpha\sigma\nu} \right) \\
& + \frac{1}{14}S_{\mu\nu\nu}S_{\mu\alpha\alpha} + \left(-\frac{17}{84}a\Gamma_{\mu\nu}^2\Gamma_{\alpha\beta}^2 - \frac{1}{21}a\Gamma_{\mu\nu}\Gamma_{\nu\alpha}\Gamma_{\mu\beta}\Gamma_{\beta\alpha} - \frac{1}{21}a\Gamma_{\mu\nu}\Gamma_{\nu\alpha}\Gamma_{\alpha\beta}\Gamma_{\beta\mu} \right. \\
& \quad \left. - \frac{5}{84}a\Gamma_{\mu\nu}\Gamma_{\alpha\beta}^2\Gamma_{\mu\nu} - \frac{13}{84}a\Gamma_{\mu\nu}\Gamma_{\alpha\beta}\Gamma_{\mu\nu}\Gamma_{\alpha\beta} - \frac{5}{21}a\Gamma_{\mu\nu}\Gamma_{\alpha\beta}\Gamma_{\beta\nu}\Gamma_{\alpha\mu} \right) \\
& + \left(-\frac{2}{21}aK_{\mu\mu\nu}\Gamma_{\nu\alpha}K_{\beta\beta\alpha} - \frac{2}{7}a\Gamma_{\mu\nu}K_{\mu\alpha\beta}K_{\nu\alpha\beta} - \frac{4}{21}a\Gamma_{\mu\nu}K_{\nu\alpha\alpha\beta}\Gamma_{\beta\mu} - \frac{2}{21}a\Gamma_{\mu\nu}K_{\alpha\alpha\mu}K_{\beta\beta\nu} \right. \\
& \quad - \frac{4}{21}S_{\mu\mu}\Gamma_{\nu\alpha}\Gamma_{\alpha\beta}\Gamma_{\beta\nu} - \frac{4}{21}S_{\mu\nu}\Gamma_{\mu\alpha}\Gamma_{\beta\nu}\Gamma_{\alpha\beta} + \frac{2}{7}S_{\mu}K_{\nu\nu\mu}\Gamma_{\alpha\beta}^2 - \frac{2}{7}S_{\mu}K_{\nu\nu\alpha}\Gamma_{\mu\beta}\Gamma_{\beta\alpha} \\
& \quad \left. + \frac{2}{21}S_{\mu}\Gamma_{\mu\nu}K_{\alpha\alpha\beta}\Gamma_{\beta\nu} \right) + \left(-\frac{1}{7}aK_{\mu\nu\nu\alpha}K_{\mu\beta\beta\alpha} + \frac{2}{21}aK_{\mu\nu\nu\alpha}K_{\alpha\beta\beta\mu} - \frac{3}{28}S_{\mu\mu\nu\nu}\Gamma_{\alpha\beta}^2 \right. \\
& \quad \left. - \frac{1}{42}S_{\mu\mu}K_{\nu\nu\alpha}K_{\beta\beta\alpha} - \frac{2}{7}S_{\mu\mu}K_{\nu\alpha\alpha\beta}\Gamma_{\nu\beta} + \frac{1}{7}S_{\mu}K_{\nu\nu\alpha\alpha\beta}\Gamma_{\mu\beta} \right) \\
& + \left(-\frac{47}{126}\Gamma_{\mu\nu}^2\Gamma_{\alpha\beta}\Gamma_{\beta\gamma}\Gamma_{\gamma\alpha} - \frac{11}{189}\Gamma_{\mu\nu}\Gamma_{\nu\alpha}\Gamma_{\mu\beta}\Gamma_{\alpha\gamma}\Gamma_{\gamma\beta} + \frac{1}{63}\Gamma_{\mu\nu}\Gamma_{\nu\alpha}\Gamma_{\mu\beta}\Gamma_{\beta\gamma}\Gamma_{\gamma\alpha} \right. \\
& \quad \left. + \frac{37}{945}\Gamma_{\mu\nu}\Gamma_{\nu\alpha}\Gamma_{\alpha\beta}\Gamma_{\beta\gamma}\Gamma_{\gamma\mu} + \frac{1}{126}\Gamma_{\mu\nu}\Gamma_{\nu\alpha}\Gamma_{\beta\gamma}\Gamma_{\alpha\mu}\Gamma_{\beta\gamma} + \frac{1}{945}\Gamma_{\mu\nu}\Gamma_{\alpha\beta}\Gamma_{\gamma\mu}\Gamma_{\nu\alpha}\Gamma_{\beta\gamma} \right) \\
& + \left(-\frac{8}{189}K_{\mu\mu\nu}K_{\nu\alpha\beta}\Gamma_{\beta\gamma}\Gamma_{\gamma\alpha} - \frac{10}{189}K_{\mu\mu\nu}K_{\alpha\sigma\nu}\Gamma_{\beta\gamma}^2 + \frac{2}{21}K_{\mu\mu\nu}K_{\alpha\alpha\beta}\Gamma_{\nu\gamma}\Gamma_{\gamma\beta} \right. \\
& \quad + \frac{4}{63}K_{\mu\mu\nu}\Gamma_{\nu\alpha}K_{\beta\beta\gamma}\Gamma_{\gamma\alpha} + \frac{5}{378}K_{\mu\mu\nu}\Gamma_{\alpha\beta}K_{\gamma\gamma\nu}\Gamma_{\alpha\beta} - \frac{61}{189}K_{\mu\nu\nu\alpha}\Gamma_{\mu\alpha}\Gamma_{\beta\gamma}^2 \\
& \quad + \frac{22}{189}K_{\mu\nu\nu\alpha}\Gamma_{\mu\beta}\Gamma_{\gamma\alpha}\Gamma_{\beta\gamma} - \frac{16}{189}K_{\mu\nu\nu\alpha}\Gamma_{\alpha\beta}\Gamma_{\beta\gamma}\Gamma_{\gamma\mu} - \frac{10}{189}K_{\mu\nu\nu\alpha}\Gamma_{\alpha\beta}\Gamma_{\gamma\mu}\Gamma_{\beta\gamma} \\
& \quad - \frac{2}{189}K_{\mu\nu\nu\alpha}\Gamma_{\beta\gamma}\Gamma_{\mu\alpha}\Gamma_{\beta\gamma} - \frac{4}{63}K_{\mu\nu\alpha}^2\Gamma_{\beta\gamma}^2 - \frac{4}{189}K_{\mu\nu\alpha}K_{\mu\alpha\beta}\Gamma_{\nu\gamma}\Gamma_{\gamma\beta} \\
& \quad - \frac{4}{189}K_{\mu\nu\alpha}K_{\mu\alpha\beta}\Gamma_{\beta\gamma}\Gamma_{\gamma\nu} - \frac{2}{63}K_{\mu\nu\alpha}K_{\beta\nu\alpha}\Gamma_{\mu\gamma}\Gamma_{\gamma\beta} + \frac{2}{63}K_{\mu\nu\alpha}\Gamma_{\nu\alpha}K_{\mu\beta\gamma}\Gamma_{\beta\gamma} \\
& \quad \left. + \frac{2}{189}K_{\mu\nu\alpha}\Gamma_{\alpha\beta}K_{\mu\nu\gamma}\Gamma_{\gamma\beta} \right) + \left(-\frac{1}{42}K_{\mu\mu\nu\nu\alpha}K_{\alpha\beta\gamma}\Gamma_{\beta\gamma} + \frac{1}{63}K_{\mu\nu\nu\alpha}K_{\mu\beta\beta\gamma}\Gamma_{\alpha\gamma} \right. \\
& \quad \left. + \frac{4}{63}K_{\mu\nu\nu\alpha}K_{\alpha\beta\beta\gamma}\Gamma_{\mu\gamma} - \frac{5}{63}K_{\mu\nu\nu\alpha}K_{\beta\beta\mu}K_{\gamma\gamma\alpha} \right) + \frac{1}{126}K_{\mu\mu\nu\alpha}K_{\beta\beta\gamma\gamma\alpha} \Big\} + \text{h.c.}
\end{aligned}$$

Here Hermitian conjugation h.c. is defined by

$$a^\dagger = a, \quad (S_{\mu\dots})^\dagger = S_{\mu\dots},$$

$$(\Gamma_{\mu\nu})^\dagger = -\Gamma_{\mu\nu}, \quad (K_{\alpha\mu\nu\dots})^\dagger = -K_{\alpha\mu\nu\dots}.$$

Our expressions for the heat-kernel coefficients $h_{4,5}$ proved to be equivalent with the results of Refs. [7, 8]. On the other hand, we have found deviations from Ref. [2] in the part containing terms with 6 and 8 indices:

$$\begin{aligned} & (24 \text{ Tr } h_4)_{\text{this Ref.}} - (24 \text{ Tr } h_4)_{\text{Ref. [2]}} \\ &= \left(-\frac{8}{15} a d_\mu^2 d_\nu d_\alpha^2 d_\nu + \frac{8}{15} a d_\mu^2 d_\nu d_\alpha d_\nu d_\alpha - \frac{8}{15} a d_\mu d_\nu^2 d_\mu d_\alpha^2 + \frac{8}{15} a d_\mu d_\nu d_\mu d_\nu d_\alpha^2 \right. \\ & \quad \left. + \frac{16}{15} a d_\mu d_\nu d_\alpha^2 d_\nu d_\mu - \frac{16}{15} a d_\mu d_\nu d_\alpha d_\nu d_\alpha d_\mu \right) \\ & - \frac{16}{105} d_\mu^2 d_\nu^2 d_\alpha^2 d_\beta^2 - \frac{16}{105} d_\mu^2 d_\nu^2 d_\alpha d_\beta^2 d_\alpha + \frac{64}{105} d_\mu^2 d_\nu^2 d_\alpha d_\beta d_\alpha d_\beta + \frac{32}{105} d_\mu^2 d_\nu d_\alpha d_\nu d_\beta^2 d_\alpha \\ & - \frac{64}{105} d_\mu^2 d_\nu d_\alpha d_\nu d_\beta d_\alpha d_\beta + \frac{16}{105} d_\mu^2 d_\nu d_\alpha d_\beta^2 d_\nu d_\alpha + \frac{32}{105} d_\mu^2 d_\nu d_\alpha d_\beta d_\nu d_\alpha d_\beta \\ & - \frac{16}{35} d_\mu^2 d_\nu d_\alpha d_\beta d_\nu d_\beta d_\alpha - \frac{16}{35} d_\mu^2 d_\nu d_\alpha d_\beta d_\alpha d_\nu d_\beta + \frac{16}{105} d_\mu^2 d_\nu d_\alpha d_\beta d_\alpha d_\beta d_\nu \\ & - \frac{32}{105} d_\mu d_\nu d_\mu d_\nu d_\alpha d_\beta d_\alpha d_\beta + \frac{64}{105} d_\mu d_\nu d_\mu d_\alpha d_\nu d_\beta d_\alpha d_\beta - \frac{32}{105} d_\mu d_\nu d_\mu d_\alpha d_\beta d_\nu d_\alpha d_\beta \\ & + \frac{32}{105} d_\mu d_\nu d_\mu d_\alpha d_\beta d_\nu d_\beta d_\alpha \end{aligned}$$

These deviations arise from the terms of the expression from Ref. [2]

$$\left(+\frac{2}{5} S_{\alpha\alpha} \Gamma_{\mu\nu}^2 + \frac{4}{105} \{ \Gamma_{\alpha\beta} \mid K_{\rho\rho\mu} \mid K_{\mu\alpha\beta} \} \right),$$

where $\{A \mid B \mid C\} \equiv ABC + CBA$. The corresponding terms of our expression are

$$\left(+\frac{2}{15} S_{\alpha\alpha} \Gamma_{\mu\nu}^2 + \frac{16}{105} K_{\mu\alpha\sigma\nu} \Gamma_{\nu\rho} \Gamma_{\rho\mu} \right).$$

We have performed the following checks of our results. First of all, we have used the identities $\partial(\text{Tr } h_n)/\partial a \simeq -h_{n-1}$ [2]. Secondly, substituting the explicit form of operators a and Γ_μ in the Nambu–Jona-Lasinio (NJL) model [9] and evaluating trace over Dirac gamma-matrices we derived the effective chiral p^4 - and p^6 -lagrangians describing the low-energy processes with mesons and photons [6]. In particular, we reproduce the structure coefficients L_i of the p^4 -lagrangians obtained in Ref. [10]. We also reproduce the effective Euler-Heisenberg lagrangian [11] describing the photon-photon scattering for particles of both spin 0 and spin 1/2 [12].

In the case $\Gamma_\mu = 0$ we can also present the next orders of the “minimal” parts of the heat-kernel coefficients:

$$\begin{aligned}
\text{Tr } h_5^{min} &= \frac{1}{120} \text{Tr} \left\{ -a^5 + 3a^2 S_\mu^2 + 2a S_\mu a S_\mu - a S_{\mu\nu}^2 - \frac{5}{3} S_\mu S_\nu S_{\mu\nu} + \frac{1}{14} S_{\mu\nu\alpha}^2 \right\}, \\
\text{Tr } h_6^{min} &= \frac{1}{720} \text{Tr} \left\{ a^6 + (-4a^3 S_\mu^2 - 6a^2 S_\mu a S_\mu) + \left(\frac{12}{7} a^2 S_{\mu\nu}^2 + \frac{9}{7} a S_{\mu\nu} a S_{\mu\nu} \right. \right. \\
&\quad \left. \left. + \frac{26}{7} a S_{\mu\nu} S_\mu S_\nu + \frac{18}{7} a S_\mu S_{\mu\nu} S_\nu + \frac{26}{7} a S_\mu S_\nu S_{\mu\nu} + \frac{9}{7} S_\mu^2 S_\nu^2 + \frac{17}{14} S_\mu S_\nu S_\mu S_\nu \right) \right. \\
&\quad \left. + \left(-\frac{3}{7} a S_{\mu\nu\alpha}^2 - \frac{11}{21} S_{\mu\nu} S_{\nu\alpha} S_{\mu\alpha} - S_\mu S_{\mu\nu\alpha} S_{\nu\alpha} - S_\mu S_{\nu\alpha} S_{\mu\nu\alpha} \right) + \frac{1}{42} S_{\mu\nu\alpha\beta}^2 \right\}, \\
\text{Tr } h_7^{min} &= \frac{1}{5040} \text{Tr} \left\{ -a^7 + \left(5a^4 S_\mu^2 + 8a^3 S_\mu a S_\mu + \frac{9}{2} a^2 S_\mu a^2 S_\mu \right) \right. \\
&\quad \left. + \left(-\frac{5}{2} a^3 S_{\mu\nu}^2 - \frac{9}{2} a^2 S_{\mu\nu} a S_{\mu\nu} - 6a^2 S_{\mu\nu} S_\mu S_\nu - \frac{7}{2} a^2 S_\mu S_{\mu\nu} S_\nu - 6a^2 S_\mu S_\nu S_{\mu\nu} \right. \right. \\
&\quad \left. \left. - \frac{7}{2} a S_\mu^2 S_\nu^2 - \frac{11}{2} a S_\mu a S_{\mu\nu} S_\nu - \frac{11}{2} a S_\mu a S_\nu S_{\mu\nu} - \frac{11}{2} a S_\mu S_\nu^2 S_\mu - \frac{17}{2} a S_\mu S_\nu a S_{\mu\nu} \right. \right. \\
&\quad \left. \left. - \frac{17}{2} a S_\mu S_\nu S_\mu S_\nu \right) + \left(\frac{5}{6} a^2 S_{\mu\nu\alpha}^2 + \frac{2}{3} a S_{\mu\nu\alpha} a S_{\mu\nu\alpha} + \frac{17}{6} a S_{\mu\nu\alpha} S_\mu S_{\nu\alpha} \right. \right. \\
&\quad \left. \left. + \frac{5}{2} a S_{\mu\nu\alpha} S_{\nu\alpha} S_\mu + \frac{5}{3} a S_{\mu\nu} S_{\mu\nu\alpha} S_\alpha + \frac{11}{3} a S_{\mu\nu} S_{\nu\alpha} S_{\mu\alpha} + \frac{17}{6} a S_{\mu\nu} S_\alpha S_{\mu\nu\alpha} \right. \right. \\
&\quad \left. \left. + \frac{5}{3} a S_\mu S_{\mu\nu\alpha} S_{\nu\alpha} + \frac{5}{2} a S_\mu S_{\nu\alpha} S_{\mu\nu\alpha} + \frac{5}{3} S_\mu^2 S_\nu^2 + \frac{35}{18} S_\mu S_{\mu\nu} S_\alpha S_{\nu\alpha} \right. \right. \\
&\quad \left. \left. + \frac{11}{6} S_\mu S_{\nu\alpha} S_\mu S_{\nu\alpha} + \frac{35}{18} S_\mu S_{\nu\alpha} S_\alpha S_{\mu\nu} + \frac{97}{18} S_\mu S_\nu S_{\mu\alpha} S_{\nu\alpha} + \frac{43}{18} S_\mu S_\nu S_{\nu\alpha} S_{\mu\alpha} \right. \right. \\
&\quad \left. \left. + \frac{35}{9} S_\mu S_\nu S_\alpha S_{\mu\nu\alpha} \right) + \left(-\frac{1}{6} a S_{\mu\nu\alpha\beta}^2 - \frac{16}{15} S_{\mu\nu} S_{\mu\alpha\beta} S_{\nu\alpha\beta} - \frac{7}{10} S_{\mu\nu} S_{\alpha\beta} S_{\mu\nu\alpha\beta} \right. \right. \\
&\quad \left. \left. - \frac{1}{2} S_\mu S_{\mu\nu\alpha\beta} S_{\nu\alpha\beta} - \frac{1}{2} S_\mu S_{\nu\alpha\beta} S_{\mu\nu\alpha\beta} \right) + \frac{1}{132} S_{\mu\nu\alpha\beta\gamma}^2 \right\}.
\end{aligned}$$

The expressions for the heat-kernel coefficients up to $\text{Tr } h_6^{min}$ have been also presented in Ref. [13], and up to $\text{Tr } h_8^{min}$ in Ref. [14]. The minimal coefficients h_n^{min} can be easily transformed into a unique minimal basis [14]: it is necessary to move all identical indices in each symbol, e.g. $S_{\mu\nu\dots}$, to the rest symbols in every term because any addition of some total derivative to the lagrangian does not alter its physical content. This way we have checked our expressions for the minimal coefficients to coincide with other papers.

4 Nonlocal corrections for the effective lagrangian

The usage of the recursive relations allows us to extend the local NJL model [9] by taking into account the finite sizes of mesons [15] in a simple way. In this case the modulus squared of Dirac operator $\widehat{\mathbf{D}}$ receives some additional contributions proportional to a small parameter α/Λ^2 characterizing the size of the nonlocal corrections:

$$\begin{aligned} \widehat{\mathbf{D}}^\dagger \widehat{\mathbf{D}} &= \beta \partial^2 + \mu^2 + 2\Gamma_\mu \partial^\mu + \Gamma_\mu^2 + a \\ &+ \frac{\alpha}{\Lambda^2} \left[b + Q_\alpha \partial^\alpha + (a+c)\partial^2 + 2(\Gamma_\mu \partial^2 + \partial_\alpha \Gamma_\mu \partial^\alpha) \partial^\mu \right] + O\left(\frac{\alpha^2}{\Lambda^4}\right), \end{aligned}$$

where $\beta = 1 + 2\alpha\mu^2/\Lambda^2$ and a , b , c and Q_μ are local operators (their explicit form can be found in Ref. [15]).

The modified recursive equation for the heat-kernel coefficients h_n (for simplicity we study only the case without vector fields) reads

$$\begin{aligned} &\frac{\alpha}{4\Lambda^2} z^2 c h_{n+1}(x, y) \\ &+ \left\{ n + z_\mu \partial^\mu + \frac{\alpha}{\Lambda^2} \left[2(a+c) \left(1 + \frac{1}{2} z_\mu \partial^\mu \right) - 2\mu^2 z_\mu \partial^\mu + \frac{1}{2} z_\mu (\partial^\mu a + 2Q^\mu) \right] \right\} h_n(x, y) \\ &+ \left\{ a + \partial^2 + \frac{\alpha}{\Lambda^2} \left[b + (\partial_\mu + 2Q^\mu) \partial^\mu + (a+c)\partial^2 \right] \right\} h_{n-1}(x, y) = 0. \end{aligned}$$

Analogously to the algorithm described above, one can obtain the recursive relations for $d_\alpha d_\beta \dots h_n(x, y)|_{z=0}$, where each iteration gives either $(n \rightarrow n-1)$, or $(m \rightarrow m-1)$, or $(m \rightarrow m-2)$, or $(n \rightarrow n-1; m \rightarrow m+1)$, or $(n \rightarrow n-1; m \rightarrow m+2)$. It is easy to see that functional $F = 3n + m$ reduces at least by 1 after each iteration. Therefore, the final result is obtained after $3n$ iterations [15].

5 Implementation of calculation in computer algebra system Reduce

We have used the Computer Algebra System (CAS) Reduce [16] extended by our package for the calculations in chiral meson theories [17]. Reduce suits well for our purposes since it is a universal and open system which can be easily extended by the users. It is widely used for our calculations of amplitudes of various meson processes and for the derivation and transformation of lagrangians from bosonization of the NJL model. It allows us to complete the general mathematical and algorithmic base of Reduce by the specific data types and operations for this field. In general it has

provided a convenient computational environment for studying chiral meson theories. A more detailed description of our package can be found in Ref. [17].

For the calculations described in this paper the following package features are the most essential:

- transformation by cyclic permutation of noncommutative operator products under the trace operation, e.g. $\text{Tr}(ABC) \equiv \text{Tr}(BCA)$;
- transformation by redefinition of dummy indices, e.g. $S_{\mu\nu}S_\mu S_\nu \equiv S_{\nu\mu}S_\nu S_\mu$;
- \LaTeX output of large expressions (on the basis of RLF1 package from Reduce library [18]).

For the work with the heat-kernel coefficients the following operations with indices have been additionally introduced:

- ordering indices in a symbol:

$$S_{\nu\mu} \rightarrow S_{\mu\nu} + \alpha\Gamma_{\mu\nu} - \Gamma_{\mu\nu}\alpha;$$

- moving indices from the symbols with too many of them:

$$\text{Tr}(S_{\alpha\mu\nu}S_\alpha S_\mu S_\nu) \rightarrow -\text{Tr}(S_{\mu\nu}[d_\alpha, S_\alpha S_\mu S_\nu]);$$

- moving apart identical indices to different symbols:

$$\text{Tr}(S_{\mu\mu}S_{\nu\nu}) \rightarrow \text{Tr}(S_{\mu\nu}S_{\mu\nu}) + \text{commutator terms.}$$

As it was already noted, these procedures are not sufficient to transform expressions to some unique form. Nevertheless, they perform the overwhelming part of the work to reduce the original expressions.

Universal Lisp-based CAS such as Reduce cannot work with large expressions which do not fit into available computer memory. At the same time the intermediate expressions can grow very much during the large-scale recursive calculations. This problem is effectively tackled by the CAS Form [19] which works with the terms of large expressions “locally” keeping only a relatively small part of the terms in the memory and storing the others on the disk. Another alternative is writing a specialized program to calculate the heat-kernel coefficients in the language C [20].

We have implemented the following simple method to resolve the problem of uncontrolled growth of expressions causing the memory overflow in Reduce: During

the work of algorithm the expression is regularly (after each iteration) looked through and a limit is imposed on the number of terms to be substituted at the next iteration. This can be done by a simple change of all differential operators d_μ , except the first n_{max} ones, to some new operator \tilde{d}_μ for which no substitutions are set. The parameter n_{max} is chosen to provide an optimal loading of memory without its overflow. This method has turned out to be effective. It allowed us to calculate the next orders of the heat-kernel coefficients.

Conclusion

The usage of the classical DeWitt algorithm allows to generate effective lagrangians for a wide range of problems in a simple way. In our previous papers [6] we obtained an effective chiral lagrangian from bosonization of the NJL model at $O(p^6)$ order using this method. It also allows to study the nonlocal corrections for the p^4 chiral lagrangian [15]. The usage of the CAS Reduce extended by a specialized package allowed us to obtain the higher-order heat-kernel coefficients, reduce these expressions to minimal basis and compare the results with some other papers.

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