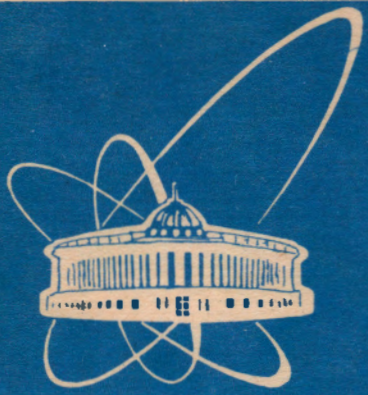

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DROPLET-LIKE SOLUTIONS TO THE NONLINEAR SCALAR FIELD EQUATIONS IN THE ROBERTSON-WALKER SPACE-TIME AND THEIR STABILITY

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1. Introduction

When the general theory of relativity (GTR) and quantum theory of field were developed, an interest to study the role of gravitational interaction in elementary particle physics arose. On this context, to obtain and study the particle-like solutions to the consistent systems of wave and gravitational fields present a major interest. To obtain and study the properties of regular localized solutions to the nonlinear classical field equations (soliton- or particle-like solutions) is connected with the hope to develop a divergence-free theory of elementary particle, which in its turn would describe the complex spatial structure of particle, observed experimentally. In doing so one should keep in mind that the nonlinear generalization of field theory is necessary irrespective of the question of divergence as the consideration of interaction between the fields inevitably leads to the advent of nonlinear terms in the field equations. Consequently, nonlinearity should be considered not only as one of the ways to eliminate difficulties of theory, but also the reflection of objective properties of field. As it is noticed by N. N. Bogoluibov and D. B. Shirkov [1], the complete description of elementary particles with all their physical characteristics (say, magnetic momentum) can give only the interacting field theory. So one can say that individual free (linear) fields present themselves as the basis to describe these particles in the framework of interacting field theory. As elementary particle is a quantum object, so the attempts to develop a classical model of particle remain preliminary but necessary stage of study for transformation to quantum theory.

In this paper the system of interacting scalar and electromagnetic fields are being considered in the Robertson-Walker Universe with the metric [2]:

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2\right], \qquad (1.1)$$

where $R(t)$ defines the size of the Universe, and k takes the values 0 and ± 1. Droplet: it is some kind of soliton-like solutions to the field equations possessing sharp boundary. Similar solution was first obtained by Werle [3]. Further, a series of work was done where the solutions with sharp boundary to the nonlinear field equations were

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being found and studied in external gravitational field as well as in the selfconsistent one [4-10]. Present paper generalizes the partial results obtained by the authors earlier. Moreover here the question of stability is considered which presents a growing interest.

2. Fundamental equations and their solutions

We will choose the Lagrangian of interacting scalar (φ) and electromagnetic ($\mathcal{F}_{\alpha\beta}$) fields in the form [4]:

$$\mathcal{L} = (1/2)\varphi_{,\alpha}\,\varphi^{,\alpha} - (1/4)\mathcal{F}_{\alpha\beta}\,\mathcal{F}^{\alpha\beta}\,\Psi(\varphi), \tag{2.1}$$

where the function $\Psi(\varphi) = 1 + \kappa\,\Phi(\varphi)$ characterizes the interaction ($\Psi(\varphi) = 1$ corresponds to the system of free fields). We will seek the static spherically symmetric solutions assuming that the scalar field φ is the function of r only, and the vector field $\mathcal{A}_\mu$ possesses one component $\mathcal{A}_0(r)$, i.e.

$$\varphi = \varphi(r), \qquad \mathcal{A}_\mu = \delta^0_\mu \mathcal{A}_0(r) = \delta^0_\mu \mathcal{A}(r).$$

It means that ($\mathcal{F}_{\alpha\beta}$) also possesses one component i.e.

$$\mathcal{F}_{\alpha\beta} = (\delta^0_\alpha\,\delta^1_\beta - \delta^0_\beta\,\delta^1_\alpha)\,\mathcal{F}_{01}(r) = \mathcal{A}'(r),$$

where $'$ denotes differentiation with respect to r.

The equations to scalar and electromagnetic fields write:

$$\partial_\nu\,(\sqrt{-g}\,g^{\mu\nu}\,\varphi_{,\mu}) + (\sqrt{-g}/2)\,\mathcal{F}_{\alpha\beta}\,\mathcal{F}^{\alpha\beta}\,\Psi_\varphi(\varphi) = 0, \quad \Psi_\varphi(\varphi) = \partial\Psi/\partial\varphi, \tag{2.2}$$

$$\partial_\nu\,(\sqrt{-g}\,\mathcal{F}^{\mu\nu}\,\Psi(\varphi)) = 0. \tag{2.3}$$

In accordance with the assumption, made above, the equation (2.3) is easily integrated at $r > 0$:

$$\mathcal{F}^{01}(r) = q\,P(\varphi)/\sqrt{-g'} = q\,P(\varphi)\sqrt{1 - k\,r^2}/R^3 r^2, \tag{2.4}$$

where $q = \text{const}$, $P(\varphi) = 1/\Psi(\varphi)$ and $-g' = -g/\sin^2\theta = \frac{R^6 r^4}{(1-k\,r^2)}$.

The equation (2.2) for $\varphi(r)$ in this case metamorphoses to the equation with "induced nonlinearity" [5]:

$$\sqrt{-g'}\left(\sqrt{-g'}\,g^{11}\,\varphi'\right)' = q^2\,g_{11}\,P_\varphi. \tag{2.5}$$

It is also easy to find the first integral and solution in quadrature for the equation (2.5):

$$\varphi' = -\sqrt{2}\,q\,\sqrt{P + C}/R r^2\,\sqrt{1 - k\,r^2}, \quad C = \text{const}, \tag{2.6}$$

$$\int d\varphi/\sqrt{P + C} = \sqrt{1 - k\,r^2}/R r + C_3. \tag{2.7}$$

The regularity condition of T^0_0 at the center leads to the fact that $C = 0$. Choosing $P(\varphi)$ in the form

$$P(\varphi) = 1/\Psi(\varphi) = J^{2-4/\sigma}\left(1 - J^{2/\sigma}\right)^2, \tag{2.8}$$

where $J = \lambda\,\varphi$, $\sigma = 2n + 1$, $n = 1, 2\cdots$, for $\varphi(r)$ one gets:

$$\varphi(r) = \frac{1}{\lambda}\left[1 - \exp\left(-\frac{2\sqrt{2}\,q\,\lambda}{R\sigma}\sqrt{1/r^2 - k} + C_3\right)\right]^{\sigma/2} \tag{2.9}$$

from which it is obvious that

$$\text{at} \quad r \to 0 \quad \varphi(0) \to 1/\lambda,$$

and at $r \to r_c = 2\sqrt{2}q\,\lambda/\sqrt{(R^2\sigma^2 C_3^2 + 8\,k\,q^2\,\lambda^2)}$, $\varphi(r_c) \to 0$.

Let us write the energy-momentum tensor for the interacting fields:

$$T^\nu_\mu = \varphi_{,\mu}\,\varphi^{,\nu} - \mathcal{F}_{\mu\beta}\,\mathcal{F}^{\nu\beta}\,\Psi(\varphi) - \delta^\nu_\mu\,\mathcal{L}. \tag{2.10}$$

From (2.10) we find the density of field energy of the system:

$$T^0_0 = \frac{3}{2}\,\frac{q^2\,P}{R^4\,r^4} \tag{2.11}$$

and total energy

$$E_J = \int T^0_0\,\sqrt{-^3g}\,d^3\mathbf{x} = \frac{3\sqrt{2}\,q\,\pi}{2\lambda\,(\sigma - 1)}. \tag{2.12}$$

Thus, we came to the conclusion that energy density T^0_0 and total energy of the configurations obtained do not depend on the conventional values of the parameter $k = 0, \pm 1$.

3. Stability problem

To study the stability of the configurations obtained we will write the linearized equations for the radial perturbations of scalar field. Assuming that

$$\varphi(r,\,t) = \varphi(r) + \xi(r,\,t), \quad \xi \ll \varphi, \tag{3.1}$$

from (2.2) in view of (2.5) we get the equation for $\xi(r,\,t)$:

$$\ddot{\xi} + 3\frac{\dot{R}}{R}\dot{\xi} - \frac{1 - k\,r^2}{R^2}\xi'' - \frac{2 - 3\,k\,r^2}{r\,R^2}\xi' + \frac{q^2\,P_{\varphi\varphi}}{R^4\,r^4}\xi = 0. \tag{3.2}$$

As far as according to the assumption the external gravitational field is cosmological one, we can consider that $R(t)$ is a slowly varying time-function: $\dot{R}(t) \approx 0$. Assuming that

$$\xi(r,\,t) \approx v(r)\exp(-i\,\Omega\,t), \quad \Omega = \omega/R, \tag{3.3}$$

from (3.2) we obtain

$$(1 - k\,r^2)\,v'' + (2/r - 3\,k\,r)\,v' + (\omega^2 - \frac{q^2\,P_{\varphi\varphi}}{R^4\,r^4})\,v = 0. \tag{3.4}$$

Let us first consider the case when $k = +1$. Then substituting $v(r) = y(x)$, where $x = 1 - 1/r^2$, from (3.4) we get the equation

$$4\,x\,y_{xx} + 2\,y_x + \left(\frac{\omega^2}{(1-x)^2} - \frac{q^2\,P_{\varphi\varphi}}{R^4}\right)y = 0, \tag{3.5}$$

which for $y(x) = u(z), \quad x = z^2$, takes the form

$$u_{zz} + \left(\frac{\omega^2}{(1 - z^2)^2} - \frac{q^2\,P_{\varphi\varphi}}{R^4}\right)u = 0. \tag{3.6}$$

Further substitution

$$\eta(\zeta) = u(z)/\sqrt{1 - z^2}, \quad z = -\mathrm{th}\,\zeta,$$

leads the equation (3.6) to the normal form of Liouville [11]

$$\eta_{\zeta\zeta} + \left(\omega^2 - 1 - \frac{q^2\,P_{\varphi\varphi}}{R^4}\mathrm{sech}^4\,\zeta\right)\eta = 0. \tag{3.7}$$

In case of $k = -1$ the equation (3.4) can analogously be transferred to the form (3.7) doing the following substitutions: $x = 1 + 1/r^2$, $x = z^2$ and $z = \mathrm{th}\,\zeta$.

At last in case of $k = 0$ from (3.4) we get

$$W'' + \left(\omega^2 - \frac{q^2\,P_{\varphi\varphi}}{R^4\,r^4}\right)W = 0, \tag{3.8}$$

where $W(r) = r \cdot v(r)$.

Using the form of $P_{\varphi\varphi}$ from (2.8), we come to the conclusion that for $\sigma \geq 5$ the expressions of the potentials

$$V_{\pm}(\varphi) = 1 + \frac{q^2\,P_{\varphi\varphi}}{R^4}\mathrm{sech}^4\,\zeta \quad \text{and} \quad V_0(\varphi) = \frac{q^2\,P_{\varphi\varphi}}{R^4\,r^4}$$

tends to $+\infty$ at $r \to 0$ as well as at $r \to r_c = \frac{2\sqrt{2}q\,\lambda}{\sqrt{(R^2\,\sigma^2\,C_3^2 + 8\,k\,q^2\,\lambda^2)}}$. It means that for $\sigma \geq 5$ for $P(\varphi)$ given by (2.8) the configuration obtained is stable for the class of perturbation, vanishing at $r = 0$ and $r = r_c$.

In stability can be assured in general introducing the variable

$$\zeta = -\int^r \frac{dr}{\sqrt{-g'\,g^{11}}} = \frac{1}{R}\int^r \frac{dr}{r^2\sqrt{1 - k\,r^2}}$$

and rewriting the equation for perturbation in the form

$$\frac{d^2\,\xi}{d\zeta^2} + (\Omega^2 - q^2\,P_{\varphi\varphi})\,\xi = 0.$$

The equation mentioned possesses at $\Omega = 0$ nonnegative solution $\xi = -\,d\varphi/\,d\zeta$, which according to the Sturm theorem corresponds to the absence of "coupled" state with $\Omega^2 < 0$.

4. Conclusion

Thus, we obtain the object with sharp boundary, described by the regular function $\varphi(r)$. In the center of the system $r = 0 \quad \varphi(0) \to 1/\lambda$, and at some critical value of radius $r = r_c$ function $\varphi(r)$ possesses trivial value. The configuration obtained, possesses limited energy density

and finite total energy. From (2.12) it is explicit that the expression for energy does not contain r, defining the size of droplet. It means that the droplets of different linear sizes up to the soliton with $r_c \to \infty$ share one and the same total energy. For different values of k, the field function $\varphi(r)$ changes it's form. It is noteworthy to notice that at $r_c \to \infty$ for $k = 0$ droplet transfers to usual solitonian solution, while in case of $k = \pm 1$ this type of transition remains absent. It should also be emphasized that the values $k = \pm 1$ enforce the stability of the configurations obtained, which is obvious from the expressions of $V_0(\varphi)$ and $V_\pm(\varphi)$.

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