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RENORMALIZATION GROUP SYMMETRY AND SOPHUS LIE GROUP ANALYSIS

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1 Simple Introduction into RG

1.1 *Group Equations*

The base of RG formalism is the Functional Equation (FEq) for $\bar{g}$ which, in UV case, has the form

$$\bar{g}(x,g) = \bar{g}\left(\frac{x}{t} , \bar{g}(t,g)\right) \ . \tag{1}$$

More popular differential equation

$$\frac{\partial \bar{g}(x;g)}{\partial \ln x} = \beta(\bar{g}) \ , \quad \beta(g) = \left. \frac{\partial \bar{g}(\xi, g)}{\partial \xi} \right|_{\xi=1} . \tag{2}$$

can be directly obtained from it by differentiating over x and putting then $t = x$. On the other hand, by differentiating (1) with respect to t at $t = 1$ we get the partial DEq of Ovsiannikov-Kallan-Symanzik type. Hence, FEq.(1) as well as FEqs for propagators and vertex functions must be considered as the most adequate and general formulation of the RG symmetry in QFT.

However, in reality group FEqs, do not content any physics at all being just the reflection of nothing else but the group composition law! Here, we mean the group of transformations based upon the operation of changing a reference point μ involved into the definition of a coupling constant g_μ.

Namely, we can regard the change of a reference coupling $g_\mu \to g_{\bar{\mu}}$ as a result of the operation of a group element T_t depending on a real continuous positive numerical parameter $t(=\mu/\bar{\mu})$ defined as

$$T_t g_\mu = g_{\mu t} = \bar{g}(t, g_\mu) \ .$$

If we set $x = \tau t$, then the l.h.s. of (1) can be achieved from g by $T_{\tau t}$, while the r.h.s. may be identified as $T_\tau T_t g$. The content of Eq. (1) now is just the group composition law,

$$T_{\tau t} = T_\tau T_t \ .$$

Thus the property of the basic RG equation (1) is the formal condition for transformations T_t to form a group.

1.2 *Renorm-Group Method*

An approximate solution of a physical problem with the RG symmetry does not usually obey this symmetry which is lost in the course of approximation. This is essential when the solution under consideration possesses a singularity as far as the singularity structure is, as a rule, destroyed by approximation.

On this basis one can put the problem of "renormalization-invariant improvement" of perturbative results. The key idea is to combine an approximate expression with group equations. The most simple and convenient way for this "marriage" (as formulated first in Ref.([1]) is the use of the Lie equations, i.e., group differential equations. The renormalization group method (RGM) is essentially based on group DEqs.

Technology of the RG Method. The idea of marrying an approximate solution to the group symmetry can be realised with the help of the group DEqs. If we determine group generators like β from these approximate solutions and then solve the evolution DEqs, we obtain *RG improved* solutions that obey the group symmetry, on the one hand, and correspond to approximate solutions used as an input, on the other hand.

The solution $\bar{g}_{\rm RG}$, thus obtained, exactly satisfies the RG symmetry, i.e., it is an exact solution of Eq. (2) and corresponds to the input $g_{\rm appr}$. For illustration remind that, starting from the simple UV perturbative input

$$\bar{g}_{appr} = g - g^2\beta_1 \ln x$$

we obtain by few lines of elementary calculation the well known geometric progression:

$$g_{RG}(x, g) = \frac{g}{1 + g\beta_1 \ln x} . \tag{3}$$

2 RG and Functional Self-similarity

2.1 *RG-transformation*

The renorm-group transformation for a given solution of some physical problem in the simplest case can be defined as a one-parameter transformation of its two characteristics, say q and g, by

$$R(l) \quad : \quad \{ \, q \to q' = q - l \ , \ \ g \to g' = G(l, g) \ \} \ , \tag{4}$$

the first being a translation and the second of a more complicated functional form. The equation

$$G\{\lambda, G(l, g)\} = G(\lambda + l, g) \ . \tag{5}$$

for the transformation function G provides the group property of the whole transformation (4). The Lie equation can be obtained by diiferentiating of Eq.(5):

$$\frac{\partial \bar{G}(l, g)}{\partial l} = \beta(G) \quad , \qquad \beta(g) = \left.\frac{\partial G(l, g)}{\partial l}\right|_{l=0} . \tag{6}$$

Performing the logarithmic change of variables accompanied by the appropriate redefinition of the transformation function

$$G \to \bar{g}(x, g) = G(\ln x, g)$$

we can obtain the *multiplicative* version of group equations, *i.e.*, Eqs.(1), (2) and

$$R_t : \quad \{x' = x/t, \quad g' = \bar{g}(t, g)\} \tag{7}$$

2.2 *Simple classical illustration*

A rather simple and physically interesting illustration is provided by one-dimensional transfer problem. (Mnatzakanian [2]) Consider a half-space filled with a homogeneous medium, on the surface of which falls from the vacuum some flow (of radiation or particles) with the intensity g_0.

Let us follow the flow as it moves inside the medium at a distance l from the boundary.

Due to homogeneity along the l coordinate the intensity of a penetrated flow $g(l)$ can be represented as a function of two essential arguments, $g(l) = G(l, g_0)$. The flow values at the the boundary point "O" and two other points "1" and "2" with coordinates $l_1 = \lambda$, $l_2 = \lambda + l$ can be connected with each other by the transitivity relations

$$g_1 = G(\lambda, g_0) \quad , \quad g_2 = G(\lambda + l, g_0) = G(l, g_1) \quad ,$$

which lead to Eq. (5), i.e., to an additive version of a group FE.

Note that the beta-function here is just the infinitesimal response at the boundary

$$\bar{G}(\epsilon, g) = g + \beta(g)\epsilon \quad ; \quad \epsilon \ll 1 \quad .$$

Suppose we can find it from some simple reasoning (without solving the kinetic Boltzmann equation). Consider two spesial cases when this response is linear (Case a) or quadratic (Case b) in the boundary flux value g,

$$\bar{G}(\epsilon, g) = g - \epsilon k g \quad \epsilon \ll 1 \quad \text{(Case a)} \quad ,$$

$$g - \epsilon \kappa g^2 \quad \epsilon \ll 1 \quad \text{(Case b)} \quad , \tag{8}$$

Calculating $\beta(g)$ and solving Eq.(6) we find

$$\bar{G}_{\mathrm{RG}}(l, g) = \left\{ g e^{-kl} \quad \text{(Case a)} \quad , \quad \frac{g}{1 + \kappa g l} \quad \text{(Case b)} \quad . \right\} . \tag{9}$$

These expressions possess the FSS property, i.e., they are solutions of Eq. (5). On the other hand, at small values of l they coincide with input approximate expressions (8). At the same time they describe solutions in the whole positive axis including the asymptotic region as $l \to \infty$.

Another simple examples from classical physics are provided by the "weak shock wave" and "elastic rod" problems (see, *e.g.*, Refs.([4]).

2.3 Functional Self-similarity

The RG transformations discussed above have a close connection to the concept of Self-Similarity(SS) in mathematical physics. The SS transformations for problems formulated by the nonlinear partial DEqs are well known since last century mainly in hydrodynamics of liquids and gases. They are simultaneous 1-parameter λ transformations defined as power scaling of independent variables $z = \{x, t, \ldots\}$ and functions $V_i(x, t, \ldots)$, e.t.c. :

$$S_\lambda : \{x \to x\lambda \ \ , t \to t\lambda^a \ \ , V_i(z) \to V_i'(z') = \lambda^{\nu_i} V_i(z') \ \} \quad .$$

To emphasize their power structure we shall use a term *Power Self-Similarity* = PSS. To relate RG with PSS let us turn to solution of the basic renorm-group FEq.(1) Its general solution depends on an arbitrary function of one argument.

However, at the moment we are interested in a special solution linear in the second argument: $\bar{g}(x, g) = gf(x)$. As it immediately follows from the basic equation (1), the function $f(x)$ should have a power form. This means that in our special case the RG transformation (7) is reduced to the PSS transformation,

$$R_t : \{x \to xt^{-1}, \ g \to gt^\nu\} = S_t \ . \tag{10}$$

Generally, in RG, instead of a power law we have an arbitrary functional dependence. Hence, we can consider transformations (4, (7) as it functional generalization of usual (i.e., power) self-similarity transformations. Hence, it is rather natural to refer to them generally as to transformations of functional scaling or **functional self-similarity**[5] rather than to RG-transformations. In short $\boxed{\text{RG} \equiv \text{FSS} \ ,}$

where FSS stands for "Functional Self-Similarity".

3 Wilson's Renormalization Group

3.1 Kadanoff-Wilson construction

In the early seventies the RG method was successfully applied by K.Wilson[6] to critical phenomena in spin lattice systems. As the phase transition in these systems is essentially a large-distance phenomenon, it could be associated with infra-red (IR) limit in QFT. Wilson exploited Kadanov's idea[7] of "blocking" neighboring spin cites, i.e., constructing some auxiliary set of spin lattice models.

RG in critical phenomena is based on the Kadanoff-Wilson procedure referred to as "decimation" or "blocking". Consider a 2-dimensional spin lattice with spacing a, an elementary spin $\vec{\sigma}$ sitting at every lattice site.

Hamiltonian describing the spin interaction of the nearest neighbours is expressed as

$$H = k\sum_i \vec{\sigma}_i \cdot \vec{\sigma}_{i\pm1} \ . \qquad (11)$$

where k is the coupling constant.

To realize the blocking or decimation, one has to perform an "averaging" over blocks consisting of n elementary sites. This is a very essential step as far as it diminishes the degree of freedom number (from N to N/n). It destroys the small-range properties of the system under consideration, in the averaging course some information being lost. However, the long-range physics (like critical phenomena of phase transition) is not affected by it and one gains the simplification of a problem. After this procedure new "effective spins" $\mathbf{\Sigma}$ arise on a new "effective" lattice. We obtain also a new effective Hamiltonian,

$$H_{\rm eff} = K_n \sum_I \mathbf{\Sigma}_I \cdot \mathbf{\Sigma}_{I\pm1} + \Delta H \ ,$$

where ΔH contains quartic and higher terms, For the long-distance properties ΔH is nonessential. Hence, we can conclude that the spin averaging over blocks and the transition to new effective blocks leads to an (approximate !) transformation,

$$k\sum_i(\vec{\sigma}\cdot\vec{\sigma}) \ \rightarrow \ K_2\sum_I(\mathbf{\Sigma}\cdot\mathbf{\Sigma}) \ , \qquad (12)$$

or, taking into account the size of "elementary block" change,

$$KW_n \ : \ \left\{a \rightarrow \sqrt{n}a \ , \quad k \rightarrow K_n \quad .\right\} .$$

The latter can be called the Kadanoff-Wilson transformation.

In general, the "new" coupling constant K_n is a function of the "old" one k and the decimation number n. It is convenient to write it down in the form $K_n = K(1/n, K)$. Then, we can formulate the KW-transformation as follows:

$$KW(n) : \left\{a \rightarrow \sqrt{n}a, \quad k \rightarrow K_n = \bar{K}\left(\frac{1}{n}, k\right)\right\} \ . \qquad (13)$$

These transformations obey the group composition law $KW_n \cdot KW_m = KW_{nm}$. Denoting now $x = 1/nm$, $t = 1/n$, we obtain

$$K(x,k) = \bar{K}(x/t, \ \bar{K}(t,k)) \qquad (14)$$

– just a RG = FSS functional equation.

Here, several comments are in order:

- The FSS=RG symmetry here is approximate (due to ΔH).
- The transformations are discrete.

- There exists no reverse transformation to $KW(n)$.

Hence, the "Kadanoff-Wilson RG" is an *approximate and discrete semi-group*. As we see, the Kadanov-Wilson motivation was quite diffeent from the original QFT one. However, this formally more simple and physically more transparent argumentation turned out to be more apprehensible and in the course of the next decade a quick proliferation of RG practice in quite diverse fields of theoretical physics took place. The "decimation" procedure was successively used by other people in polymers, percolation, non-coherent radiation transfer, dynamical chaos and some other physical systems.

3.2 *Paths of the RG expansion*

A bit different line of reasoning was used in **turbulence**. Here, a closer correspondence with QFT has been employed. The corresponding construction for the RG analysis in the turbulence problem has been obtained along the following steps (see Refs. ([8]):

1. Define the generating functional.
2. Write the path integral representation.
3. Find the equivalence of the considered system to some QFT model.
4. Construct the system of Schwinger-Dyson equations.
5. Apply the Feynman diagram technique.
6. Perform the finite renormalization procedure.
7. Then the RG ideology and equations are obtained.

The physics of the RG transformation in the turbulence problem is related to the change of high-frequency cut-off k_{max} value and simultaneous transformation of some effective k_{max}-dependent parameters like, *e.g.*, Reynolds number.

We see that RG expanded in different fields in two different ways:

- the direct analogy with the Kadanov-Wilson construction (averaging over some set of degrees of freedom) in polymers, non-coherent transfer and percolation, i.e., constructing a set of models for a given physical problem.
- search for the exact symmetry (FSS = RG) by proof of the equivalence with QFT.

Now we can conclude that the answer to the question:

Are there different renormalization groups?

is – YES:
As we saw

- In QFT and macroscopic examples, RG $\equiv$ FSS symmetry is an exact symmetry of a solution formulated in its natural variables.
- In turbulence, continuous spin-field models and some others, it is a symmetry of an equivalent QFT model.
- In polymers, percolation, *etc.* , (with Kadanov-Wilson blocking), the RG transformation is a

  transformation between different auxiliary models

  (specially constructed for this purpose).

  Here RG transformation is acting inside of the set of models. Thus, the FSS symmetry "exists" only inside this set of models.

Hence, as we have seen, there is no essential difference in mathematical formalism. There exists, however, a **profound difference in physics**. In what follows we shall use the term " renorm–group " only in the first (i.e., non-Wilsonian) sence.

4 Renorm-group and Lie analysis

4.1 *Including Boundary Parameters into Group Analysis*

The standard group analysis ascending to the last century papers[9] by Marius Sophus Lie reveals (see, e.g., Refs. [10], [11] the problem of discovering the symmetry of differential equations (DEq). However, it does not consider transformations involving parameters (like, e.g., coupling constant) entering into these equations. Also, it does not deal with symmetry of their solution and, particularly, with transformations involving parameters entering into the solution via the initial or boundary conditions.

Due to this, it is quite interesting to look for a regular method of revealing such RGS in different types of physico-mathematical problems including ones that cannot be formulated by a finite system of DEqs.

In problems described by the DEqs such a regular algorithm can be constructed [13] by combining the standard Lie analysis [10] with Ambartsumian's [15] invariant embedding procedure. When the embedding of the boundary DEqs problem yields an integral equation, one faces the problem of enlarging a classical group algorithm on integro-differential systems. As far as in this direction some progress has been recently achieved, one hopes that the above-mentioned combination procedure could be rather constuctive in this case.

On the other hand, the embedding of the Cauchy problem for the system of ordinary differential equations (ODEqs) returns us to the basics of the theory of these equations. Here, it turns out to be fruitful to treat parameters (like, e.g., a coupling constant) standing in equations as new variables involved into the group transformation and/or invariant embedding procedure.

Differential formulation of the RG symmetry employs an infinitesimal operator (the tangent field) R that generally combines some symmetry of the initial problem with a symmetry of its solution involving boundary parameters. The invariant embedding includes these parameters into the number of variables participating in group transformation. Hence, the object of the Lie group analysis is now a new system of equations composed of the initial and embedding equations. The last ones are constucted on the basis of the initial equations (for equations parameters) and of the boundary conditions (for boundary parameters). The symmetry group $\mathcal{G}$ of this new enlarged system can be found now by the standard group algorithm by solving the defining equation for coordinates of correspondingly modified infinitesimal operator X describing the invariance of a new differential manifold.

Now the RG itself can be obtained from $\mathcal{G}$ by its appropriate narrowing/contraction on the solution.

To illustrate, take the boundary problem for ordinary first order DEq

$$u_t = f(t,u)\ ,\ t \geq \tau\ ; \quad u = x,\ t = \tau\ . \tag{15}$$

for the function $u = u(t)$ of one variable. The embedding equation for the boundary problem (15) is of the form:

$$u_\tau + f(\tau, x)u_x = 0\ . \tag{16}$$

with the function u considered a a function of three variables

$$u = u(t, \tau, x). \tag{17}$$

Eq.(16) can be obtained from (15) by its appropriate differentiating and is equivalent to the initial boundary problem. Meanwhile Eq.(15) coincides with characteristics of equation for (16). A unification of Eqs.(15) and (16) into a joint system (new differential manifold)

$$u_t - f(t,u) = 0\ ,\ u_\tau + f(\tau, x)u_x = 0\ . \tag{18}$$

for the function $u(t, \tau, x)$ of three arguments adds the parameters τ and x into the set of variables participating in the group transformations. Their involvement is reflected by the fact that the infinitesimal operator of the continuous (point) symmetry group for the system (18)

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial \tau} + \xi^3 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} \tag{19}$$

is defined by its coordinates ξ, η, depending on four variables $\{t, \tau, x, u\}$ with equations of its continuation on first partial derivatives u_t, u_τ, u_x, that explicitely account the new functional dependence (17) as described by the system (18). Then, the operator X should be narrowed on any accessible approximate solution up to R, the infinitesimal renormgroup operator.

Quite analogously one can involve the parameters, figuring in the initial equation, into the group analysis. The simple illustration is provided by the *modified Burgers equation*

$$u_t - au_x^2 - \nu u_{xx} = 0 \ , \tag{20}$$

$$\text{at} \quad t \geq 0 \ ; \quad -\infty < x < +\infty, \quad \text{and} \quad u = f(x) \ , \ t = 0 \ ,$$

with the dissipation ν and nonlinearity a parameters. Its solution by perturbation expansion in powers of $a \ll 1$ is strightforward

$$u = v(t, x, \nu) + av_1(t, x, \nu) + 0(a^2) \ . \tag{21}$$

with starting approximation v taken as a solution of a linearised problem.

The group of symmetry for differential manifold of Eq.((20) can then be found by the standard algorithm [9, 10, 11] with one important amendment - the parameters a and ν should be treated as independent arguments of $u = u(t, x, a, \nu)$ in the procedure of constructing the defining group equation (as well as under its solving). In other words, the full set of group variables now looks like $\{t, x, a, \nu, u\}$.

4.2 *Recent Results*

1. **Generation of infinite set of harmonics** of the fundamental laser frequency by inhomogeneous laser plasma [12] With the account of hierarchy of magnitudes of potential and nonpotential fields in plasma the one-parameter point renormgroup for the most intensive potential electric field in the vicinity of laser plasma critical density was constructed. It enables to obtain an approximate solution of the problem of generation of high harmonics of radiation without restrictions on the value of nonlinearity parameter (i.e. on the intensity of laser radiation). The space-time behaviour of solutions of nonlinear field equations which describes the structure of potential plasma field and non-potential field of radiation was found. Conversion factors (transformation coefficients) of the fundamental laser frequency radiation in multiple harmonics were calculated and their temperature dependencies were analyzed.

   For the most intensive potential fields at critical density of the laser plasma the renorm-group symmetry has been formulated. This enables us to write down an approximate solution for an arbitrary type/power? of nonlinearity.

   The structure of potential and nonpotential radiation field has been found; coefficients of transformation of the laser beam basic frequency into arbitrary higher harmonics have been determined.

2. **General expressions for nonlinear dielectric permittivities** (NDP) of hot inhomogeneous collisionless plasma were calculated starting from special expressions for NDP in cold plasma [13]. In this case the group of admitted Lorentz transformations for vectors of electric and magnetic fields strength and partial current and charge densities in plasma was used as a renormgroup with noncanonical parameter of renormgroup transformations in the form of velocity of plasma particles kinetic motion.

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