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KALUZA-KLEIN OR FIBRE BUNDLES?

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1 Introduction

The unified 5D geometric theory of gravity and electromagnetism first was proposed by Kaluza in 1921 ([1]). Later this theory was developed by O.Klein ([2]), Mandel([3]), V.A.Fock([4]), Rainich ([5]) and Einstein and Bergmann ([6]) in 1938. Its foundation is the metrics of 5D space including as its components 4D Riemannian space-time metrics and the electromagnetic vector-potentials. In 1965 B.de Witt ([7]) proposed to use the (r+4)D extension of 5D Kaluza-Klein theory as a geometric treatment of a classical gauge field theory. By this extension one can unify the nonabelian gauge fields with gravity. In the B.de Witt scheme electrodynamics was considered an abelian one-parametric gauge theory. In 1968 R.Kerner ([8]) used the de Witt's prescription to write manifestly all geometric relations and the classical gauge fields equations. It became clear (see [9]) that a simple method of producing of the classical Yang-Mills equations in its usual form in de Witt's scheme is absent. The same conclusions are true of the others also (the lagrangian forms, the correct relations between the vector-potentials and stress tensor of the gauge fields, etc.). Without a number of supplementary conditions and hand-prescriptions the (r+4)D geometric Kaluza-Klein theory doesn't correspond to the usual variational gauge fields theory. Nevertheless (r+4)D Kaluza-Klein theory is widely being used in the modern quantum theory of the gauge fields and its supersymmetric extensions and the strings theories.

In contrast to (r+4)D extensions of Kaluza-Klein theory,in the geometric gauge fields theory being based on the fibre bundle spaces geometry the main concept of the theory is a connection but not a metrics. Therefore the gauge fields vector-potentials are included in the components of a connection but not a metrics of the space. So the gauge fields stress tensor is considered a curvature tensor of fibre bundle space but not a part of connection components as occur in Kaluza-Klein theory. The difference between these geometric ways is disparity in the order of the derivatives of a space-time metrics being used for the geometric treatment of the gauge fields. Moreover the fibre bundle space may be not supplied with a metrics at all. These features of the two geometric ways can be taken into account when quantization of theory is producing.

2 The structure of the Kaluza-Klein theory extension in (r+4)D

The essence of the Kaluza-Klein theory extension in (r+4)D consist in following.

1. The space is Riemannian (r+4)D manifold V_{r+4}, where r - gauge group Lie G_r parameters number, three coordinates are spacelike coordinates and one coordinate is timelike.

2. The symmetry group of theory is the general covariant transformations of all coordinates of V_{r+4}.

3. The main field variable is (r+4)D metric tensor $g_{\alpha\beta}$, where $\alpha, \beta = 1, 2, ..., r+4$.

4. The main equations of the theory are (r+4)D Einstein's equations which are postulated in this theory.

$$P_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} P = \kappa Q_{\alpha\beta}, \qquad (1)$$

where $P_{\alpha\beta}$ - (r+4)D curvature tensor Ricci, P - (r+4)D curvature scalar, $Q_{\alpha\beta}$ - the fields sources.

5. The metric tensor $g_{\alpha\beta}$ has the form (in [7]):

$$g_{\alpha\beta} = \begin{pmatrix} g_{ab} & A_{a\nu} \\ A_{b\mu} & g_{\mu\nu} + A_{\mu c} A^c_\nu \end{pmatrix},$$

$$g^{\alpha\beta} = \begin{pmatrix} g^{ab} + A^a_\lambda A^{b\lambda} & -A^{a\nu} \\ -A^{b\mu} & g^{\mu\nu} \end{pmatrix},$$

where $a, b = 1, 2, ..., r; \mu, \nu, \lambda = r+1, r+2, r+3, r+4$.

In the 5D Kaluza-Klein theory (see for example [10]) the metric tensor has the form:

$$G_{\alpha\beta} = \begin{pmatrix} \tilde{g}_{ik} + (1+\chi) g_i g_k & (1+\chi) g_k \\ (1+\chi) g_i & 1+\chi \end{pmatrix}$$

$$G^{\alpha\beta} = \begin{pmatrix} \tilde{g}^{ik} & -\tilde{g}^{ik} g_k \\ -\tilde{g}^{ik} g_i & \frac{1}{1+\chi} + \tilde{g}^{ik} g_i g_k \end{pmatrix},$$

where $i, k = 1, 2, 3, 4$ - space-time indexes,$\alpha, \beta = 1, 2, 3, 4, 5$; $x^5 = \frac{S}{mc}$; S, m - action and mass of particle; $g_i = \frac{e}{mc^2} A_i = \sqrt{\frac{\kappa}{2\pi}} A_i$, e - electron charge, κ - gravitational constant, c - light velocity, A_i - electromagnetic vector-potential, χ - scalar field, $(1+\chi) = G_{55} = N$ - normalizing multiplier.

Though the 5D metric components $G^{ik} = \tilde{g}^{ik}$ almost coincide with the GR 4D metric components g^{ik} the automatic transition from 5D Einstein equations to 4D Einstein equations in Kaluza-Klein theory is absent for the simple reason that the gravitational potentials in GR are the covariant components of metrics g_{ik} but not the contravariant components g^{ik}. The 5D metric components $G_{ik}=\tilde{g}_{ik}$ $+(1+\chi) g_i g_k$ don't coincide with the GR 4D metric components g_{ik} and contain an electromagnetic field g_i in addition to a gravitational field. It is also necessary to note that even $\tilde{g}_{ik}=(1+\chi) g_{ik}$ and $\tilde{g}^{ik} = \frac{1}{(1+\chi)} g^{ik}$ are different from the GR metrics components g_{ik} and g^{ik} and contain a scalar field χ. Just from this point an idea of a scalar gravitation arose. Moreover the Einstein's gravitational potentials g_{ik} are considered the effective potentials in Kaluza-Klein theory but not the true potentials. The true gravitational potentials in Kaluza-Klein theory are $\tilde{g}_{ik}$ and $\tilde{g}^{ik}$ including a scalar field. This field is experimentally inseparable from the gravitational field and mix with the electromagnetic field also.

For a comparison of Kaluza-Klein theory with GR and 4D Maxwell theory the components of 5D Einstein equations can be separated into the following groups:

$$P^{ik} - \frac{1}{2}g^{ik}P = \kappa Q^{ik} \tag{2}$$

$$P_5^i = \kappa Q_5^i \tag{3}$$

$$P_{55} - \frac{1}{2}g_{55}P = \kappa Q_{55} \tag{4}$$

If one suppose Q^{ik}=T^{ik} (T^{ik} - stress-energy tensor of matter) the equations (2) may be transformed to 4D Einstein equations. But a coincidence of these equations take place only if the scalar and electromagnetic fields are absent.

If one suppose Q_5^i=j^i (j^i - electromagnetic field sources) the equations (3) may be transformed to 4D Maxwell equations in the 4D Riemannian space. But a coincidence of these equations take place only if the scalar field is absent.

The equations (4) has not a clear physical sense and may be considered the supplementary conditions.

The symmetry group of Kaluza-Klein theory and its generalizations is 5D (or (r+4)D) general covariant coordinates transformations of 5D Riemannian space (or (r+4)D-space). By the proper conditions it may be separated into 4D general covariant transformations of 4D Riemannian space-time coordinates and a gauge transformations.

In the 5-optics this separation is: the transformations of 4D space-time coordinates (i, k =1,2,3,4)

$$x^{i'} = x^i + f(x^1, x^2, x^3, x^4, S) \tag{5}$$

corresponding to 4D coordinates transformations in GR, and the transformations of x^5-coordinate corresponding to the action S transformations naming the gauge transformations

$$S' = S + f(x^1, x^2, x^3, x^4, S) \tag{6}$$

So far as any dependence of the physical phenomena from a fifth coordinate is not observable experimentally a supplementary assumption was proposed. This assumption consists in that 5D space is topologically closed in a fifth coordinate. Consequently 5D space of Kaluza-Klein theory is really 4D surface of a 5D cylinder of radius b. In 1938 Einstein and Bergmann proposed to connect the constant b

with the Planc's constant h. Such 5-optics unified the classical and quantum physics as two limit cases of the 5D classical gravity and electromagnetism theory. Moreover the periodic conditions for the wave functions of the particles admit a geometrization of the mass of these particles in Proca's equations. Then the constant b becomes proportional to the particle mass (Fock, 1926).

From a topological closure of the fifth coordinate it follows that S-transformations obey the periodic constraints.

$$f^i(x^1, x^2, x^3, x^4, S+h) = f^i(x^1, x^2, x^3, x^4, S) \tag{7}$$

$$f(x^1, x^2, x^3, x^4, S+h) = f(x^1, x^2, x^3, x^4, S) \tag{8}$$

From (8) it follows that 5D Lorentz group is not a subgroup of 5D general covariant coordinates transformations. Consequently a transition from 5D general relativity to 5D special relativity is impossible in contrast to a 4D case.

In a classical limit $h \to 0$ 5D general covariant coordinates transformations become their subgroup

$$x^{i'} = x^i + f^i(x^1, x^2, x^3, x^4)$$

$$S' = S + f(x^1, x^2, x^3, x^4)$$

This subgroup may be separated into

a) subgroup of 4D general covariant coordinates transformations under condition S=$const$

$$x^{i'} = x^i + f^i(x^1, x^2, x^3, x^4)$$

$$S' = S$$

b) subgroup of gauge transformations under condition x^i=$const$

$$x^{i'} = x^i$$

$$S' = S + f(x^1, x^2, x^3, x^4)$$

If the nature laws are formulated by 5D covariant equations for 5D-tensors their gauge covariance is manifest. But it's necessary to keep the gauge invariance when the fifth coordinate is separated and the transition to 4D space-time take place. 5D general covariant coordinates transformations destroy the separation of 5D Einstein equations into the components (2) - (4).

In a special coordinate system under A_5 =1 the transformations (6) lead up to a usual gauge transformations of vector-potential covariant components A_μ:

$$A_i{}' = A_i - A_5\frac{\partial f}{\partial x^i}, \quad A_5{}' = A_5$$

But in contrast with usual electrodynamics the contravariant vector-potential components A^i do not transform simultaneously and $A^{i'}$ =A^i under (6). The analogous properties have the transformations of additional r coordinates in the extended Kaluza-Klein theories.

Thus 5D (or (r+4)D) general covariance of Kaluza-Klein theories have rather declared than real character. The true possibility to use it in these theories do not exist because all transitions to the usual formulae (being obtained by variational methods) are fulfiled under special conditions fixing a freedom of transformations choice.

3 The structure of the fibre bundle space gauge fields theory

The essence of the geometric gauge fields theory based on the fibre bundle spaces geometry consists in following.

1. The space of this theory is the fibre bundle space. Locally this space may be considered the direct product of two spaces. In our case it is a product of 4D Riemannian space-time and the gauge group rD space. The global structure of a fibre bundle space is given by following definition. It is a differentiable c^v-manifold E on which there is specified an equivalence relation R such that: a) the quotient space $B = E/R$, or the basic space, is a differentiable manifold of n dimensions; b) the projection p, i.e., the canonical mapping of the manifold E onto the base B, corresponding to the definition of B as the quotient space, is a c^v- differentiable mapping that is everywhere of rank n. Under these conditions, the structure of a differentiable fibre space on E is determined by the following collection of elements: B - the base; F - the standard fibre; a c^v-differentiable manifold; G_r - a Lie group, which acts in a c^v- differentiable manner on F (the group of automorphisms of F); p - the projection of E onto the base B; and Φ - the family of homeomorphisms of the topological product $U \times F$ onto the inverse image $p^{-1}(U)$, where U is an open set of the space B, a number of conditions on Φ being satisfied. If F is the group space G_r, we call E the principal fibre space; if F is the tangent space to the base, we speak of the tangent bundle; when F is the representation space of G_r, we say that E is the associated fibre space.

   A connection in a homogeneous fibre bundle determines a mapping of fibres one onto another when they are transported along different paths in the base.

   In the geometric fibre bundle space gauge fields theory 4D Riemannian space-time is identified with the base of fibre bundle space and the fibre is identified with the space of r-parametric gauge Lie group G_r. The gauge fields vector-potential A^a_μ (a =1,2,... r) becomes the connection in the fibre bundle space. Then the gauge fields stress tensor $F^a_{\mu\nu}$ coincides with the fibre bundle space curvature tensor ([9]). The relation between A^a_μ and $F^a_{\mu\nu}$ has a usual form corresponding to the variational formulae:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \frac{1}{2} f^a_{bc}(A^b_\mu A^c_\nu - A^b_\nu A^c_\mu)$$

2. The symmetry group of this geometric theory locally is direct product of 4D GR coordinate transformations in the base and group G_r acting in the fibre. The result is a local Lie group.

3. The main field variable is the gauge field vector-potential A^a_μ if the gauge field is not a gravity and $g_{\mu\nu}$ for the gravity being considered a gauge field.

4. The main equations of the theory are not postulated a'priori but are producing by the variational principle after the transformational properties of A^a_μ under local group G_r and 4D GR coordinates transformations are given. These equations automatically coincide with the Yang-Mills equations in 4D Riemannian space-time if the gauge field is not gravity and with 4D Einstein equations for the gravity as the gauge field.

   Such situation corresponds to the lower derivatives of fields variables in a lagrangian. But taking into account the higher derivatives one may to generalize these equations ([11]).

5. The metric tensor in the base coincides with usual 4D GR metrics $g_{\mu\nu}$. The metric tensor in the fibre is a Cartan group metrics. The metric tensor of the global fibre bundle space may not exist.

It is necessary to note that the difference between a gravity and other gauge fields is only illusory. Really all gauge fields vector-potentials are furnished by two indices: one index corresponds to 4D base coordinates (Greek index) and another index (Latin) corresponds to the gauge group parameters. When 4D GR coordinates transformations are considered the gauge transformations corresponding to local gauge group G_4 (local displacements group in 4D manifold) both indices of the gauge vector-potential become Greek and one may consider the vector-potential the tensor of rank two. Hence one need not urge the necessity of $g_{\mu\nu}$ being a base metrics. The Einstein equations are obtaining by variational principle for $g_{\mu\nu}$ in this scheme independently from a $g_{\mu\nu}$ role in the base space ([12]).

In the fibre bundle space geometric approach all fundamental interactions are unified by a single principle of constructing the every of them theory. To produce the lagrangian and the equation of theory one can give: 1) the field variables, 2) two kinds of symmetry groups (in base and in fibre), 3) two kinds of the field variable transformations (in correspondence with two kinds of symmetry groups), 4) the derivatives order of the field variables in the lagrangian.

4 Conclusions

So what is Kaluza-Klein theory and its generalizations?

In these theories we have one space (sufficiently large dimensional), one symmetry group (large dimensional general covariant group of the coordinate transformations) and one equation (large dimensional Einstein equation). The 4D gravity and the gauge fields, their symmetry groups and equations are shared out from the components of above-mentioned large dimensional space, general covariant group and Einstein equations.

What are the gauge fields in a fibre bundle space geometry terms?

The space of this theory is two spaces being unified by the special construction of a fibre bundle space. One space is the 4D general relativity space-time. Another space is the gauge Lie's group space. Everyone point of GR space furnished by a copy of the gauge group space. So we have a possibility to represent the internal properties of point particles in 4D Riemannian space-time by the properties of Lie's group of the gauge transformations of the field variables (but not the coordinates transformations as occur in Kaluza-Klein theory). Thus in a fibre bundle space geometrical gauge fields theory we have two kinds of symmetry groups which act in the each from two kinds of spaces and two kinds of equations: 4D Einstein equations with the gauge fields sources and the gauge fields equations (of Yang-Mills type) in 4D GR space-time. The localization of gauge group manifests in the fibre bundle space structure.

What is a difference between these ways?

In Kaluza-Klein theory and its extensions the space is a configuration space but not a physical real space which is universal for all particles. It is universal only for the particles with the same ratio e/m (charge to mass ratio). Moreover the decision of the problem of fields finding under given sources and inverse problem of sources finding under given fields produce the different configuration spaces. These configuration spaces may be identify only under special supplementary conditions. Each particle can see the Universe with its own eyes and there is no Universe which are seen identically for all particles. Einstein's gravity is seen as effective but not veritable gravity.

In the fibre bundle space gauge fields theory it is postulated that the universal physical space exists. It is 4D space-time of GR and Einstein's gravity is a real veritable gravity forming the Riemannian structure of this space. But each point of this space is furnished by its own structure reflecting its individual properties. As the internal properties of points it is possible to consider the higher derivatives of field functions, internal freedom degrees, etc. Riemannian 4D space-time is interpreted as the physical space in which experiments are usually fulfiled. The fibre corresponds to the space which properties represent the internal freedom degrees space or the space of measurement instruments states localized in each 4D

space-time point. Thus in the fibre bundle space gauge fields theory we have a clear physical interpretation of the mathematical (geometrical) space-time constructions. In the Kaluza-Klein theory the space is formal mathematical space and it has not a physical interpretation.

The symmetry group of Kaluza-Klein theory is the formal extension of 4D GR symmetry group. After gauge transformations being shared out from 5D GR coordinates transformations these 5D GR transformations can not be using as the symmetry group of theory because we can keep the separation of 5D GR symmetry group as well as 5D Einstein equations into the 4D components.

In the fibre bundle space gauge fields theory the symmetry group unifies the symmetry group of 4D space-time and internal symmetry group acting in the fibre in each point of 4D space-time. This unified symmetry is the symmetry group of theory and its equations.

In the Kaluza-Klein theory the 4D Einstein equations and gauge fields equations can not follow from 5D Einstein equations automatically without a number of supplementary conditions.

In the fibre bundle space gauge fields theory all equations of theory follow from the single variational principle and are being produced by the single variational procedure. To produce them it is necessary to give the field variable and its transformation properties under the group symmetry of theory. The same is true for GR also. GR is the gauge theory of symmetrical tensor $g_{\mu\nu}$ and gauge group coinciding with 4D GR coordinates transformations ([12]). Thus the problem of unification of 4D GR with other gauge fields does not exist in this scheme. All fundamental interactions are on an equal footing. The usual Einstein equations, usual Yang-Mills equations and other gauge fields equations are producing automatically by the variational procedure without any supplementary conditions.

Thus two above-mentioned geometric ways in the classical gauge fields theory lead to the different ways of quantum gauge fields theory. But now a collection of the different quantum schemes is not ordered. That is the reason of different geometrical ways mixing up under the gauge fields quantization. There is necessity in distinction them.

It is need to note that both geometrical approach are the extension of Felix Klein ideas. But the Kaluza-Klein theory base on the optics-mechanics analogy following from the formal properties of differential equations. The fibre bundle space theory extends the ideas of F.Klein's Erlangen program (1872) where the generalized spaces idea was formulated and a connection between geometry and the measurement instruments properties was demonstrated.

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Коноплева Н.П.

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Калуза-Клейн или расслоенные пространства?

Анализируются и сравниваются между собой два основных геометрических подхода в современной теории калибровочных полей. Первый из них является обобщением единой теории гравитации и электромагнетизма Калузы-Клейна. Второй обобщает картановскую формулировку римановой геометрии и ОТО, развившуюся в настоящее время в геометрию расслоенных пространств. Показано, что два вышеупомянутых геометрических пути построения классической теории калибровочных полей неэквивалентны и ведут к различным схемам квантования калибровочных теорий.

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Kaluza-Klein or Fibre Bundles?

Two basis geometric ways in the modern theory of the gauge fields are analyzed and compared. The first way is an extension of the Kaluza-Klein unified theory of gravity and electromagnetism. The second way extends the Cartan's formulation of Riemannian geometry and GR which is transformed now to the fibre bundle spaces geometry. It is shown that two above-mentioned geometric ways in the classical gauge fields theory are nonequivalent and lead to the different schemes of the quantum gauge fields theory.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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