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ON ABELIZATION OF FIRST CLASS CONSTRAINTS

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I Introduction

It is the purpose of this note to describe a practical method for the conversion of non - Abelian constraints into the Abelian form in the theories with first class constraints. For explanation of the practical importance of this procedure let us briefly recall the general principles of the description of the standard Dirac theory of Hamiltonian systems with constraints [1] - [4].

For the sake of simplicity as usually we will discuss the main ideas using a mechanical system, i.e. systems with a finite number of degrees of freedom, with having in mind that the direct extension of the results to a field theory in general is possible only in the local sense.

Suppose that the system with $2n$ - dimensional phase space Γ acquires the following set of irreducible first class constraints $\varphi_\alpha(p,q)$, $(\alpha = 1,2,\ldots,m)$

$$\begin{aligned} \varphi_\alpha(p,q) &= 0, \\ \{\varphi_\alpha(p,q), \varphi_\beta(p,q)\} &= f_{\alpha\beta\gamma}(p,q)\varphi_\gamma(p,q) \end{aligned} \tag{1.1}$$

This means that the dynamics of our system is constrained on the certain submanifold [1]of the total phase space or, in another words, not all canonical coordinates are responsible for the dynamics . The generalized Hamiltonian dynamics is described by the extended Hamiltonian which is a sum of canonical Hamiltonian $H_C(p,q)$ and a linear combination of constraints with arbitrary multipliers $u_\alpha(t)$

$$H_E(p,q) = H_C(p,q) + u_\alpha(t)\varphi_\alpha(p,q) \tag{1.2}$$

The arbitrariness of the functions u_α reflects the presence in the theory of coordinates whose dynamics is governed in an arbitrary way . According to the principle of gauge invariance for physical quantities, these coordinates do not affect them and thus can be treated as ignorable. The main problem that arise is identification of these coordinates . If there are in the theory only Abelian constraints

$$\{\varphi_\alpha(p,q), \varphi_\beta(p,q)\} = 0, \tag{1.3}$$

one can find these ignorable coordinates as follows. In this case, it is always possible [5] - [8] to define a canonical transformation to a new set of canonical

[1]In the next we will symbolize by notation Γ_c this $2n-m$ — dimensional submanifold of Γ, $\Gamma_c \subset \Gamma$.

coordinates

$$\begin{aligned} q_i &\mapsto Q_i = Q_i(q_i,p_i), \\ p_i &\mapsto P_i = P_i(q_i,p_i), \end{aligned} \tag{1.4}$$

so that m of the new P 's ($\overline{P}_1,\ldots,\overline{P}_m$) become equal to the abelian constraints (1.3)

$$\overline{P}_\alpha = \varphi_\alpha(q_i,p_i) \tag{1.5}$$

In the new coordinates we have the following system of canonical equations

$$\begin{aligned} \dot{Q}^* &= \{Q^*, H_{Ph}\}, \\ \dot{P}^* &= \{P^*, H_{Ph}\}, \\ \dot{\overline{P}} &= 0, \\ \dot{\overline{Q}} &= u(t), \end{aligned} \tag{1.6}$$

with arbitrary functions $u(t)$ and with the physical Hamiltonian

$$H_{Ph} \equiv H_C(p,q)\Big|_{\varphi_\alpha=0} \equiv H_C(P^*,Q^*)\Big|_{\overline{P}_\alpha=0}$$

The physical Hamiltonian H_{Ph} depends only on the remaining $n-m$ pairs of the new canonical coordinates $(Q_1^*, P_1^* \ldots, Q_{n-m}^*, P_{n-m}^*)$ wich are a gauge invariant physical variables. This means that the coordinates $\overline{Q}_\alpha$ conjugated to the momenta $\overline{P}$ are ignorable coordinates and the canonical system admits explicit separation of phase space into sectors: physical and nonphysical one

$$2n\left\{\begin{pmatrix} q_1 \\ p_1 \\ \vdots \\ q_n \\ p_n \end{pmatrix}\right. \quad\mapsto\quad \begin{array}{ll} 2(n-m)\left\{\begin{pmatrix} Q^* \\ P^* \end{pmatrix}\right. & \begin{array}{c} Physical \\ sector \end{array} \\[2em] 2m\left\{\begin{pmatrix} \overline{Q} \\ \overline{P} \end{pmatrix}\right. & \begin{array}{c} Nonphysical \\ sector \end{array} \end{array} \tag{1.7}$$

The straight generalization of this method to the non - Abelian case is unattainable; identification momenta with constraints is forbidden due to the non - Abelian character of constraints (1.1). However, there is a general proof of a possibility of a local replacement of the constraints (1.1) by the equivalent set of constraints forming an abelian algebra [2], [4], [8], [9], [10].

This general observation — *Abelization statement* reads —

For a given set of m first class constraints it is always possible to chose locally m new equivalent constraints

$$\varphi_\alpha(p,q) = 0 \Longleftrightarrow \Phi_\alpha(p,q) = 0 \qquad (1.8)$$

that define the same constraint surface Γ_C *so the Poisson brackets between the new constraints strongly vanishes, i.e.*

$$\{\Phi_\alpha(p,q), \Phi_\beta(p,q)\} = 0. \qquad (1.9)$$

Thus, one can deal with this equivalent set of Abelian constraints to construct the reduced phase space, the space of physical degrees of freedom. To reveal ignorable coordinates, we need an explicit form of the new Abelian constraints $\Phi_\alpha(p,q)$. We would like to emphasize that in all proofs of the abelization it has been assumed that the constraints form the functional group [2], [10]. In the present paper, we shall point out two alternative schemes of realization of the abelization procedure: via constraints resolution and via "generalized" canonical transformation for general non - Abelian constraints of type (1.1). The generalized canonical transformations [12] are those preserving the form of all constraints of the theory as well as the canonical form of the equations of motion. It will be shown that in a constructive fashion it is possible to convert constraints into the Abelian form with the help of the Dirac equivalence linear transformation

$$\boxed{\Phi_\alpha(p,q) = \mathcal{D}_{\alpha\gamma}\varphi_\gamma(p,q)} \qquad (1.10)$$

with the nonsingular matrix $\mathcal{D}$

$$det||\mathcal{D}_{\alpha\gamma}||\Big|_{constraints} \neq 0$$

The main point of our result is that this matrix $\mathcal{D}$ can be determined by *linear* first order differential equations.

The remaining part of this note is the proof of this statement and the application to the special example: non - Abelian Christ and Lee model [13]

II Abelization : alternative schemes

A Abelization via constraint resolution

The direct way of Abelization of constraints is as follows [10], [11]. Under the assumption that $\varphi_\alpha(p,q)$ are m independent functions one can resolve the constraints (1.1) for m of p's.

$$p_\alpha = F_\alpha(\underline{p}, q) \qquad (2.11)$$

where $\underline{p}$ denotes the remaining p's. Let us pass to a new equivalent to the $\varphi_\alpha(p,q)$ constraints

$$\Phi_\alpha(p,q) = p_\alpha - F_\alpha(\underline{p}, q) \qquad (2.12)$$

By the explicit computing one can convince that the Poisson brackets of the new constraints

$$\{\Phi_\alpha(p,q), \Phi_\beta(p,q)\}$$

are independent of p_α, but as it is clear that they are again the first class the unique possibility is that their Poisson brackets with each other must vanish identically . Thus, after transformations to the new constraints $\Phi_\alpha(p,q)$ we can realize the above - mentioned canonical transformation (1.4) such that m of the new P 's become equal to the modified constraints Φ_α (2.12)

$$\overline{P}_\alpha = \Phi_\alpha(q_i, p_i) \qquad (2.13)$$

with the corresponding conjugate ignorable coordinates $\overline{Q}_\alpha$.

B Abeliazation of constraints via Dirac' s transformation

In this section, it will be demonstrated that due to the freedom in the representation of constraint surface Γ_c

$$\begin{aligned} \varphi_\alpha(p,q) &= 0, \\ \{\varphi_\alpha(p,q), \varphi_\beta(p,q)\} &= f_{\alpha\beta\gamma}(p,q)\varphi_\gamma(p,q). \end{aligned} \qquad (2.14)$$

one can always pass with the help of Dirac 's transformation from these first class non - Abelian constraints to the equivalent ones

$$\Phi_\alpha(p,q) = \mathcal{D}_{\alpha\beta}(p,g)\varphi_\beta(p,q), \qquad (2.15)$$

so that the new constraints will be Abelian

$$\{\Phi_\alpha(p,q), \Phi_\beta(p,q)\} \;=\; 0. \tag{2.16}$$

According to (2.16) the matrix $\mathcal{D}_{\alpha\beta}$ must satisfy the set of the *nonlinear* differential equations

$$\{\mathcal{D}_{\alpha\gamma}(p,g)\varphi_\gamma(p,q), \mathcal{D}_{\beta\sigma}(p,g)\varphi_\sigma(p,q)\} \;=\; 0. \tag{2.17}$$

Such a formulation of the abelization statement means a possibility to find a particular solution for this very complicated system of *nonlinear* differential equations. Beyond the question eq. (2.17) in this form does not contain any practical value but, as it will be shown here, there is a particular solution to this equation which can be represented as

$$\mathcal{D} = \underbrace{\mathcal{D}^1(p,q)\cdots\mathcal{D}^m(p,q)}_{m} \tag{2.18}$$

where each matrix $\mathcal{D}^k$ has a form of the product of k's $m \times m$ matrices

$$\mathcal{D}^k = \mathcal{R}^{a_k+k}(p,q)\prod_{i=k-1}^{0}\mathcal{S}^{a_k+i}(p,q) \tag{2.19}$$

$(a_k \equiv \frac{k(k+1)}{2})$

$$\mathcal{R}^{a_k+k} = \left(\begin{array}{cc} \overbrace{\boxed{\;\mathrm{I}\;}}^{k} & \overbrace{0}^{m-k} \\ 0 & \boxed{\mathrm{B}^{a_k+k}} \end{array}\right) \tag{2.20}$$

$$\begin{array}{c} \begin{array}{cc} \quad\;\; k & \qquad\qquad m-k \end{array} \\ \mathcal{S}^{a_k+i} = \left(\begin{array}{ccccc|ccccc} 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & \cdots & C^{a_k+i}_{k+1} & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & C^{a_k+i}_{k+2} & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \vdots & C^{a_k+i}_{m-1} & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & \cdots & C^{a_k+i}_{m} & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right) \\ \underbrace{\qquad\qquad}_{k-i} \end{array} \tag{2.21}$$

and the corresponding matrix elements satisfy a set of first order *linear* differential equations (see below (2.25) - (2.28)). Just the linear character of these equations allows one to speak about a practical usage of the proposed method of abelization. As it will be explained below, the constraints which are obtained as a result of the action of k's matrices (constraints at the $a_k + k$ -th step)

$$\Phi^{a_k+k}_\alpha = \left(\mathcal{D}^k \cdot \mathcal{D}^{k-1}\cdots\mathcal{D}^1\right)_{\alpha\beta}\Phi^0_\beta \tag{2.22}$$

obey the algebra where k constraints have zero Poisson brackets with any one. From the algebraic standpoint proposed the abelization method is nothing but an iterative procedure of constructing "equivalent" algebras $\mathcal{A}^{a_i}$ of constraints $\Phi^{a_i}_\alpha$. In a_m steps the m - dimensional non - Abelian algebra is converted into an equivalent Abelian one in such a manner that at the a_k - th step the obtained algebra $\mathcal{A}^{a_k}$ possesses a center with k elements $\mathcal{Z}_k[A] = (\Phi^{a_k}_1, \Phi^{a_k}_2, \ldots, \Phi^{a_k}_k)$.

$$\boxed{\mathcal{A}^0 \underbrace{\xrightarrow{\mathcal{S}^1} \mathcal{A}^1 \xrightarrow{\mathcal{R}^2}}_{\mathcal{D}^1} \mathcal{A}^2 \underbrace{\xrightarrow{\mathcal{S}^3} \mathcal{A}^3 \xrightarrow{\mathcal{S}^4} \mathcal{A}^4 \xrightarrow{\mathcal{R}^5}}_{\mathcal{D}^2} \mathcal{A}^5 \ldots \underbrace{\xrightarrow{\mathcal{S}^{a_k}} \mathcal{A}^{a_k} \ldots \xrightarrow{\mathcal{R}^{a_k+k}}}_{\mathcal{D}^k} \mathcal{A}^{a_k+k} \ldots} \tag{2.23}$$

The matrix $\mathcal{D}^k$ converts the algebra $\mathcal{A}^k$ into the algebra $\mathcal{A}^{k+1}$ in which the center contains one element more than the previous one.

The proof of the validity of the representation (2.19) and the equations for the matrices $\mathcal{S}$ and $\mathcal{R}$ are obtained by induction. Suppose that $\Phi^{a_k}_\alpha$ - are a set of constraints (obtained as a result of the action of $k-1$ matrices $\mathcal{D}^i$) with algebra having the center $\mathcal{Z}_k[A] = (\Phi_1^{a_k}, \Phi_2^{a_k}, \ldots \Phi_k^{a_k})$ than a matrix $\mathcal{D}^k$ from (2.18) perform the transformation to the new constraints

$$\Phi_\alpha^{a_{k+1}-1} = \mathcal{D}^k{}_{\alpha\beta}\Phi_\beta^{a_k+1} \tag{2.24}$$

which form the algebra with the center $\mathcal{Z}_{k+1}[A] = (\Phi_1^{a_{k+1}}, \Phi_2^{a_{k+1}}, \ldots, \Phi_k^{a_{k+1}}, \Phi_{k+1}^{a_{k+1}})$ if the matrices $\mathcal{S}, \mathcal{R}$ are the solutions to the following set of linear differential equations:

$$\left.\begin{array}{rcl} \{\Phi_1^{a_k+i-1}, S_{\alpha_k}^{a_k+i}\} & = & 0 \\ \vdots & & \vdots \\ \{\Phi_{k-1}^{a_k+i-1}, S_{\alpha_k}^{a_k+i}\} & = & 0 \end{array}\right\} \Longrightarrow \{\Phi_{\overline{\alpha}_k}^{a_k+i-1}, S_{\alpha_k}^{a_k+i}\} = 0 \tag{2.25}$$

$$\{\Phi_k^{a_k+i-1}, S_{\alpha_k}^{a_k+i}\} = f_{k\alpha_k\gamma_k}^{a_k+i-1} S_{\gamma_k}^{a_k+i} - f_{k\alpha_k i+1}^{a_k+i-1} \tag{2.26}$$

$$\left.\begin{array}{rcl} \{\Phi_1^{a_k+k-1}, B_{\alpha_k\beta_k}^{a_k+k}\} & = & 0 \\ \vdots & & \vdots \\ \{\Phi_{k-1}^{a_k+k-1}, B_{\alpha_k\beta_k}^{a_k+k}\} & = & 0 \end{array}\right\} \Longrightarrow \{\Phi_{\overline{\alpha}_k}^{a_k+k-1}, B_{\alpha_k\beta_k}^{a_k+k}\} = 0 \tag{2.27}$$

$$\{\Phi_k^{a_k+k-1}, B_{\alpha_k\beta_k}^{a_k+k}\} = -f_{k\gamma_k\beta_k}^{a_k+k-1} B_{\alpha_k\gamma_k}^{a_k+k} \tag{2.28}$$

where $\alpha_k = k+1, \ldots, m$, $\overline{\alpha}_k = 1, 2, \ldots, k-1$ and $f_{\alpha\gamma\beta}^{a_k+i}$ are the structure functions of constraints algebra A^{a_k+i} at the a_k+i -th step.

Let us begin in a consecutive order the construction of algebras with the center containing $1, 2, \ldots, m$ elements. For determination of the new algebra with one central element (let φ_1) one can act in the following way :

a) from the beginning exclude φ_1 from the left hand side of eq. (2.14);

b) then realize abelization with all others

To achieve first, we can perform the transformation with the matrix $\mathcal{S}^1$

$$\Phi_\alpha^1 = \mathcal{S}_{\alpha\beta}^1 \varphi_\beta$$

of type (2.21)

$$\mathcal{S}^1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ C_2 & 1 & 0 & \cdots & 0 \\ C_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_m & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{2.29}$$

In the expanding form it is

$$\begin{array}{rcl} \Phi_1^1 & = & \Phi_1^0 = \varphi_1 \\ \Phi_{\alpha_1}^1 & = & \varphi_{\alpha_1} + C_{\alpha_1}^1 \varphi_1 \end{array} \tag{2.30}$$

The new constraints algebra remains the algebra of first class

$$\begin{array}{rcl} \{\Phi_1^1, \Phi_{\alpha_1}^1\} & = & f_{1\alpha_1 1}^1 \Phi_1^1 + f_{1\alpha_1\gamma_1}^1 \Phi_{\gamma_1}^1 \\ \{\Phi_{\alpha_1}^1, \Phi_{\beta_1}^1\} & = & f_{\alpha_1\beta_1 1}^1 \Phi_1^1 + f_{\alpha_1\beta_1\gamma_1}^1 \Phi_{\gamma_1}^1 \end{array} \tag{2.31}$$

and the new structure functions $f_{\alpha\beta\gamma}^1$ are determined through the old one $f_{\alpha\beta\gamma}$ and the transformation functions $C_{\alpha_1}^1$ as follows

$$f_{1\alpha_1 1}^1 = f_{1\alpha_1 1} + f_{1\alpha_1\gamma_1} C_{\gamma_1}^1 + \{\Phi_1^0, C_{\alpha_1}^1\} \tag{2.32}$$

$$\begin{array}{rcl} f_{\alpha_1\beta_1 1}^1 & = & \frac{1}{2}\left(f_{\alpha_1\beta_1 1} - f_{\alpha_1\beta_1\gamma_1} C_{\gamma_1}^1 + \{C_{\alpha_1}^1, C_{\beta_1}^1\}\Phi_1^0 \right) - \\ & - & f_{1\alpha_1 1}^1 C_{\beta_1}^1 + \{\Phi_{\alpha_1}^0, C_{\beta_1}^1\} - (\alpha_1 \leftrightarrow \beta_1) \end{array} \tag{2.33}$$

$$f_{\alpha_1\beta_1\gamma_1}^1 = f_{\alpha_1\beta_1\gamma_1} + C_{\alpha_1}^1 f_{1\beta_1\gamma_1} - C_{\beta_1}^1 f_{1\alpha_1\gamma_1} \tag{2.34}$$

$$f_{1\alpha_1\gamma_1}^1 = f_{1\alpha_1\gamma_1} \tag{2.35}$$

Let us now choose the transformation functions $C_{\beta_1}^1$ so that the Poisson bracket of first constraints Φ_1^1 with all other modified constraints do not contain it

$$\{\Phi_1^1(p,q), \Phi_{\alpha_1}^1(p,q)\} = \sum_{\gamma \neq 1} f_{1\alpha_1\gamma}^1(p,q)\Phi_\gamma^1(p,q). \tag{2.36}$$

These $m-1$ requirements $f_{1\alpha_1 1}^1 = 0$, according to (2.32), mean that the transformation function $C_{\alpha_1}^1$ must satisfy the following set of linear nonhomogeneous differential equations

$$\{\Phi_1^0, C_{\alpha_1}^1\} = -f_{1\alpha_1 1} + f_{1\alpha_1\gamma_1} C_{\gamma_1}^1 \tag{2.37}$$

Note that the problem of existence of solution to such a set of equations is studied very well (see e.g. [15]) Suppose, we find some particular solution $C^1_{\alpha_1}$ for (2.37), then one can determine all structure functions of the modified algebra according to eq.(2.32).

$$\begin{aligned}
f^1_{1\alpha_1 1} &= 0 && (2.38)\\
f^1_{\alpha_1\beta_1 1} &= f_{\alpha_1\beta_1 1} - f_{\alpha_1\beta_1\gamma_1}C^1_{\gamma_1} + \{C^1_{\alpha_1}, C^1_{\beta_1}\}\Phi^0_1 \\
&+ \{\Phi^0_{\alpha_1}, C^1_{\beta_1}\} + \{\Phi^0_{\beta_1}, C^1_{\alpha_1}\} && (2.39)\\
f^1_{\alpha_1\beta_1\gamma_1} &= f_{\alpha_1\beta_1\gamma_1} + C^1_{\alpha_1} f_{1\beta_1\gamma_1} - C^1_{\beta_1} f_{1\alpha_1\gamma_1} && (2.40)\\
f^1_{1\alpha_1\gamma_1} &= f_{1\alpha_1\gamma_1} && (2.41)
\end{aligned}$$

Now let us again keep first constraint unchanged and perform the Dirac transformation on the remaining part of constraints Φ_{α_1}, $\alpha_1 = 2, 3, ..., m$

$$\begin{aligned}
\Phi^2_1 &= \Phi^1_1 = \Phi^0_1 = \varphi_1 \\
\Phi^2_{\alpha_1} &= B^2_{\alpha_1\beta_1}\Phi^1_{\beta_1} && (2.42)
\end{aligned}$$

with the requirement that the new constraints have zero Poisson brackets with the first one Φ^1_1

$$\{\Phi^2_1, \Phi^2_{\alpha_1}\} = 0. \qquad (2.43)$$

One can verify that this requirement means that the transformation functions $B_{\alpha_1\beta_1}$ are the solution to the equation

$$\{\Phi^1_1, B^2_{\alpha_1\beta_1}\} = -f_{1\gamma_1\beta_1}B^2_{\alpha_1\gamma_1} \qquad (2.44)$$

With the help of a solution of eq. (2.44) the modified algebra has the following structure functions

$$\begin{aligned}
f^2_{1\alpha_1 1} &= 0, && (2.45)\\
f^2_{\alpha_1\beta_1 1} &= B^2_{\alpha_1\delta_1}B^2_{\beta_1\sigma_1}f^1_{\delta_1\sigma_1 1}, && (2.46)\\
f^2_{\alpha_1\beta_1\gamma_1} &= \left[\{B^2_{\alpha_1\delta_1}, B^2_{\beta_1\sigma_1}\}\Phi^1_{\sigma_1} + \{B^2_{\alpha_1\delta_1}, \Phi^1_{\sigma_1}\}B^2_{\beta_1\sigma_1} - \right.\\
&- \left.\{B^2_{\beta_1\delta_1}, \Phi^1_{\sigma_1}\}B^2_{\alpha_1\sigma_1} + B^2_{\alpha_1\kappa_1}B^2_{\beta_1\sigma_1}f^1_{\kappa_1\sigma_1\delta_1}\right](B^2)^{-1}_{\delta_1\rho_1} && (2.47)
\end{aligned}$$

Thus, as a result of two transformations $\mathcal{D}^1 = \mathcal{S}^1\mathcal{R}^2$ we obtain the modified algebra $\mathcal{A}^2$ of constraints Φ^2_α with the central element Φ^2_1

$$\begin{aligned}
\{\Phi^2_1, \Phi^2_{\alpha_1}\} &= 0 && (2.48)\\
\{\Phi^2_{\alpha_1}, \Phi^2_{\beta_1}\} &= f^2_{\alpha_1\beta_1 1}\Phi^2_1 + f^2_{\alpha_1\beta_1\gamma_1}\Phi^2_{\gamma_1} && (2.49)
\end{aligned}$$

The structure functions $f^2_{\alpha_1\beta_1\gamma}$ of algebra (2.48),(2.49) possess the significant property : due to the fact that Φ^2_1 is the central element the structure functions obey the following property:

$$\{\Phi^2_1, f^2_{\alpha_1\beta_1\gamma}\} = 0 \qquad (2.50)$$

To verify this, it is enough to calculate the Poisson bracket of Φ^2_1 with (2.49) and use the Jacobi identity .

To extend the center of the new algebra $\mathcal{A}^2$ by Φ^2_2, we will act by analogy with the previous case

a) exclude Φ^2_1 and Φ^2_2 from the left hand side of eq. (2.49)

b) then achieve the abelization with all others

To carry out the first point of this program we will deal with two consecutive transformations $\mathcal{S}^3$ and $\mathcal{S}^4$. Let us require that the first transformation $\mathcal{S}^3$ of type (2.21)

$$\begin{aligned}
\Phi^3_1 &= \Phi^2_1 \\
\Phi^3_2 &= \Phi^2_2 \\
\Phi^2_{\alpha_2} &= \Phi^2_{\alpha_2} + C^3_{\alpha_2}\Phi^2_1, \quad \alpha_2 = 3, \ldots, m && (2.51)
\end{aligned}$$

lead to the new algebra of constraints so that Φ^3_1 is again the central element

$$\{\Phi^3_1, \Phi^2_{\alpha_1}\} = 0 \qquad (2.52)$$

and the Poisson brackets of the second constraint Φ^3_2 with all other modified constraints does not contain Φ^3_1

$$\{\Phi^3_2, \Phi^2_{\alpha_2}\} = \sum_{\gamma\neq 1} f^3_{2\alpha_2\gamma}\Phi^3_\gamma. \qquad (2.53)$$

This requirement leads to the following equations

$$\begin{aligned}
\{\Phi^3_1, C^3_{\alpha_2}\} &= 0 \\
\{\Phi^2_2, C^3_{\alpha_2}\} &= f^2_{2\alpha_2\gamma_2}C^3_{\gamma_2} - f^2_{2\alpha_2 1} && (2.54)
\end{aligned}$$

What about the existence of the solutions to these equations. One can verify that the integrability condition for the system of differential equations (2.54) is nothing else but (2.50). In full analogy with the previous case one can express the new structure functions $f^3_{\alpha,\beta\gamma}$ through $f^2_{\alpha,\beta\gamma}$ and verify that they obey the following property

$$\{\Phi^3_1, f^3_{2\alpha_2\gamma}\} = 0 \qquad (2.55)$$

Now one can realize the transformation $\mathcal{S}^4$ of type (2.21)

$$\begin{aligned} \Phi^4_1 &= \Phi^3_1 \\ \Phi^4_2 &= \Phi^3_2 \\ \Phi^4_{\alpha_2} &= \Phi^3_{\alpha_2} + C^4_{\alpha_2}\Phi^2_2 \end{aligned} \qquad (2.56)$$

so that Φ^4_1 is again central element

$$\{\Phi^4_1, \Phi^2_{\alpha_1}\} = 0 \qquad (2.57)$$

and the Poisson brackets of the second constraint Φ^2_4 with all other modified constraints do not contain Φ^4_1 and Φ^4_2

$$\{\Phi^4_2, \Phi^4_{\alpha_2}\} = \sum_{\gamma\neq 1,2} f_{2\alpha_2\gamma}\Phi^3_\gamma. \qquad (2.58)$$

This requirement leads to the following equations :

$$\begin{aligned} \{\Phi^4_1, C^4_{\alpha_2}\} &= 0 \\ \{\Phi^3_2, C^4_{\alpha_2}\} &= f^3_{2\alpha_2\gamma_2}C^4_{\gamma_2} - f^3_{2\alpha_2 2} \end{aligned} \qquad (2.59)$$

This system is consistent in virtue of equations (2.55) . For the new structure function one can again to verify that

$$\{\Phi^4_1, f^4_{2\alpha_2\gamma}\} = 0 \qquad (2.60)$$

as for the previous step (see eq.(2.55). Now for abelization of two constraints Φ^4_1, Φ^5_2 it is enough to perform last transformation with matrix $\mathcal{R}$ of type (2.20) if its elements are the solution to the equations

$$\begin{aligned} \{\Phi^4_1, B^5_{\alpha_2\beta_2}\} &= 0 \\ \{\Phi^4_2, B^5_{\alpha_2\beta_2}\} &= -f^4_{2\gamma_2\beta_2}B^5_{\alpha_2\gamma_2} \end{aligned} \qquad (2.61)$$

As a result

$$\{\Phi^5_1, \Phi^5_\alpha\} = \{\Phi^5_2, \Phi^5_\alpha\} = 0$$

Note that (2.60) provides the existence of the solution to eq.(2.61). As a result, one can easy verify that the new structure functions possess the property

$$\{\Phi^5_{\overline{\alpha}_2}, f^5_{\alpha_2\beta_2\gamma}\} = 0, \quad \overline{\alpha}_2 = 1,2. \qquad (2.62)$$

Thus, in five steps (for summary, see Table 1.). we obtain an equivalent to initial algebra $\mathcal{A}^5$ with two central elements $\Phi^5_{\overline{\alpha}_2}$ Now let us suppose that by acting in such a way we get the algebra $\mathcal{A}^{k-1}$

$$\{\Phi^{a_k-1}_\alpha, \Phi^{a_k-1}_\beta\} = f^{a_k-1}_{\alpha\beta\gamma}\Phi^{a_k-1}_\gamma \qquad (2.63)$$

with the center composed by $k-1$ elements

$$\begin{aligned} \mathcal{Z}_{k-1} &= (\Phi^{a_k-1}_1, \Phi^{a_k-1}_2, \ldots \Phi^{a_k-1}_{k-1}) \\ \{\mathcal{Z}_{k-1}, \Phi^{a_k-1}_\alpha\} &= 0 \end{aligned}$$

and the structure functions of this algebra have the property

$$\{\Phi^{a_k-1}_{\overline{\alpha}_k}, f^{a_k-1}_{\alpha_k\beta_k\gamma}\} = 0, \quad \overline{\alpha}_k = k-1,\ldots,m \qquad (2.64)$$

Now by direct calculations it is easy to verify that via the action of transformation of the matrix $\mathcal{D}^k$ with elements which are the solutions to eqs. (2.25) and (2.28), we obtain the algebra $\mathcal{A}^k$ with the center composed by k elements

$$\begin{aligned} \mathcal{Z}_{k-1} &= (\Phi^{a_k-1}_1, \Phi^{a_k-1}_2, \ldots, \Phi^{a_k-1}_{k-1}) \\ \{\mathcal{Z}_{k-1}, \Phi^{a_k-1}_\alpha\} &= 0. \end{aligned} \qquad (2.65)$$

The conditions (2.64) play the role of the integrability conditions for the system of eqs.(2.25), (2.28).

For completion we need only to in prove the following property of structure functions

$$\{\Phi^{a_k+k}_{\overline{\alpha}_{k+1}}, f^{a_k+k}_{\alpha_{k+1}\beta_{k+1}\gamma}\} = 0 \qquad (2.66)$$

To verify this one can consider the algebra of constraints

$$\{\Phi^{a_k+k}_{\alpha_{k+1}}, \Phi^{a_k+k}_{\beta_{k+1}}\} = f^{a_k+k}_{\alpha_{k+1}\beta_{k+1}\overline{\gamma}_{k+1}}\Phi^{a_k+k}_{\overline{\gamma}_{k+1}} + f^{a_k+k}_{\alpha_{k+1}\beta_{k+1}\gamma_{k+1}}\Phi^{a_k+k}_{\gamma_{k+1}} \qquad (2.67)$$

and compose the Poisson bracket of (2.67) with $\Phi^{a_k+k}_{\overline{\alpha}_{k+1}}$. Taking into account that $\Phi^{a_k+k}_{\overline{\alpha}_{k+1}}$ are the central elements of the algebra $\mathcal{A}^k$, we immediately get the desired result (2.66) with the help of Jacobi identity.

III Christ and Lee model

In this section we will apply the above described procedure to the well known example – nonabelian Christ and Lee model described by the Lagrangian

$$\mathcal{L}(\mathbf{x},\dot{\mathbf{x}},\mathbf{y}) = \frac{1}{2}(\dot{\mathbf{x}} - [\mathbf{y},\mathbf{x}])^2 - V(\mathbf{x}^2)$$

where $\mathbf{x}$ and $\mathbf{y}$ - are the three- dimensional vectors, $(x_1, x_2, x_3), (y_1, y_2, y_3)$.

It is easy to verify that except for three primary constraints

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{y}}} = 0,$$

there are two independent constraints

$$\begin{aligned} \Phi_1^0 &= x_2p_3 - x_3p_2 \\ \Phi_2^0 &= x_3p_1 - x_1p_3 \end{aligned} \tag{3.1}$$

with the algebra

$$\{\Phi_1^0, \Phi_2^0\} = -\frac{x_1}{x_3}\Phi_1^0 - \frac{x_2}{x_3}\Phi_2^0 \tag{3.2}$$

The abelization procedure for this simple case consists in two stages At first step, the transformation $\mathcal{S}^1$ reduces to

$$\begin{aligned} \Phi_1^1 &= \Phi_1^0 \\ \Phi_2^1 &= \Phi_2^0 + C\Phi_1^0 \end{aligned} \tag{3.3}$$

and equation (2.25) looks like

$$\{\Phi_1^0, C\} = \frac{x_2}{x_3}C + \frac{x_1}{x_3} \tag{3.4}$$

One can write down a particular solution to this equation

$$C(x) = \frac{x_1}{x_3}\arctan\left(\frac{x_2}{x_3}\right) \tag{3.5}$$

So, as a result of first step we get a new algebra

$$\{\Phi_1^1, \Phi_2^1\} = -\frac{x_2}{x_3}\Phi_2^1 \tag{3.6}$$

Now let us perform the second transformation $\mathcal{R}^2$

$$\begin{aligned} \Phi_1^2 &= \Phi_1^1 \\ \Phi_2^1 &= B\Phi_2^1 \end{aligned} \tag{3.7}$$

with the function B satisfing the equation of type (2.28)

$$\{\Phi_1^1, B\} = \frac{x_2}{x_3} \tag{3.8}$$

A particular solution to this equation reads

$$B(x) = \ln\left(\frac{\sqrt{x_2^2 + x_3^2}}{x_3}\right) \tag{3.9}$$

Thus, the equivalent to the initial abelian constraints have the form

$$\begin{aligned} \Phi_1^2 &= x_2p_3 - x_3p_2 \qquad (3.10) \\ \Phi_2^2 &= \ln\left(\frac{\sqrt{x_2^2 + x_3^2}}{x_3}\right)\left[(x_2p_3 - x_3p_2) + \frac{x_1}{x_3}\arctan\left(\frac{x_2}{x_3}\right)(x_3p_1 - x_1p_3)\right] \end{aligned}$$

IV Concluding remarks

We have discussed the iterative procedure of converting first class constraints in an arbitrary singular theory to the Abelian form of constraints. The final goal of dealing with a valuable form of abelization is the construction of the reduced phase space in the complicated non - Abelian theory . The application of our procedure to the SU(2) Yang - Mills will be done in separate forthcoming publication.

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Table 1. Abelization stages for the algebra with two central elements

	CONSTRAINTS	ALGEBRA	CONDITIONS
Φ^0_α	$\Phi^0_\alpha = \varphi_\alpha$	$\{\Phi^0_\alpha, \Phi^0_\beta\} = f^0_{\alpha\beta\gamma}\Phi^0_\gamma$	
Φ^1_α	$\Phi^1_1 = \Phi^0_1$ $\Phi^1_{\alpha_1} = \Phi^0_{\alpha_1} + C^1_{\alpha_1}\Phi^0_1$	$\{\Phi^1_1, \Phi^1_\alpha\} = f^1_{1\alpha\gamma_1}\Phi^1_{\gamma_1}$	$\{\Phi^0_1, C^1_{\alpha_1}\} = f_{1\alpha_1\gamma_1}C^1_{\gamma_1} - f_{1\alpha_1 1}$
Φ^2_α	$\Phi^2_1 = \Phi^1_1 = \Phi^0_1$ $\Phi^2_{\alpha_1} = B^2_{\alpha_1\beta_1}\Phi^1_{\beta_1}$	$\{\Phi^2_1, \Phi^1_\alpha\} = 0$	$\{\Phi^1_1, B^2_{\alpha_1\beta_1}\} = -f^1_{1\gamma_1\beta}B^2_{\alpha_1\gamma_1}$
1	$\{\Phi^2_1, \Phi^2_\alpha\} = 0$		
Φ^3_α	$\Phi^3_1 = \Phi^2_1 = \Phi^1_1$ $\Phi^3_2 = \Phi^2_2$ $\Phi^3_{\alpha_2} = \Phi^2_{\alpha_2} + C^3_{\alpha_2}\Phi^1_1$	$\{\Phi^3_1, \Phi^3_\alpha\} = 0$ $\{\Phi^3_2, \Phi^3_\alpha\} = f^3_{1\alpha\gamma_1}\Phi^2_{\gamma_1}$	$\{\Phi^3_1, C^3_{\alpha_2}\} = 0$ $\{\Phi^2_2, C^3_{\alpha_2}\} = f^2_{2\alpha_2\gamma_2}C^2_{\gamma_2} - f^2_{2\alpha_2 1}$
Φ^4_α	$\Phi^4_1 = \Phi^3_1 \ldots = \Phi^1_1$ $\Phi^4_2 = \Phi^3_2 = \Phi^2_2$ $\Phi^4_{\alpha_2} = \Phi^3_{\alpha_2} + C^4_{\alpha_2}\Phi^3_2$	$\{\Phi^4_1, \Phi^4_\alpha\} = 0$ $\{\Phi^4_2, \Phi^4_\alpha\} = f^4_{2\alpha\gamma_2}\Phi^4_{\gamma_2}$	$\{\Phi^4_1, C^4_{\alpha_2}\} = 0$ $\{\Phi^3_2, C^4_{\alpha_2}\} = f^3_{2\alpha_2\gamma_2}C^4_{\gamma_2} - f^3_{2\alpha_2 2}$
Φ^5_α	$\Phi^5_1 = \Phi^4_1 \cdots \Phi^1_1$ $\Phi^5_2 = \Phi^4_2 \cdots \Phi^2_2$ $\Phi^5_{\alpha_2} = B^5_{\alpha_2\beta_2}\Phi^4_{\beta_2}$	$\{\Phi^5_1, \Phi^5_\alpha\} = 0$ $\{\Phi^5_2, \Phi^5_\alpha\} = 0$	$\{\Phi^4_1, B^5_{\alpha_2\beta_2}\} = 0$ $\{\Phi^4_2, B^5_{\alpha_2\beta_2}\} = -f^4_{2\gamma_2\beta_2}B^5_{\alpha_2\gamma_2}$
2	$\{\Phi^5_1, \Phi^5_\alpha\} = \{\Phi^5_2, \Phi^5_\alpha\} = 0$		

Table 2. Abelization stages for the algebra with k central elements

	CONSTRAINTS	ALGEBRA	CONDITIONS
k-1	$\{\Phi^{a_k-1}_1, \Phi^{a_k-1}_\alpha\} = \{\Phi^{a_k-1}_2, \Phi^{a_k-1}_\alpha\} = \ldots = \{\Phi^{a_k-1}_{k-1}, \Phi^{a_k-1}_\alpha\} = 0$		
	$\Phi^{a_k}_1 = \Phi^{a_k-1}_1 \cdots = \Phi^1_1$ $\Phi^{a_k}_2 = \Phi^{a_k-1}_2 \ldots = \Phi^2_2$ $\vdots$ $\Phi^{a_k}_k = \Phi^{a_k-1}_k$ $\Phi^{a_k}_{\alpha_k} = \Phi^{a_k-1}_\alpha + C^{a_k}_{\alpha_k}\Phi^1_1$	$\{\Phi^{a_k}_1, \Phi^{a_k}_\alpha\} = 0$ $\{\Phi^{a_k}_2, \Phi^{a_k}_\alpha\} = 0$ $\{\Phi^{a_k}_k, \Phi^{a_k}_\alpha\} = f^{a_k}_{k\alpha\gamma_1}\Phi^{a_k}_{\gamma_1}$	$\{\Phi^{a_k}_1, C^{a_k}_{\alpha_k}\} = 0$ $\{\Phi^{a_k}_2, C^{a_k}_{\alpha_k}\} = 0$ $\{\Phi^{a_k-1}_k, C^{a_k}_{\alpha_k}\} = f^{a_k-1}_{k\alpha_k\gamma_k}C^{a_k}_{\gamma_k} - f^{a_k-1}_{k\alpha_k 1}$
	$\vdots$ $\vdots$	$\vdots$ $\vdots$	$\vdots$ $\vdots$
	$\Phi^{a_k+k}_1 = \cdots = \Phi^1_1$ $\Phi^{a_k+k}_2 = \cdots = \Phi^2_2$ $\vdots$ $\Phi^{a_k+k}_k = \Phi^{a_k-1}_k$ $\Phi^{a_k}_{\alpha_k} = B^{a_k+k}_{\alpha_k\beta_k}\Phi^{a_k+k-1}_{\beta_k}$	$\{\Phi^{a_k+k}_1, \Phi^{a_k+k}_\alpha\} = 0$ $\{\Phi^{a_k+k}_2, \Phi^{a_k+k}_\alpha\} = 0$ $\vdots$ $\{\Phi^{a_k+k}_k, \Phi^{a_k+k}_\alpha\} = 0$	$\{\Phi^{a_k+k}_1, B^{a_k+k}_{\alpha_k\beta_k}\} = 0$ $\vdots$ $\{B^{a_k+k}_{\alpha_k\beta_k}, \Phi^{a_k+k}_k\} = f^{a_k+k-1}_{k\gamma_k\beta_k}B^{a_k+k}_{\alpha_k\gamma_k}$
k	$\{\Phi^{a_{k+1}-1}_1, \Phi^{a_{k+1}-1}_\alpha\} = \{\Phi^{a_{k+1}-1}_2, \Phi^{a_{k+1}-1}_\alpha\} = \ldots = \{\Phi^{a_{k+1}-1}_k, \Phi^{a_{k+1}-1}_\alpha\} = 0$		

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Об абелизации связей первого рода

Развит систематический метод перехода от связей первого рода к эквивалентному набору абелевых связей с помощью преобразования эквивалентности Дирака. Предложено представление для матрицы преобразования связей, позволяющее свести проблему абелизации к задаче нахождения частного решения системы линейных дифференциальных уравнений в частных производных первого порядка.

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On Abelization of First Class Constraints

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