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WEYL CONNECTION AND NON-ABELIAN GAUGE FIELD*

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Барбашов Б.М., Пестов А.Б. E2-94-90
Связность Вейля и неабелево калибровочное поле

Показано, что конгруэнтный перенос, введенный Вейлем в 1921 году, определяет неабелево калибровочное поле. Для этого поля предложены простейшие калибровочно-инвариантные уравнения. Обсуждается связь с геометрией Римана — Картана.

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Barbashov B.M., Pestov A.B. E2-94-90
Weyl Connection and Non-Abelian Gauge Field

It is shown that the congruent transference introduced by Weyl in 1921 defines a non-Abelian gauge field. The simplest gauge-invariant equations are proposed for this field. Connection with the Riemann — Gartan geometry is discussed.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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1 Introduction

As soon as the general relativity was developed, Hermann Weyl undertook an attempt at unifying the gravity and electromagnetism [1]. Inasmuch as the invariance under the group of general transformations of coordinates (the group of diffeomorphisms [2]) determines the laws of gravitational interactions, Weyl suggested that the scale invariance he introduced should correspond to electromagnetism. The Weyl theory was not further developed because upon the construction of quantum mechanics it became conventional that just local phase invariance, rather than the Weyl geometrical invariance, is related to the electromagnetic field. However, the Weyl's idea has led to that what is at present called "the gauge theory". Besides, in 1921, Weyl in his book [3] introduced the law of parallel transport, he called the congruent transference (kongruente Verpflanzung)and, as will be shown below, he thus introduced a non–Abelian gauge field for the first time. In this paper, we will try to show that this important result by Weyl is of undoubted interest both for the nontrivial unification of space–time and gauge symmetry and for numerous attempts of giving a physical interpretation for the Riemann–Cartan geometry.

2 Weyl Connection

The Weyl connection giving the congruent transference is of the form [3],[4]

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} - F_{jkl}\, g^{li}, \qquad (1)$$

where $\{^i_{jk}\}$ are Christoffel symbols of the Riemann connection of the metric g_{ij}:

$$\{^i_{jk}\} = \frac{1}{2} g^{il} (\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}) \qquad (2)$$

and F_{jkl} are components of the third rank tensor that is skew–symmetric in the last two indices

$$F_{jkl} + F_{jlk} = 0. \qquad (3)$$

From (1) it follows that the vector components under the congruent transference change according to the law

$$dv^i = -\{^i_{jk}\}\, dx^j\, v^k + F_{jkl} g^{li}\, dx^j\, v^k, \qquad (4)$$

that consists of the displacement belonging to the Riemann geometry (the first term in(4)) and rotation defined by the metric g^{ij} and bivector of rotation $F_{jkl}\, dx^j$.

Congruent transference does not vary the vector length because $d(g_{ij} v^i v^j) = 0$ in accordance with the law (4). We denote the covariant derivative with respect to the connection Γ^i_{jk} by ∇_i. Since

$$\nabla_i g_{jk} = g_{jk;i} + F_{ijl} g^{lm} g_{mk} + F_{ikl} g^{lm} g_{mj},$$

where the semicolon means the covariant derivative with respect to (2), using (3) we obtain

$$\nabla_i g_{jk} = 0. \qquad (5)$$

This is just the general characteristics of the Weyl connection. Now we will analyse the group–theoretical meaning of that connection.

3 Gauge Group

Let S^i_j be components of the tensor field S of type (1,1) obeying the condition $\det(S^i_j) \neq 0$. Then, there exists the tensor field S^{-1} with components T^i_j satisfying the condition

$$S^i_k T^k_j = \delta^i_j.$$

The tensor field S can be considered as a linear transformation

$$\bar{v}^i = S^i_j\, v^j \qquad (6)$$

in the space of vector fields; whereas S^{-1}, as an inverse transformation. As the length of a vector under the congruent transference remains constant, of the transformations (6) we pick out those leaving the vector length constant. From the condition $g_{ij} \bar{v}^i \bar{v}^j = g_{ij} v^i v^j$ it follows that

$$g_{kl} S^k_i S^l_j = g_{ij}. \qquad (7)$$

Transformations of the form (6) and (7) form a group that is defined by the quadratic form $\varphi = g_{ij} v^i v^j$. It is natural to denote this group by $O_g(r, s)$, where r and s are indices of the inertia of the quadratic form φ, and $r + s = n$.

We will demonstrate that if the vector v^i undergoes the congruent transference (4), the transformed vector $\bar{v}^i$ will also undergo the congruent transference. From (4) and (6) we obtain

$$d\bar{v}^i = (dS^i_j)\, v^j + S^i_j\, dv^j =$$

$$= (\partial_l S^i_j T^j_k - S^i_j \{^j_{lm}\} T^m_k)\, dx^l\, \bar{v}^k + S^i_j (F_{lpm} g^{jm}) T^p_k\, dx^l\, \bar{v}^k.$$

Since

$$S^i_{j;l} T^j_k = \partial_l S^i_j T^j_k + \{^i_{lk}\} - S^i_j \{^j_{lm}\} T^m_k,$$

then

$$d\bar{v}^i = -\{^i_{lk}\}\, dx^l\, \bar{v}^k + (S^i_{j;l} T^j_k + S^i_j F_{lpm} g^{mj} T^p_k)\, dx^l\, \bar{v}^k.$$

Therefore,

$$d\bar{v}^i = -\{^i_{lk}\}\, dx^l\, \bar{v}^k + \bar{F}_{lkm} g^{mi}\, dx^l\, \bar{v}^k,$$

where

$$F_{lkm} = F_{lij} T^i_k T^j_m + g_{ij} T^i_{m;l} T^j_k. \qquad (8)$$

From (6) it follows that the tensor F_{lkm} obeys equation (3), and, consequently, the Weyl connection

$$\Gamma^i_{jk} = \{^i_{jk}\} - F_{jkl} g^{li}$$

defines the congruent transference, as well.

The transformation (8) can be written as a relation between the connections Γ^i_{jk} and Γ^i_{jk} in the following form

$$\Gamma^i_{jk} = S^i_l \Gamma^l_{jm} T^m_k + S^i_l \partial_j T^l_k = \Gamma^i_{jk} + S^i_l \nabla_j T^l_k. \qquad (9)$$

Indeed,

$$S^i_l \nabla_j T^l_k = S^i_l T^l_{k;j} - S^i_l (F_{jmp} g^{pl}) T^m_k + F_{jkm} g^{mi}.$$

Introducing the matrix notation $\Gamma_j = (\Gamma^i_{jk})$, $S = (S^i_j)$, $S^{-1} = (T^i_j)$, we can write the relation (9) in the conventional form

$$\bar{\Gamma}_j = S\,\Gamma_j\,S^{-1} + S\,\partial_j S^{-1}. \tag{10}$$

Let $B_{ij} = (B_{ijl}{}^k)$ be the Riemann tensor of the connection (1)

$$B_{ij} = \partial_i\Gamma_j - \partial_j\Gamma_i + [\Gamma_i, \Gamma_j] \tag{11}$$

and $\bar{B}_{ij} = (\bar{B}_{ijl}{}^k)$ be the Riemann tensor of the connection $\bar{\Gamma}_j$. Then from (10) and (11) it follows that the tensors B_{ij} and $\bar{B}_{ij}$ are connected by a homogeneous transformation

$$\bar{B}_{ij} = S B_{ij} S^{-1}. \tag{12}$$

From (1) and (11) it follows that the Riemann tensor can be written in the form

$$B_{ijkl} = R_{ijkl} + H_{ijkl}, \tag{13}$$

where R_{ijkl} is the tensor of Riemann curvature of the metric g_{ij}, and

$$H_{ijkl} = F_{jkl;i} - F_{ikl;j} + F_{imk}\,F_{jlp}\,g^{mp} - F_{jmk}\,F_{ilp}\,g^{mp}. \tag{14}$$

Note that in a flat space–time the tensor of Riemann curvature equals zero, and in this case in Cartesian coordinates we have

$$B_{ij} = H_{ij} = \partial_i F_j - \partial_j F_i + [F_i, F_j],$$

where $F_i = (F_{ijk} g^{lk})$.

Thus, the tensor field F_{ijk} entering into the Weyl connection is a gauge field, whereas the tensor H_{ijkl} is the tensor of strength of that field. It is to be stressed that the gauge group in the case under consideration is defined by the metric, whereas the gauge field has a geometrical meaning (the Weyl congruent transference) and no extra internal or isotopic space is to be introduced.

4 Field Equations

The field F_{ijk} can be described by gauge–invariant equations derived by the variational method from the conventional Lagrangian

$$L = -\frac{1}{4} Tr(B_{ij} B^{ij}). \tag{15}$$

By variation we obtain the following equations

$$D_i(\sqrt{-g}B^{ij}) = 0, \tag{16}$$

where g is the determinant of the metric tensor and

$$\frac{1}{\sqrt{-g}} D_i(\sqrt{-g}B^{ij}) = \frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}B^{ij}) + [\Gamma_i, B^{ij}].$$

The gauge–invariant tensor of the energy–momentum corresponding to the Lagrangian (15) can easy be obtained by the method proposed in ref. [5]. So, we have

$$T_{ij} = Tr(B_{ik} B_{jl}\, g^{kl}) + g_{ij}\, L. \tag{17}$$

If equations (16) hold valid, T_{ij} obeys the equation$T^{ij}{}_{;i} = 0$. Varying the action $A = \int \sqrt{-g} L d^4x$ with respect to the metric g^{ij} we obtain the tensor $\Theta_{ij} = \delta A/\delta g^{ij}$ that is not gauge–invariant. Therefore, the equations $\Theta_{ij} = 0$ can be considered as equations fixing the gauge. The reason for Θ_{ij} being not gauge invariant is as follows: When varying in g^{ij} we change the metric, and because the groups $O_g(r,s)$. and $O_{g+\delta g}(r,s)$ do not coincide, the variation in metric is not a gauge–invariant operation.

5 Conclusion

The interpretation given here for the Weyl congruent transference actually leads to the Lagrangian (15) quadratic in the strength tensor and, consequently, to equations (16) for the non–Abelian gauge field determining this transference. It is to be stressed that in the case under consideration, no abstract gauge space is to be introduced.

As it is known, the attempts of physical interpretation of the Riemann– Cartan geometry run into problems [6] that can, probably, be overcome in the Weyl approach that opens new possibilities for studies because the geometry and gauge principle are there related in a natural way. And finally we note that the connection can be established between the Weyl non–Abelian gauge field and torsion that is the central object of study in the Riemann-Cartan geometry, however, this connection is not gauge–invariant.

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