объединенный
институт
ядерных
исследований
дубна

E2-94-530

A.L.Koshkarov[1]

ON SEARCH FOR NON-SELF-DUAL SOLUTIONS OF YANG — MILLS EQUATIONS BY MEANS OF GENERALIZATION OF DUALITY EQUATIONS

Submitted to «Теоретическая и математическая физика»

[1]University of Petrozavodsk,
e-mail address: (koshkar@mainpgu.karelia.su)

1994

1 Introduction

Well known solutions of Yang-Mills (YM) equations such as instantons and monopoles are obtained in solving of the famous duality equations [1]. One can attempt to generalize the duality equations and maybe to find new solutions.

To motivate suggested approach let us consider a known "toy" model: one-dimensional particle with mass m is in a two-peak potential

$$V(x) = \lambda(x^2 - \eta^2)^2.$$

In this model the nontrivial classical solution appears in going to imaginary time. It is the instanton [2]. BPST-instanton [1] is analogue of this nonrelativistic instanton. In real time there exist, in certain sense, similar solutions. Let particle be at the point $x = 0$ in the state of unstable balance (by the way it is the simplest nonrelativistic sphaleron!). It can remove down from mountain and come back, that is described by solution

$$x(t) = \pm\frac{\eta\sqrt{2}}{\cosh(\omega t/\sqrt{2})}, \quad \omega^2 = 8\lambda\eta^2. \tag{1}$$

One gets from energy integral

$$\frac{1}{2}\dot{x}^2(t) + V(x) = E \tag{2}$$

if $E = V_{x=0}$ and gives a finite action, if potential is evaluated from the level $V_0 = \lambda\eta^4$.

Is it possible to find similar solutions in the YM-theory?

The basic idea is to generalize duality equations inspecting those as analogue of the first integral (2) in the pseudoeuclidean Minkovsky space and using static unstable solutions (sphalerons) [3]. Then we have hope to find the analogue of solution (1).

2 Generalizing of duality equations

How can we generalize the duality equations? The simplest one could look as (indices are missed for simplicity)

$$*F \pm iF = *\mathcal{F} \pm i\mathcal{F}. \qquad (3)$$

Here $\mathcal{F}$ - is any solution of YM-equations, F - looked for field. Gauge group will not be concrete until later. One can consider the equation (3) as the analogue of the first integral of the "toy" model, and as well as generalization of duality equations. This equation has an important property of invariance under dual transformations.

However, will the solutions of equation (3) be as well ones of YM-equations? At first glance they will. Let us act to both rhs. and lhs. of (3) by covariant derivative operator D. Taking into account Bianchi's identity and the fact that $\mathcal{F}$ is solutions it is easy to get

$$DF = 0,$$

i. e. field F is a solution of YM-equation. In reality this treatment is mistaken. Error is that the operator D contains, except derivative one, also field potentials. And therefore the covariant derivatives are different for fields F and $\mathcal{F}$, and previous treatment does not pass through.

What can we do? Let us a little modify equations (3)

$$*F + iF = (*\mathcal{F} + i\mathcal{F})\phi, \qquad (4)$$

where ϕ - unknown scalar function of x^2. Further we are acting to both sides of (4) by operator D, associated with field F. Then

$$iDF = D[(*\mathcal{F} + i\mathcal{F})\phi].$$

Require to satisfy condition

$$D[(*\mathcal{F} + i\mathcal{F})\phi] = 0. \qquad (5)$$

Then the field F will obey the YM equations. It will be shown below equation (5) will reduce to scalar that, as well as equation (4).

Here are a few notes about equations (4,5).

1. Equations are invariant under the duality transformation. Technically the dual-invariant equations are easy reduced to scalar ones since both sides of such eqs. are proportional Hooft's tensors $\eta_{\mu\nu}$ performing as basis.

2. It would be better to find a real, nonsingular, finite action solution of eqs. (4,5). However, there are no such solutions in the pure YM-theory [4] and therefore probably one must consider something as Higgs' model in this approach.

3. Formally one can substitute in (4) as $\mathcal{F}$ any known solution of YM equations, e.g.: instanton, Wu-Yang solution, etc. But consideration of the toy model prompts that one must take as $\mathcal{F}$ unstable static solution, i.e. sphaleron [3].

Generalization of duality equation (in the euclidean space) was considered by Yatsun [5]. Within the framework model with scalar fields he received the dual-invariant equation of kind

$$*F + F = f(\phi)\eta.$$

where η-Hooft's tensor, ϕ-scalar field. As distinct from that our equation involves known solution $\mathcal{F}$, that is one of the essential points in our approach.

3 Choice of gauge group

We would not like to use gauge group $SU(2)$ as a simplest case although it is usually accepted because we are working in the Minkovsky space. It is convenient to take that connected with motion space group, namely $SO(1,3)$ group. Below there we'll explain motives of this choice. And now it is convenient to begin more systematic research of eqs. (4,5) from the $SO(4)$-gauge solution in the euclidean space (e.g., look at [6]):

$$A^{\alpha\beta}_{\mu}(x) = 2\frac{x^{\beta}\delta^{\alpha}_{\mu} - x^{\alpha}\delta^{\beta}_{\mu}}{x^2 + \lambda^2}; \; F^{\alpha\beta}_{\mu\nu}(x) = \frac{-4\lambda^2}{(x^2 + \lambda^2)^2}\delta^{\alpha\beta}_{\mu\nu}; \; \delta^{\alpha\beta}_{\mu\nu} = \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu},$$

$$\mathcal{D}_{\mu}F_{\mu\nu} = \partial_{\mu}F_{\mu\nu} - i[A_{\mu}, F_{\mu\nu}] = 0, F_{\mu\nu} = F^{\alpha\beta}_{\mu\nu}T_{\alpha\beta}, T_{\alpha\beta}\epsilon SO(4)$$

It is not the instanton solution; instantons appear from here by factorization on $SU(2)$-group. Going to Minkovsky space $R_{1,3}$ as a base let us take as the gauge group $SO(1,3)$, and we suggest that group acts in the intrinsic space.

By analogy with $SO(4)$-solution $SO(1,3)$-solution is given by

$$A_\mu^{\alpha\beta}(x) = -2\frac{x^\alpha o_\mu^\beta - x^\beta \delta_\mu^\alpha}{x^2+\lambda^2}, F_{\mu\nu}^{\alpha\beta}(x) = -\frac{4\lambda^2}{(x^2+\lambda^2)^2}\delta_{\mu\nu}^{\alpha\beta}, F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu -$$

$$-i[A_\mu, A_\nu] = F_{\mu\nu}^{\alpha\beta} T_{\alpha\beta}, \mathcal{D}_\mu F^\mu{}_\nu = \partial_\mu F^\mu{}_\nu - i[A_\mu, F^\mu{}_\nu] = 0, T_{\alpha\beta}\epsilon SO(1,3);$$

$$[T^{\alpha\beta}, T_{\gamma\delta}] = \frac{i}{4}\{\varepsilon^{\alpha\beta\rho\lambda}\varepsilon_{\gamma\delta}{}^\sigma{}_\lambda - \varepsilon^{\alpha\beta\sigma\lambda}\varepsilon_{\gamma\delta}{}^\rho{}_\lambda\}T_{\rho\sigma}; x^\alpha \epsilon R_{1,3}; g^{00} = \varepsilon^{0123} = -g^{ii} = 1.$$

Here one must be careful with indices place (both gauge indices and Lorents ones) since their position in the formulae is essential. Further we notice that in spite of the Lorents group is not a direct production $SU(2)\times SU(2)$, its Lie algebra has the structure of that. And since the fields lie in the Lie algebra then passing over from basis $T_{\alpha\beta}$ to another (with generators J^a, I^a):

$$J^a = iT^{0a} + \frac{1}{2}\varepsilon^{0abc}T_{bc}, \qquad [J^a, J^b] = -i\varepsilon^{0ab}{}_c J^c$$

$$I^a = -iT^{0a} + \frac{1}{2}\varepsilon^{0abc}T_{bc}, [I^a, I^b] = -i\varepsilon^{0ab}{}_c I^c, [J^a, I^b] = 0,$$

we are getting $su(2)$-solutions in the Minkovsky space correspondingly, self- and anti-self-dual:

$$F_{\mu\nu}^{\alpha\beta} \to F_{\mu\nu}^a = \pm iF_{\mu\nu}^{0a} + \frac{1}{2}\varepsilon^{0a}{}_{bc}F_{\mu\nu}^{bc} = -\frac{4\lambda^2}{(x^2+\lambda^2)^2}[\pm i\delta_{\mu\nu}^{0a} + \frac{1}{2}\varepsilon^{0a}{}_{bc}\delta_{\mu\nu}^{bc}] =$$

$$= -\frac{4\lambda^2}{(x^2+\lambda^2)^2} \times \begin{cases} \eta_{\mu\nu}^a, & *\eta_{\mu\nu}^a = i\eta_{\mu\nu}^a \\ \bar\eta_{\mu\nu}^a, & *\bar\eta_{\mu\nu}^a = -i\bar\eta_{\mu\nu}^a \end{cases}, \tag{6}$$

where

$$\eta_{\mu\nu}^a = i\delta_{\mu\nu}^{0a} + \varepsilon^{0a}{}_{\mu\nu}, \quad \bar\eta_{\mu\nu}^a = -i\delta_{\mu\nu}^{0a} + \varepsilon^{0a}{}_{\mu\nu}$$

are the pseudoeuclidean analogues of Hooft tensors. Solution (6) is analogous to BPST-instanton [1], however it, as distinct from last, has such bad properties as unreality and singularity.

4 Reduction of basic equations

There will be received two scalar equations on two functions from two basic tensor equations in this section. First a few formulae to use further are given. Let values $C_{\mu\nu}$ and $S\mu\nu$ be defined as

$$C_{\mu\nu} = x^\rho x^\sigma[\eta_{\mu\rho}, \eta_{\mu\sigma}], \quad S_{\mu\nu} = ix^\rho(x_\mu\eta_{\nu\rho} - x_\nu\eta_{\mu\rho}), \quad \eta_{\mu\nu} = \eta_{\mu\nu}^a J_a.$$

Then using explicit expression for symbols $\eta_{\mu\nu}^a$, one can prove validity of formulae

$$C_{\mu\nu} = it\eta_{\mu\nu} + S_{\mu\nu}, \quad t = x^2 = x^\rho x_\rho \tag{7}$$

$$* S_{\mu\nu} + iS_{\mu\nu} = t\eta_{\mu\nu} \tag{8}$$

(rather hard to prove). After that trivially to get

$$* C_{\mu\nu} + iC_{\mu\nu} = -t\eta_{\mu\nu}. \tag{9}$$

Also it is valid

$$[\eta_{\rho\mu}, \eta^\rho{}_\nu] = 2i\eta_{\mu\nu}. \tag{10}$$

Now we can pass over to transform the basic equations (4,5) which in indices are given by

$$* F_{\mu\nu} + iF_{\mu\nu} = (*\mathcal{F}_{\mu\nu} + i\mathcal{F}_{\mu\nu})\phi(t) \tag{11}$$

$$D_\mu[(*\mathcal{F}^\mu{}_\nu + i\mathcal{F}^\mu{}_\nu)\phi] = 0 \tag{12}$$

Now there are unknown values $\phi(t)$ and $F_{\mu\nu}(x)$ in the equations. At first we'll engage in equation (12). Let the vector-potential of known field $\mathcal{F}_{\mu\nu}$ is given by $\mathcal{A}_\mu = \psi(t)x^\rho\eta_{\mu\rho}$ with known function $\psi(t)$. By means of formulae (7,8,9) we find

$$* \mathcal{F}_{\mu\nu} + i\mathcal{F}_{\mu\nu} = -i(2t\psi' - t\psi^2 + 4\psi)\eta_{\mu\nu} = -iu_\psi\eta_{\mu\nu}. \tag{13}$$

Then we substitute this expression to equation (12) and have

$$\eta^\mu{}_\nu(\phi\partial_\mu u_\psi + u_\psi\partial_\mu\phi) = i\phi u_\psi[A_\mu, \eta^\mu{}_\nu]. \tag{14}$$

Now we take into account that the field $\mathcal{F}_{\mu\nu}$ obeys YM equations. It means that equation $\mathcal{D}_\mu(*\mathcal{F}^\mu{}_\nu + i\mathcal{F}^\mu{}_\nu) = 0$ is satisfied. It gives

$$\eta^\mu{}_\nu \partial_\mu u_\psi = i u_\psi [\mathcal{A}_\mu, \eta^\mu{}_\nu] \cdot$$

Taking into account this relation, equation (14) becomes

$$\eta^\mu{}_\nu \partial_\mu \phi = i\phi [A_\mu - \mathcal{A}_\mu, \eta^\mu{}_\nu] \,. \qquad (15)$$

Right-hand side prompts that it is convenient to introduce unknown function $f(t)$ instead of field A_μ (spherically-symmetrical ansatz)

$$A_\mu - \mathcal{A}_\mu = f x^\rho \eta_{\mu\rho} = a_\mu \,. \qquad (16)$$

After simple transformations and application of formula (10) we see that the right-hand side of (15) is proportional to η^μ_ν. Hence we have simply scalar equation instead of vector one (12)

$$(\ln \phi)' = -f \cdot \qquad (17)$$

Now we turn to equation (11). We have known its rhs, let us engage with lhs. By using (16) we find

$$F_{\mu\nu} = \mathcal{F}_{\mu\nu} + f_{\mu\nu} - i\{[\mathcal{A}_\mu, a_\nu] - [\mathcal{A}_\nu, a_\mu]\},$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - i[a_\mu, a_\nu]$. Then we evaluate $*F_{\mu\nu}$ and further by means of formulae (7,8,9,10,13) get

$$*F_{\mu\nu} + iF_{\mu\nu} = -i\{u_f + u_\psi - 2tf\psi\}\eta_{\mu\nu} \cdot$$

Now it is seen that both rhs. and lhs. of (11) are proportional to $\eta_{\mu\nu}$, and finally instead of (11) we have

$$2tf' - tf^2 + 2f(2-\psi) = (\phi - 1)(2t\psi' - t\psi^2 + 4\psi) . \qquad (18)$$

Thus we have obtained two nonlinear differential equations of the first order (17,18) on functions $\phi(t), f(t)$.

Unfortunately, one can get the scalar equations by different ways.

For example instead of unknown field $F_{\mu\nu}$ (or A_μ) one can introduce another unknown function $f(t)$ not so as in (16), but other-wise, e.g.

$$A_\mu = f x^\rho \eta_{\mu\rho} \cdot \qquad (19)$$

Then eqs. (11,12) will be reduced to those on functions f, ϕ

$$2tf' - tf^2 + 4f = (2t\psi' - t\psi^2 + 4\psi)\phi \qquad (20)$$

$$(\ln \phi)' = \psi - f \cdot \qquad (21)$$

Maybe, these equations are a little more simple than (17,18).

5 Discussion

The obtained equations appeared to be rather complicated. Are there any nontrivial solutions ? It is not quite clear. One can try by substituting here known solutions (e.g. instanton, monopole,etc.) to look for integrable case. Unfortunately, at first glance there are no such cases. Really it has been told above one must take the sphaleron as $\mathcal{F}_{\mu\nu}$ (in the suitable model) that maybe exactly corresponds to integrable occurrence. The only argument for that is the analogy with nonrelativistic model. Again, unfortunately the analytical sphaleron solutions are absent. Since in this approach we are attempting to exploit classical unstability (it is being found time-dependent solution by way of decay of unstable static solution) probably it is of interest to discuss from topological point of view.

One may of course take the instanton as the known solution. In this case for example the system of eqs. (20,21) has a trivial solution $\phi = 1, f = \psi_{inst}$. And are there another solutions? It is not clear. For system (17,18) even it is not clear about trivial solution, since unknown field is being looked for on the "background" of known one. Let us write the systems of equations for case $\psi = \psi_{inst} = 2/(t+c)$. Eqs.(17,18) are given by

$$(\ln \phi)' = -f$$

$$2tf' - tf^2 + 2f\frac{t+c-1}{t+c} = (\phi - 1)\frac{8c}{(t+c)^2} \cdot$$

And (20,21) are

$$u_f = \frac{8c}{(t+c)^2}\phi \quad , \quad (\ln\phi)' = \frac{2}{t+c} - f \cdot$$

The real finite action solutions in the Minkovsky space require to modify the path integral method in quasi-classical approximation [7]. If those solutions really exist they could be used to evaluate high energy amplitudes as well as widely applied there instantons and sphalerons.

The work has been supported in part by RFFR (project N 93-02-3972)

Author is grateful for useful discussion to B.M.Barbashov, V.V.Nesterenko, A.M.Chervyakov, M. Shaposhnikov, M.Chaichian, C.Montonen, A.Popov, and especially I. Volobuev.

References

1. A.Polyakov, Pis'ma JETP **20** (1974) 430; G.Hooft, Nucl. Phys.**B79** (1974) 276; A.Belavin, A.Polyakov, A.Schwarz, Y.Tyupkin, Phys. Lett. **B59** 1975 85.

2. A.Vainshtein, V.Zakharov, V.Novikov, M.Shifman, Uspehi Fiz. Nauk **136** (1982) 553.

3. N.Manton, Phys. Rev. **D28** (1983) 2019; A.Ringwald, Preprint CERN-TH.6135/91.

4. S.Coleman, Comm. Math. Phys..**55** (1977) 113.

5. V.Yatsun, Lett. Math. Phys. **v.11** (1986) 153.

6. N.Konopleva, V.Popov, Kalibrovochnye polya (in russ.) 1980, Moscow, Atomizdat.

7. A.Koshkarov, Preprint JINR P2-93-321, (in russ.) Dubna, 1993.

Received by Publishing Department
on December 26, 1994.

Кошкаров А.Л. E2-94-530
О поиске невакуумных несамодуальных решений
уравнений Янга — Миллса

На основе обобщения уравнений дуальности в псевдоевклидовом пространстве выводится система двух уравнений на две скалярные функции. Обсуждается возможность существования невакуумных, несамодуальных, вещественных, с конечным действием, зависящих от времени решений уравнений Янга — Миллса.

Работа выполнена в Лаборатории теоретической физики им. Н.Н.Боголюбова ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 1994

Koshkarov A.L. E2-94-530
On Search for Non-Self-Dual Solutions of Yang — Mills Equations
by Means of Generalization of Duality Equations

Two nonlinear scalar equations on two functions have been obtained by means of generalization of duality equations. Possibility of existence of nonvacuum, non-self-dual, real, finite action, timedependent solutions is discussed.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 1994