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WHAT DESCRIPTION OF RADIATION FOLLOWS FROM M.BORN SOLUTION OF 1909?

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1 Introduction

A time-like hyperbola specified as

$$z^2 - t^2 = const \tag{1}$$

describes a world line of an object moving with constant acceleration. Also, it consitutes an orbit of one-parametric group of Lorentz transformations. Thus, the world line of a charge moving with constant acceleration possesses symmetry with respect to the transformations and, since it forms a source of electromagnetic field produced by the charge, the field also possesses this symmetry. M.Born used this idea when constructing the field of a charge moving with constant acceleration in 1909 [1]. In our recent work [2] the symmetry was applied to constructing a coordinate system which makes it possible to separate variables in Maxwell equations and obtain the field as an exact solution of the equations. In standard Lorentzian coordinates $\{t, z, \rho, \phi\}$ with t being the time and the reminder meaning usual coordinates of circular cylinder, the field has the form

$$A_t = \frac{z}{\zeta^2}\left(\frac{\zeta^2+\rho^2+a^2}{\sqrt{[(\zeta-a)^2+\rho^2]\,[(\zeta+a)^2+\rho^2]}} - 1\right) \tag{2}$$

$$A_z = -\frac{t}{\zeta^2}\left(\frac{\zeta^2+\rho^2+a^2}{\sqrt{[(\zeta-a)^2+\rho^2]\,[(\zeta+a)^2+\rho^2]}} - 1\right)$$

where $\zeta = \sqrt{z^2 - t^2}$.

This solution describes in fact the field of a pair of charges with world lines being two conjugated hyperbolas (1). Electromagnetic field produced by the charge with positive values of z is non-zero only for $z > -t$, because all the rest space-time points have not causal connections with that of the charge world line. Correspondingly, electromagnetic field produced by the charge with negative values of z is non-zero only for $z < t$. Consequently, both the fields are zero in the domain $-t > |z|$ and non-zero for $t > |z|$. The domain $z \geq |t|$ $(-z \geq |t|)$ contains only the field produced by the charge with positive (negative) values of z.

Consider now the field produced by only one charge, whose world line has positive z's. It is zero for $z < -t$, and has the form (1) for $z \geq |t|$. Its form in the domain $t > |z|$ is unknown and does not play any part in further considerations. In the present work we assume that there presents only one charge and use the form (2) for the field components only in the domain $z \geq |t|$. The aim of the present work is to study what form of radiation field emitted under small changes of the charge acceleration follows from the Born solution.

2 Statement of the problem

Since the Born solution describes a field which does not contain any wave part [1, 2] the radiation can be emitted only at the moments when the charge acceleration suffers a change. In the simplest case the acceleration is piecewise-constant with single step-like change. The corresponding world line is smooth and consists of two halves of different hyperbolas such that the acceleration suffers a change in the point of the junction. To study the phenomenon it suffices to consider only infinitesimal changes of the acceleration.

Consider a motion with piecewise-constant acceleration. The world line of the motion consists of two halves of hyperbolas (1) with different values of a and lying, in general, in different planes. The fields corresponding to these motions differ in shape and orientation. The Born solution (2) describes the field up to the moment of the acceleration change. Applying another solution to the later moments leads to a wrong result describing a non-smooth field which suffers an infinitesimal discontinuity whereas the field produced in reality is smooth. Evidently, the discontinuity is to be erased by adding a source-free field equal to the difference at the moment of junction and to zero before it. To see that such a source-free field exists and is unique, consider the discontinuity in a frame corresponding to zero particle velocity at the moment of junction. Both the fields have only electric intensities whose divergences are equal to the charge density. Consequently, their difference constituting the discontinuity in question, is an electric intensity with zero divergence and, hence, can be considered as a initial value of some electromagnetic wave. Now, inserting the electric intensity into Maxwell equations and eliminating the undefined magnetic intensity leads to a wave equation for the field to be added. The discontinuity cancels by the only solution. Apparently, the solution describes some purely radiative field which thus can be considered as radiation emitted by the charge. In the present work we evaluate the field differences for an arbitrary infinitesimally small change of the charge acceleration.

3 The coordinate system

The coordinate sysem in which the uniformly accelerated motion has the simplest form, is built as follows. Let $\{t, z, \rho, \varphi\}$ be a Lorentzian coordinate system with t being the time and $\{z, \rho, \varphi\}$ forming the standard circular cylinder coordinate system in the Euclidean 3-space $t = const$. Then, a uniformly accelerated frame moving along the z-axis may be introduced as a foliation of the domain $z \geq 0$ by the semispaces $t - z = const$. The

accelerated coordinates

$$t = \zeta \sinh \xi, \quad z = \zeta \cosh \xi$$
$$\zeta = \sqrt{z^2 - t^2}, \quad \xi = arctanh(t/z)$$

form a natural representation of this frame. Now the 3-planes $\xi = const$ are endowed with circular cylinder coordinate system $\{\zeta, \rho, \varphi\}$. Any time-like curve specified by constant values of the spatial coordinates is a world line of a point moving with acceleration equal to ζ^{-1}.

It must be pointed out that the 3-planes $\xi = const$ play the role of "space" for an observer whose world line is just described. Indeed, usually one defines the notion of "space" for an inertial observer as a typical Euclidean 3-plane orthogonal to his world line. However, there is a significant difference between the notions of "space" for inertial and non-inertial observers because the 3-planes orthogonal to a straight world line are parallel whereas that to a curve are not and, hence, the "spaces" of non-inertial observers intersect. By definition, the "spaces" should form a chronologically ordered family otherwise the causality principle is broken in his "space". Threfore, the "space" of a non-inertial observer is bounded with the 2-plane of intersections. In the case of uniformly accelerated frame the "spaces" are bounded with the plane $\zeta = 0$. More general, if a point-like object moves with acceleration equal to $|\vec{\kappa}|$ then its rest frame spreads over a semispace bounded by a plane lying at the distance $|\vec{\kappa}|^{-1}$ from the object and orthogonal to the acceleration vector.

The coordinate system in which representation of the electromagnetic field of a charge moving with constant acceleration takes the simplest form, is obtained from the system $\{\xi, \zeta, \rho, \varphi\}$ introduced above, by a conformal transformation in the $\xi = const$, $\varphi = const$ semiplanes. In terms of complex variables

$$a + \zeta + \imath\rho \equiv s; \quad u + \imath v \equiv w \tag{3}$$

this transformation has the following form:

$$w = \ln\left(\frac{2a}{s} - 1\right); \quad s = \frac{2a}{1 + e^w}. \tag{4}$$

As well-known, this transformation defines the bi-spherical coordinate system $\{u, v, \varphi\}$ in the $\xi = const$ semispaces. The semispace boundary is specified as $u = 0$ plane and we assume that the u coordinate takes only non-negative values in the semispace [2].

The charge world line intersects the 3-planes in the pole of this coordinate system and his coordinates are $s = a$ or $u = \infty$. When matching two such systems their foci are to coincide, whereas the planes specified as $u = 0$ are different because they constitute boundaries of semispaces concerned to different uniform accelerated frames.

4 Matching the coordinate systems

As the field of a charge moving with constant acceleration is found in a special coordinate system whose shape depends on the acceleration vector, we should introduce two such systems for $t > 0$ and $t < 0$ domains. Their foci coincide at $t = 0$ and constitute a smooth world line of the charge and the boundaries of their domains at $t = 0$ are different. Indeed, the perpendicular dropped from the focus to the bounding plane is collinear with the vector of acceleration and its length is inverse to the value of the acceleration. Therefore, there exist two main cases to be considered. In the first case the acceleration changes only in value and in the second case it does only in direction. Therefore we label the corresponding increments of coordinates and the field components with subscripts $\|$ and $\perp$. For infinitesimal changes of the acceleration all the rest cases are reducible to the two main ones. As the fields at $t > 0$ and $t < 0$ have the same form in both the coordinate systems it is natural first to match the coordinate systems.

In the first case there are two spatial coordinate systems in the 3-plane $t = 0$ specified by the expressions (3) and (4) with different values of the parameter a. Let δa be the small increment of the parameter caused by small change of acceleration. This change causes a small deformation of the coordinate system. Small deformation of the coordinate system causes a small change of any object considered in the system. As well-known, the change is equal ot its Lie differential with respect to the deformation [3, 5]. The deformation is expressed in changes of values of coordinates u and v for any point of the space. Therefore, we start with evaluating changes of the coordinates and then find out the Lie derivatives of the field. Since Lie derivatives of the field are different in the first and the second cases we denote them as $L_{\|}$ and $L_{\perp}$ respectively.

Small differences of the values of u and v coordinates at any point may be obtained from the fact that increment of the s coordinate is zero:

$$0 = \delta s = -\frac{2\delta a}{1+e^w} + \frac{2ae^w\delta w}{(1+e^w)^2}.$$

Thus, increment of the w coordinate is equal to

$$\delta w = \frac{1+e^{-w}}{a}\delta a = -\frac{\partial w}{\partial a}.$$

Apparently, the differences of u and v coordinates constitute the real and imaginary parts of δw respectively. Finally, one can rewrite the results in the following form:

$$L_{\|}u = a^{-1}(1+e^{-u}\cos v); \quad L_{\|}v = a^{-1}e^{-u}\sin v. \tag{5}$$

To match the coordinate systems in the second case we introduce Cartesian coordinates $\{x, y, \eta\}$

$$x = \frac{a\sin v\cos\varphi}{\cosh u + \cos v}; \quad y = \frac{a\sin v\sin\varphi}{\cosh u + \cos v}; \tag{6}$$
$$\eta = a\left(\frac{\sinh u}{\cosh u + \cos v} - 1\right)$$

and the vector $\vec{a}$ with components $(0, 0, a)$ in these coordinates. The charge world line intersects the space in the point $\vec{r} = 0$ where $\vec{r}$ has components (x, y, η) and it is easy to check out that

$$e^{2u} = \frac{|\vec{r}+2\vec{a}|^2}{|\vec{r}|^2}; \quad \cos v = -\frac{(\vec{r}+2\vec{a})\cdot\vec{r}}{|\vec{r}+2\vec{a}||\vec{r}|}. \tag{7}$$

The vector $\vec{a}$ is collinear to the charge acceleration and has the length equal to the power minus one of the acceleration. As was pointed out above, the vector $\vec{a}$ represents the perpendicular dropped from the focus to the bounding plane and its increment is orthogonal to it. Let the small change of the vector $\vec{a}$ is denoted as $\delta\vec{a}$. For any given vector $\delta\vec{a}$ orthogonal to $\vec{a}$ the φ coordinate may be established such that the $\delta\vec{a}$ vector has only one Cartesian component:

$$\delta\vec{a} = (\delta_{\perp}a, 0, 0)$$

As follows from the formulas (6), the expression of $\vec{r}^{\ 2}$ in terms of bi-spherical coordinates has the form

$$\vec{r}^{\ 2} = \frac{2a^2e^{-u}}{\cosh u + \cos v}.$$

Now, evaluating the increments of the expressions (7) with $\delta\vec{r} = 0$ leads to the following results:

$$-2e^{-2u}L_{\perp}u = \frac{2x}{\vec{r}^{\ 2}} = 2a^{-1}e^{u}\sin v\cos\varphi;$$

$$-\sin vL_{\perp}v = \sin v\cdot a{-}1(1+e^{-u}\cos v)\cos\varphi,$$

Extracting the Lie derivatives from the last two expressions one finds that they and the formulas (5) form Cauchy-Riemann–like equations for Lie derivatives of the coordinates:

$$L_{\|}u = a^{-1}(1+e^{-u}\cos v); \quad L_{\|}v = -a^{-1}e^{-u}\sin v \tag{8}$$
$$L_{\perp}u = -a^{-1}e^{-u}\sin v\cos\varphi; \quad L_{\perp}v = -a^{-1}(1+e^{-u}\cos v)\cos\varphi.$$

These relations make it possible to find out the Lie derivatives of 1-forms like du. Indeed, since the operation of Lie derivation commutates with that of exterior derivation [5], the explicit form of Ldu is

$$L_{\|}du = a^{-1}d(1+e^{-u}\cos v); \quad L_{\perp}du = -a^{-1}d(e^{-u}\sin v\cos\varphi). \tag{9}$$

5 Matching the fields

As was shown in the work [2], the electric intensity produced by a charge moving with constant acceleration can be expressed in its rest frame as the following 1-form:

$$E = E_i dx^i = \frac{q}{a}(\cosh u + \cos v)du \qquad (10)$$

where q denotes the charge value. Increments of the intensities are Lie derivatives $L_{\|}E$ and $L_{\perp}E$ of the 1-form E on the deformation of the coordinate system. As was pointed out above, both the expressions to be constructed represent sourceless electromagnetic fields. Consequently, they are to satisfy the equation

$$d*(LE) = 0 \qquad (11)$$

i.e., $div\ \vec{E} = 0$ in the standard denotions. Below this equation will be used when checking the results out.

When evaluating the expression $L_{\|}E$ one must take into account the explicit dependence of the intensity on the parameter a (10), i.e. to add the usual partial derivative $\partial E/\partial a$ to the expression. Derivating the 1-form E (10) and substituting the Lie derivatives (8) and (9) one obtains

$$L_{\|}E = qa^{-2}\{[-(\cosh u + \cos v) + (1 + e^{-u}\cos v)\sinh u + e^{-u}\sin^2 v -$$

$$-e^{-u}\cos v(\cosh u + \cos v)]du - e^{-u}\sin v(\cosh u + \cos v)dv].$$

This result can be simplified by the following identity:

$$(1 + e^{-u}\cos v)\sinh u + e^{-u}\sin^2 v = (1 - e^{-u}\cos v)(\cosh u + \cos v)$$

After that the result takes the following form:

$$L_{\|}E = -qa^{-2}e^{-u}(\cosh u + \cos v)(2\cos v du + \sin v dv) \qquad (12)$$

The analogous evaluation in the second case yields

$$L_{\perp}E = \qquad (13)$$

$$= qa^{-2}e^{-u}(\cosh u + \cos v)(2\sin v\cos\varphi du + \cos v\cos\varphi dv + \sin v\sin\varphi d\varphi).$$

6 Checking out and the final form of the results

As was pointed out above, the results may be checked out by inserting them into the equation (11). To do this, we start with constructing the conjugated 1-forms $*(L_{\|}E)$ and $*(L_{\perp}E)$. The Levi-Civita symbol components

corresponding to the three-dimensional bi-spherical metric [4]

$$ds^2 = \frac{du^2 + dv^2 + \sin^2 v d\varphi^2}{(\cosh u + \cos v)^2}$$

have the following form:

$$\varepsilon^u{}_{v\varphi} = \varepsilon_u{}^v{}_\varphi = \frac{a\sin v}{\cosh u + \cos v}; \quad \varepsilon_{uv}{}^\varphi = \frac{a}{\sin v(\cosh u + \cos v)}$$

Now, transforming the components of the 1-forms (12) and (13) one obtains the following 2-forms:

$$*(L_{\|}E) = qa^{-2}e^{-u}(-\sin^2 v du\wedge d\varphi + 2\sin v\cos v dv\wedge d\varphi)$$

$$*(L_{\perp}E) = qa^{-2}e^{-u}(\sin\varphi du\wedge dv + \sin v\cos v\cos\varphi du\wedge d\varphi$$

$$+2\sin^2 v\cos\varphi dv\wedge d\varphi).$$

It is easy to check out that both of them satisfy the equation (11), for example:

$$d*(L_{\perp}E) =$$

$$= qa^{-2}e^{-u}[-2\sin^2 v - \frac{d}{dv}(\sin v\cos v) + 1]\cos\varphi du\wedge dv\wedge d\varphi$$

By definition, the vector $\vec{a}$ is related to the acceleration vector $\vec{\kappa}$ as follows:

$$\vec{\kappa} = a^{-2}\vec{a}.$$

Denoting increments of the charge acceleration components as $\delta\vec{w}_{\|}$ and $\delta\vec{w}_{\perp}$ one can derive the following relations for their increments:

$$\delta\vec{w}_{\|} = -a^{-2}\delta a_{\|}; \quad \delta\vec{w}_{\perp} = -a^{-2}\delta a_{\perp}.$$

Now it is possible to find out the strength increment related to that of acceleration. Denoting them as $\varepsilon_{\|}$ and $\varepsilon_{\perp}$ respectively one obtains the final result in the form

$$\varepsilon_{\|} = q\delta w_{\|}e^{-u}(\cosh u + \cos v)(2\cos v du + \sin v dv)$$

$$\varepsilon_{\perp} = q\delta w_{\perp}e^{-u}(\cosh u + \cos v)(2\sin v\cos\varphi du - \cos v\cos\varphi dv + \sin v\sin\varphi d\varphi).$$

Apparently, for an arbitrary change of the charge acceleratrion the intensity increment is sum of ε's. It is seen that the radiation field grows as r^2 near the charge. This means that in a small enough neighbourhood of the charge the radiation field is negligibly small and cannot yield large corrections for the charge self-energy.

References

[1] Pauli, W., *Theory of Relativity* Pergamon, London 1958, p. 92

[2] Turakulov, Z.Ya. *Turkish J. Phys.* 18,479 (1994), Z.Y. Turakulov, J. Geom. Phys., 14,305 (1994)

[3] Yano K., *The Theory of Lie Derivatives and its Applications* North-Holland, Amsterdam, 1955 p.10

[4] Morse, P.M. and Feshbach, H. *Methods of Theoretial Physics* McGrow-Hill, 1953

[5] Schutz, B.F.L. *Geometrical Methods of Mathematical Physics* Cambridge University Press, New York, 1982

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Какое описание излучения следует из решения М.Борна 1909 года?

Точное решение уравнений Максвелла для поля заряда, движущегося с постоянным ускорением, найденное впервые М.Борном в 1909 году, отличается от потенциалов Лиенара-Вихерта для того же источника поля. Главное различие между физическими результатами, следующими из этих двух выражений поля, состоит в том, что решение Борна не содержит волновой части. Хотя это решение описывает отсутствие излучения в данном конкретном случае ускоренного движения [1], оно приводит к некоторым новым результатам в классической теории излучения. В настоящей работе рассматривается случай движения с кусочно-постоянным ускорением. Показано, что сшивка полей, соответствующих разным значениям ускорения, порождает некоторое новое поле, допускающее интерпретацию, как поле испущенного излучения.

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What Description of Radiation Follows from M.Born Solution of 1909?

Exact solution of the Maxwell equations for electromagnetic field of a charge moving with constant acceleration found first by M.Born in 1909 differs from Lienard-Wiechert potential of the same source. The main difference between physical results obtained from these two expressions for the field is that the Born solution does not contain any wave part. Although the solution displays absence of radiation in this special case of accelerated motion [1] it leads to some new results in the classical theory of radiation. In this work the case of motion with piecewise constant acceleration is considered. It is shown that matching the fields corresponding to different values of acceleration yields some new field which can be interpreted as that of radiation emitted.

The investigation has been performed at the Laboratory of Particle Physics, JINR.

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