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TO THE PROBLEM OF $1/N_c$ APPROXIMATION IN THE NAMBU — JONA-LASINIO MODEL

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К проблеме $1/N_c$-приближения
в модели Намбу — Иона–Лазинио

Исследуется уравнение щели для составляющей массы кварка в $U(2) \times U(2)$ Намбу — Иона–Лазинио модели в $1/N_c$-приближении. Показано, что учет скалярных изовекторных мезонов играет важную роль для правильного описания кварковых масс в этом приближении. Кратко обсуждается роль тождеств Уорда при вычислении $1/N_c$-поправок к мезонным вершинным функциям.

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Ebert D., Nagy M., Volkov M.K. E2-94-488
To the Problem of $1/N_c$ Approximation
in the Nambu — Jona–Lasinio Model

In this article, the gap equation for the constituent quark mass in the $U(2) \times U(2)$ Nambu — Jona–Lasinio model for the $1/N_c$ approximation is investigated. It is shown that taking into account scalar isovector mesons plays an important role for the correct description of quark masses in this approximation. The role of the Ward identity in calculations of $1/N_c$ corrections to the meson vertex functions is shortly discussed.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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The NJL model in the leading $1/N_c$ approximation, Hartree approximation, allows us to obtain a relatively complete picture of low-energy meson physics [1-5] (N_c is the number of quark colors). However, in the last time, there have been undertaken some attempts to describe next to the leading $1/N_c$ approximation in the NJL model and to consider mesons not in the tree diagrams only but also in the loop diagrams [6-12]. Interesting results have been obtained in this direction for the description of the behaviour of the thermodynamical potential and the bulk of thermodynamical quantities in the vicinity of the critical temperature. It has been shown that mesonic degrees of freedom play the dominant role at low T, and the quark degrees of freedom are most relevant at high T.

Thus, it seems to be very useful to continue these investigations, to study more carefully the $1/N_c$ approximation in the NJL model by using different methods. Here, we consider the perturbation theory and calculate $1/N_c$ corrections to the gap equation. We will show how to correctly use the perturbation theory for the description of constituent quark mass in the $1/N_c$ approximation. Our results are remarkably different from those obtained in the series of previous papers (see e.g. [7]). It will be shown that the inclusion of scalar isovector mesons $a_0(980)$ plays an important role in the description of the $1/N_c$ approximation.

We consider the NJL model for the $U(2) \times U(2)$ chirally symmetric case [1-2]

$$L(\bar{q}, q) = \bar{q}(i\hat{\partial} - m^0)q + \frac{G}{2}[(\bar{q}\lambda^a q)^2 + (\bar{q}i\gamma_5\lambda^a q)^2], \tag{1}$$

where q are the fields of u and d quarks, m^0 is the current quark mass, $\lambda^0 = 1$ is the unique matrix and $\lambda^a = \tau^a$ $(a = 1, 2, 3)$ are the Pauli matrices.

After the introduction of meson fields by using the technique of generating functional [1-3] and performing the integration over quark fields in the functional integral, we come to the Lagrangian

$$L'(\tilde{\sigma}, \phi) = -\frac{\tilde{\sigma}_a^2 + \phi_a^2}{2G} - i\mathrm{Tr}\ln S^{-1}(x - y), \tag{2}$$

where $\tilde{\sigma}_a$ and ϕ_a are the scalar and pseudoscalar meson fields, respectively, $\tilde{\sigma}_0 = \sigma_0 - m + m_0$, $\tilde{\sigma}_a = \sigma_a$ $(a = 1, 2, 3)$,($< \sigma_0 >_0 = 0$)

$$S^{-1}(x,y) = [i\hat{\partial}_x - m + \sigma_a\lambda^a + i\gamma_5\lambda^a\phi_a]\delta^4(x-y)$$

To get the σ-model, it is enough to consider the divergent quark loops, depicted in Fig.1, and to perform the renormalization of meson fields [1-3]

As a result, we obtain the meson Lagrangian of the following type:

$$L''(\sigma,\phi) = \frac{1}{4}\mathrm{Tr}\left\{(\partial_\mu\bar{\sigma})^2 + (\partial_\mu\bar{\phi})^2 + 2g\left(\frac{m-m_0}{G} - 8mI_1(m,\Lambda)\right)\bar{\sigma} - \right.$$
$$\left. - g^2\left(\frac{1}{G} - 8I_1(m,\Lambda)\right)(\bar{\sigma}^2 + \bar{\phi}^2) - g^2\left[\bar{\sigma}^2 - 2\frac{m}{g}\bar{\sigma} + \bar{\phi}^2\right]^2\right\} -$$
$$-i\mathrm{Tr}\ln\left\{1 + \frac{g}{i\hat{\partial} - m}[\bar{\sigma} + i\gamma_5\bar{\phi}]\right\}', \quad (4)$$

where the prime in the last term means that we heve here the convergent quark loop diagrams, $g = [4I_2(m,\Lambda)]^{-1/2}$, $\bar{\sigma} = \sigma^a\lambda_a$, $\bar{\phi} = \phi^a\lambda_a$, and $I_1(m,\Lambda)$ and $I_2(m,\Lambda)$ are divergent integrals (Λ is the cut-off parameter)

$$I_n(m,\Lambda) = -i\frac{N_c}{(2\pi)^4}\int^\Lambda \frac{d^4k}{(m^2-k^2)^n}. \quad (5)$$

From (5) we can see that the coupling constant g^2 has the order $1/N_c$. Remind that the coupling constant G also has the order $1/N_c$.

From the condition $\frac{\delta L''(\sigma,\phi)}{\delta\sigma}|_{\sigma,\,\phi=0} = 0$ (absence of linear in σ terms in $L''(\sigma,\phi)$) we obtain the gap equation

$$m = m^0 + 8mGI_1(m,\Lambda)\ . \quad (6)$$

How does the gap equation change, if we permit the existence of meson propagators inside the quark loops? ($1/N_c$ approximation). From Fig.1 one can easily see that in this case, in addition to the tadpole 1a there appear complementary terms (linear in σ) from the diagram 1c which lead to the appearance of additional terms in the gap equation (6) (see Fig.2)

$$m = m^0 + 8mGI_1(m,\Lambda) + \Delta =$$
$$= m^0 + 2G\frac{iN_c}{(2\pi)^4}\mathrm{Tr}\int^\Lambda \frac{d^4k}{\hat{k}-m} + 2G\frac{iN_c}{(2\pi)^4}\mathrm{Tr}\int^\Lambda d^4k\frac{1}{\hat{k}-m}\Sigma(k)\frac{1}{\hat{k}-m} + ... \quad (7)$$

The last two terms in (7) can be written in the form of one tadpole with the modified quark mass:

$$m = m^0 + 2G\frac{iN_c}{(2\pi)^4}\mathrm{Tr}\int^\Lambda \frac{d^4k}{\hat{k}-m-\Sigma(k)}\ , \quad (7a)$$

where $\Sigma(k)$ is the quark self-energy

$$\Sigma(k) = 3\Sigma_\pi(k) + \Sigma_{\sigma_0}(k) + 3\Sigma_{a_0}(k)\ , \quad (8)$$

$$\Sigma_\pi(k) = i\frac{g^2}{(2\pi)^4}\int^{\bar{\Lambda}} d^4q\frac{\hat{q}-m}{(m^2-q^2)(M_\pi^2-(k-q)^2)}, \quad (9)$$

$$\Sigma_{\sigma_i}(k) = i\frac{g^2}{(2\pi)^4}\int^{\bar{\Lambda}} d^4q\frac{\hat{q}+m}{(m^2-q^2)(M_{\sigma_i}^2-(k-q)^2)}. \quad (10)$$

Here M_π and M_{σ_i} are the masses of pions and σ-particles ($\sigma_i = \sigma_0, a_0^0, a_0^+, a_0^-$), respectively. [1]

The gap equation (7a) can be written in the form of the Schwinger - Dyson equation for the new quark mass $\bar{m}(p) = m + \Sigma(p)$. For this purpose, we add the term $\Sigma(p)$ to both the sides of equation (7a) and write it in the form

$$\bar{m}(p) = m^0 + 2G\frac{iN_c}{(2\pi)^4}\mathrm{Tr}\int^\Lambda \frac{d^4k}{\hat{k}-\bar{m}(k)} + \Sigma(p). \quad (11)$$

The gap equation (7) and the Schwinger - Dyson equation (11) noticeably differ from each other. Indeed, the equation (7) contains $\Sigma(k)$ inside of the quark loop (see Fig.2) and the equation (11) contains $\Sigma(p)$ as a quark-loop-independent term. However, in order to prove the low-energy theorems in the $1/N_c$ approximation (the Goldstone theorem, the Goldberger- Treiman relation, the conservation of the quark axial current) we have to use the gap equation (7) rather than the equation (11).

From equation (7) we can find the correction δm to the quark mass m_H, obtained in the Hartree approximation, after taking account of the first order in $1/N_c$ expansion. That is why we write the mass m in the form

[1] In the general case the cut-off parameters Λ and $\bar{\Lambda}$ are not equal to each other .Here, we assume that $\Lambda = \bar{\Lambda} = 1.2$GeV.

$$m = m_H + \delta m \tag{12}$$

and expand the second term in the r.h.s. of (7) over δm, conserving the terms of first order over $1/N_c$

$$m_H + \delta m = m_0 + (m_H + \delta m) 8G \left[I_1(m_H, \Lambda) + \delta m \frac{\delta I_1}{\delta m}|_{m=m_H} \right] + \Delta(m_H, \Lambda) . \tag{13}$$

By using the formulae

$$\frac{\delta I_1(m,\Lambda)}{\delta m} = -2m I_2(m,\Lambda) = -\frac{m}{2g^2} \tag{14}$$

and the gap equation in the Hartree approximation (see formula (6))

$$m_H = m_0 + 8G m_H I_1(m_H, \Lambda),$$

we find for δm the following expression:

$$\delta m = Z^{-1} \Delta(m_H, \Lambda) , \tag{15}$$

where

$$Z = 16 G m_H^2 I_2(m_H, \Lambda) + \frac{m_0}{m_H} = \left(\frac{2m_H}{g}\right)^2 G + \frac{m_0}{m_H} . \tag{16}$$

For the parameters we use here [1b]: $m_H = 280$ MeV, $m_0 = 3.3$ MeV, $\Lambda = 1.2$ GeV, $G = 5.4$ GeV^{-2},and $g^2 \approx 2\pi$, we get

$$Z^{-1} = 3.6, \quad \delta m = 3.6\ \Delta(m_H, \Lambda).$$

Now we have to determine the term $\Delta(m,\Lambda)$.For that let us calculate the functions $\Sigma_\pi(p,\Lambda)$ and $\Sigma_\sigma(p,\Lambda)$. One can easily evaluate the integrals in formulae (9) and (10) and get the following expressions:

$$\Sigma_\pi(p,\Lambda) = \frac{g_\pi^2}{(4\pi)^2} \int_0^1 dx\ (m - x\hat{p}) \left[\ln\left(\frac{\Lambda^2}{m^2} + 1\right) + \right.$$
$$\left. + \ln \frac{1 + \bar{b}_\pi x + \bar{c}x^2}{1 + b_\pi x + c x^2} - \left(1 + \frac{m^2}{\Lambda^2}\right)^{-1} \frac{1}{1 + \bar{b}_\pi x + \bar{c} x^2} \right] =$$
$$= \frac{g_\pi^2}{(4\pi)^2} \left[m C_1^\pi(p,\Lambda) - \hat{p} C_2^\pi(p,\Lambda) \right], \tag{17}$$

$$\Sigma_{\sigma_i}(p,\Lambda) = -\frac{g^2}{(2\pi)^4} \int_0^1 dx\ (m + x\hat{p}) \left[\ln\left(\frac{\Lambda^2}{m^2} + 1\right) + \right.$$
$$\left. + \ln \frac{1 + \bar{b}_{\sigma_i} x + \bar{c}x^2}{1 + b_{\sigma_i} x + c x^2} - \left(1 + \frac{m^2}{\Lambda^2}\right)^{-1} \frac{1}{1 + \bar{b}_{\sigma_i} x + \bar{c} x^2} \right] =$$
$$= -\frac{g^2}{(4\pi)^2} \left[m C_1^{\sigma_i}(p,\Lambda) + \hat{p} C_2^{\sigma_i}(p,\Lambda) \right], \tag{18}$$

where

$$b_i = \frac{M_i^2 - m^2 - p^2}{m^2}, \quad c = \frac{p^2}{m^2}, \quad \bar{b}_i = \frac{M_i^2 - m^2 - p^2}{a}, \bar{c} = \frac{p^2}{a}, a = m^2 + \Lambda^2 ,$$

$$C_1^i = \ln\left(\frac{\Lambda^2}{m^2} + 1\right) + \left(1 + \frac{\bar{b}_i}{2\bar{c}}\right) \ln(1 + \bar{b}_i + \bar{c}) - \left(2 + \frac{b_i}{c}\right) \ln \frac{M_i}{m} +$$
$$+ \left(1 - \frac{\bar{b}_i^2}{2\bar{c}}\right) \bar{I}_0 + \left(\frac{b_i^2}{2c} - 2\right) I_0 , \tag{19}$$

$$C_2^i = -\frac{1}{2}\left(\frac{\bar{b}_i}{\bar{c}} - \frac{b_i}{c}\right) + \frac{1}{2} \ln\left(\frac{\Lambda^2}{m^2} + 1\right) + \frac{1}{2}\left(1 - \frac{\bar{b}_i^2}{2\bar{c}^2}\right) \ln(1 + \bar{b}_i + \bar{c}) -$$
$$- \left[1 + \frac{1}{c}\left(1 - \frac{b_i^2}{2c}\right)\right] \ln \frac{M_i}{m} - \frac{\bar{b}_i}{2\bar{c}} \left(1 - \frac{\bar{b}_i^2}{2\bar{c}}\right) \bar{I}_0 + \frac{b_i}{2c}\left(2 - \frac{b_i^2}{2c}\right) I_0 , \tag{20}$$

$$I_0 = \int_0^1 \frac{dx}{1 + b_i x + c x^2}, \qquad \bar{I}_0 = \int_0^1 \frac{dx}{1 + \bar{b}_i x + \bar{c} x^2} .$$

The term $\Delta(m,\Lambda)$ is described by divergent inyegrals. From the formulae (19) and (20) we can see that the coefficients $C_j^i(p,\Lambda)$ are slowly changing functions of the momentum p. Therefore we can use the estimates for these coefficients at $p^2 = -\Lambda^2$ and then calculate the divergent integrals in the equation (7) with the cut-off parameter Λ.

In our case of the group U(2)×U(2), to the three pions there correspond four scalar mesons in the scalar sector (scalar isoscalar $\sigma_0(700)$ and three scalar isovectors $a_0(980)$). [2]

[2] The isoscalar partner of pions appears only in the U(3)×U(3) group in the form of a η meson. Therefore, we will not consider it here.Scalar mesons have the masses: $m_{\sigma_0} = 700 MeV$ and $m_{a_0} = 980 MeV$.

Table 1 gives the coefficients C_1^i and C_2^i evaluated for all these mesons at $p^2 = -\Lambda^2$.

Table 1

$C_1^{\sigma_0}$	$C_2^{\sigma_0}$	$C_1^{a_0}$	$C_2^{a_0}$	C_1^{π}	C_2^{π}
0.52	0.19	0.45	0.16	0.85	0.46

Then, for $\Delta(m,\Lambda)$ we obtain

$$\Delta(m,\Lambda) = -3\frac{Gg^2m^3}{2(2\pi)^4}[\frac{\Lambda^2}{m^2}(C_1^{\sigma_0} + 2C_2^{\sigma_0} + 3(C_1^{a_0} + 2C_2^{a_0})$$

$$+3(2C_2^{\pi} - C_1^{\pi})) - \ln(\frac{\Lambda^2}{m^2} + 1)(3C_1^{\sigma_0} + 4C_2^{\sigma_0} + 3(3C_1^{a_0} + 4C_2^{a_0})$$

$$+3(4C_2^{\pi} - 3C_1^{\pi})) + 2(C_1^{\sigma_0} + C_2^{\sigma_0} + 3(C_1^{a_0} + C_2^{a_0}) + 3(C_2^{\pi} - C_1^{\pi}))]$$

$$= -0.12m.$$

$$\delta m = Z^{-1}\Delta = -0.4\ m\ .$$

As a result, the mass of a constituent quark decreases by 40 % . If we consider only one scalar meson $\sigma_0(700)$, the corrections noticeably decrease, amounting to 18 % ($\delta m = -0.18\ m$).

These calculations have only a qualitative character because to obtain more accurate results it is necessary to redefine all the model parameters (m, Λ, G) in the $1/N_c$ approximation. However, the derivation of the very accurate results is not our task in this short note. We would like only to emphasize here the following two important facts:

1) The gap equation (7) and the Schwinger-Dyson equation (11) are different in form in the $1/N_c$ approximation of the NJL model. In order to test the low-energy theorems in the $1/N_c$ approximation, it is necessary to use the gap equation (7).

2) In the realistic case the inclusion of scalar isovector mesons a_0 can very strongly change the $1/N_c$ corrections.

The first point is very important in calculations in the $1/N_c$ approximation and at present several works appear where a inadequate approximation is used. In paper [7] , the equation (11) has been employed with a constant mass $\bar{m}$ and without expanding of the tadpole term around the mass m_H. On the other hand, in the recent work [12], the gap equation (7) has been used but with an additional term Σ in the right-hand side.However, to prove the low-energy theorems, they omitted this term, i.e. they used our equation (7).

One of the interesting tasks is the construction of chirally symmetric perturbation theory for the $1/N_c$ expansion. The positive results in this direction have been obtained by G.S.Guralnik with coauthors already in 1976 [13]. They showed that in the $1/N_c$ approximation for the NJL model with one scalar and one pseudoscalar mesons the pion mass was equal to zero when the current quark mass was vanishing. Therefore, the pion remains the Goldstone particle in this approximation as well.

When this work has been fulfilled, we found out that a very interesting paper appeared just now [12].In this work, a chirally symmetric self-consistent $1/N_c$ approximation scheme to the NJL model was developed.The authors used the correct $1/N_c$ approximation for the gap equation and demonstrated explicitly that their scheme fulfills all the chiral symmetry theorems - the Goldstone theorem, Goldberger-Treiman relation and the conservation of the quark axial current.

This paper is very close to ref. [13]. In contrast with our work they considered the $SU(2) \times SU(2)$ chiral symmetric Lagrangian with only one scalar isoscalar meson and the case when the current quark mass was equal to zero.

It is interesting to consider the changes of the meson coupling constants g and g_π in the $1/N_c$ approximation. As we have shown in the Appendix, the scalar meson coupling constant g does not change in the $1/N_c$ approximation. A more complicated situation took place for the coupling constants g_π and the Goldberger-Treiman identity (see [12]).

In conclusion, we would like to say that the papers [12-13] and this one give the full picture of the chirally symmetric $1/N_c$ approximation in the NJL model.

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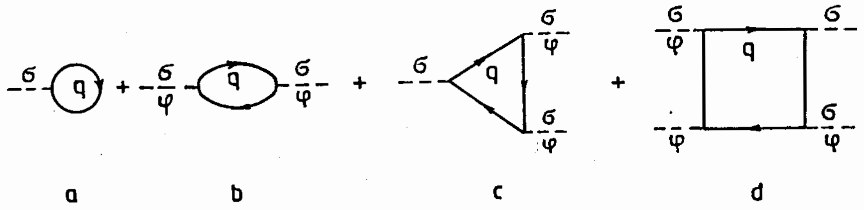

Fig.1 The quadratically (1a, 1b) and logarithmically (1c, 1d) divergent quark loop diagrams in the NJL model.

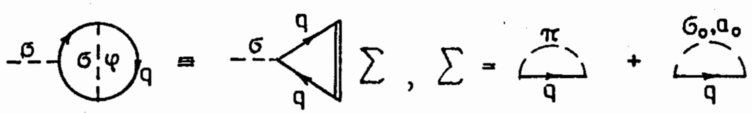

Fig.2 The additional tadpole diagram in the $1/N_c$ approximation. The Σ is the self-energy part of the quark propagator with pion and scalar meson internal lines.

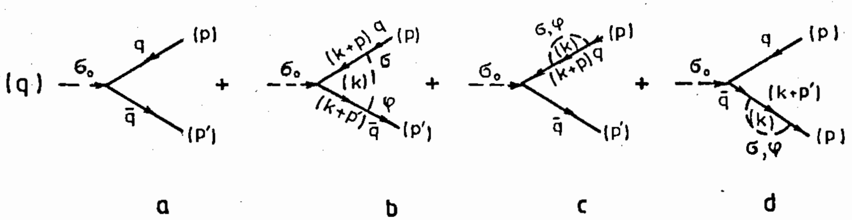

Fig.3 The scalar vertex diagrams in the $1/N_c$ approximation .

APPENDIX

The scalar vertex function and Ward identity

Let us show that $1/N_c$ corrections to the scalar coupling constant g are equal to zero. For this aim we consider diagrams depicted in Fig.3. The scalar vertex function for σ_0 meson in $1/N_c$ approximation takes the form

$$\Gamma^{(1/N_c)}(p,p'|q) = g_\sigma + \Gamma^b_{\sigma_0}(p,p'|q) + \Gamma^{(c+d)}_{\sigma_0}(p,p'|q) + 3\Gamma^b_{a_0}(p,p'|q) + 3\Gamma^{(c+d)}_{a_0}(p,p'|q) +$$
$$+3\Gamma^b_\pi(p,p'|q) + 3\Gamma^{(c+d)}_\pi(p,p'|q). \qquad (A.1)$$

Now consider the case when $q=0$, $p=p'$. Then

$$\Gamma^{(b)}_{\sigma_0}(p,p|0) = -i\frac{g^3}{(2\pi)^4}\int \frac{d^4k}{(\hat{k}+\hat{p}-m)^2(M^2_{\sigma_0}-k^2)}, \qquad (A.2)$$

$$\Sigma_{\sigma_0}(p+k) = -i\frac{g^2}{(2\pi)^4}\int \frac{d^4k}{(\hat{k}+\hat{p}-m)(M^2_{\sigma_0}-k^2)}, \qquad (A.3)$$

$$\Gamma^{(c+d)}_{\sigma_0}(p,p|0) = g\frac{\Sigma_{\sigma_0}(p)-\Sigma_{\sigma_0}(m)}{\hat{p}-m}|_{\hat{p}=m} = g\frac{\delta\Sigma_{\sigma_0}(p)}{\delta\hat{p}}|_{\hat{p}=m} =$$
$$= i\frac{g^3}{(2\pi)^4}\int \frac{d^4k}{(\hat{k}+\hat{p}-m)^2(M^2_{\sigma_0}-k^2)} = -\Gamma^{(b)}_{\sigma_0}(p,p|0) \qquad (A.4)$$

The similar situation takes place for Γ_{a_0} and Γ_π. As a result all contributions of the diagrams depicted in Fig.3b-d cancel each other and finally we got

$$\Gamma^{(1/N_c)}(p,p|0) = g_\sigma\ . \qquad (A.5)$$

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