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DIFFERENTIAL CALCULUS ON THE QUANTUM SPHERE AND DEFORMED SELF-DUALITY EQUATION

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Зупник Б.М. E2-94-487
Дифференциальное исчисление на квантовой сфере
и деформированное уравнение самодуальности

Обсуждается левоковариантное 3-мерное дифференциальное исчисление на квантовой сфере $SU_q(2) / U(1)$. Спинорные $SU_q(2)$-гармоники рассматриваются как координаты квантовой сферы. Мы рассматриваем калибровочную теорию для квантовой группы $SU_q(2) \times U(1)$ на деформированном евклидовом пространстве $E_q(4)$. Предложено q-обобщение формализма гармонических калибровочных полей. Этот формализм используется для гармонической (твисторной) интерпретации уравнения q-самодуальности на $E_q(4)$. Мы рассматриваем представление нулевой кривизны и общую конструкцию q-самодуальных решений с помощью аналитического препотенциала.

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Zupnik B.M. E2-94-487
Differential Calculus on the Quantum Sphere
and Deformed Self-Duality Equation

We discuss the left-covariant 3-dimensional differential calculus on the quantum sphere $SU_q(2) / U(1)$. The $SU_q(2)$-spinor harmonics are treated as coordinates of the quantum sphere. We consider the gauge theory for the quantum group $SU_q(2) \times U(1)$ on the deformed Euclidean space $E_q(4)$. A q-generalization of the harmonic-gauge-field formalism is suggested.

This formalism is applied for the harmonic (twistor) interpretation of the quantum-group self-duality equation (QGSDE). We consider the zero-curvature representation and the general construction of QGSDE-solutions in terms of the analytic prepotential.

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1 Introduction

The 2-dimensional sphere S^2 is the simplest example of homogeneous space and can be treated as $SU(2)/U(1)$ coset space. S^2 plays an important role in the twistor program of Penrose [1] and, particularly, in the twistor interpretation of self-duality equation [2]-[4]. The harmonic approach [4],[5] is a specific version of the twistor formalism based on using the spinor harmonics as coordinates on S^2.

In the present talk, we make an attempt to construct a q-deformed harmonic formalism in the framework of the quantum-group concept [6],[7]. Noncommutative geometry of quantum spheres has been considered in Refs[7]-[9]. We shall use the left-invariant 3D differential calculus on the quantum group $SU_q(2)$ [10], [11] to study geometry on the quantum sphere $SU_q(2)/U(1) = S^2_q$. Global functions on S^2_q can be defined as the subset of $SU_q(2)$-functions with a zero $U(1)$-charge; so we shall consider the $SU_q(2)\times U(1)$-covariant relations for the basic geometrical objects on S^2_q.

Quantum harmonics will be considered as matrix elements $u^i_\pm$ of the $SU_q(2)$-matrix u. An operator of external derivation d_u on $SU_q(2)$ can be decomposed in terms of three invariant operators corresponding to the different generators of a deformed Lie algebra. We discuss the analogous decomposition of Maurer-Cartan equations on $SU_q(2)$.

The deformed harmonic formalism can be used for analysis of the self-duality equation on the quantum Euclidean space $E_q(4)$. The noncommutative coordinates x of $E_q(4)$ satisfy the $SU^L_q(2) \times SU^R_q(2)$-covariant commutation relations. In this approach, quantum harmonics are connected with the left $SU_q(2)$-group.

We use the noncommutative algebra of differential complexes [12]- [14] as a basis of the quantum-group gauge theory. The quantum-group self-duality equation (QGSDE) on $E_q(4)$ can be formulated with the help of a duality operation on the curvature 2-form. We present the deformed analog of the classical BPST-instanton solution.

Quantum harmonics allow us to interpret QGSDE as a zero-curvature equation for some harmonic decomposition of the connection form. We discuss harmonic solutions of QGSDE by analogy with the classical harmonic formalism [4],[5].

2 Quantum harmonics and 3D-differential calculus on the quantum group $SU_q(2)$

We shall use the R-matrix approach [7] for definition of the unitary quantum group $U_q(2) = SU_q(2) \times U(1)$ where q is a real deformation parameter. Let T^i_k $(i,k = 1,2)$ be elements of a quantum matrix T satisfying the standard RTT-relations (in the notations of Ref[14])

$$RTT' = TT'R \qquad (2.1)$$
$$(T)^{ik}_{lm} = T^i_l \delta^k_m, \quad (T')^{ik}_{lm} = \delta^i_l T^k_m$$

The symmetrical $R, \bar{R}$ and $P^{(\pm)}$ matrices obey the following relations

$$R^2 = I + \lambda R, \qquad \bar{R}R = I, \qquad \bar{R} = R - \lambda I \qquad (2.2)$$
$$P^{(+)} + P^{(-)} = I, \quad P^{(a)}P^{(b)} = \delta^{ab}P^{(b)}, \quad R = qP^{(+)} - q^{-1}P^{(-)}$$

where $\lambda = q - q^{-1}$, $a, b = +, -$.

It is convenient to use a covariant expression for the q-generalization of an antisymmetrical symbol

$$\varepsilon_{ik}(q) = \sqrt{q(ik)}\, \varepsilon_{ik} = -q(ik)\varepsilon_{ki}(q) \qquad (2.3)$$
$$q(12) = [q(21)]^{-1} = q, \qquad q(11) = q(22) = 1 \qquad (2.4)$$
$$\varepsilon_{ik}(q)\varepsilon^{kl}(q) = \delta^l_i \, ,$$

where ε_{ik} is an ordinary antisymmetrical symbol $(\varepsilon_{ik} = \varepsilon^{ki})$.

R-matrix elements can be written in terms of δ and $\varepsilon(q)$ symbols

$$R^{ik}_{lm} = q\delta^i_l\delta^k_m + \varepsilon^{ki}(q)\varepsilon_{ml}(q) \qquad (2.5)$$

Eq(1) for the $U_q(2)$ group is equivalent to the following relations:

$$\varepsilon_{ml}(q)T^l_j\, T^m_n = \varepsilon_{nj}(q)\, D(T) \qquad (2.6)$$
$$\varepsilon^{ml}(q)T^i_l\, T^k_m = \varepsilon^{ki}(q)\, D(T) \qquad (2.7)$$

where $D(T) = \mathrm{Det}_q(T)$ is the quantum determinant

$$D(T) = -\frac{q}{1+q^2}\varepsilon_{ki}(q)\varepsilon^{ml}(q)T^i_l\, T^k_m \qquad (2.8)$$

Write also the covariant relations for the inverse quantum matrix $S(T) = T^{-1}$

$$S(T^i_k) = S^i_k = \varepsilon_{kl}(q)\, T^l_j\, \varepsilon^{ji}(q)\, D^{-1}(T) \qquad (2.9)$$
$$S^i_l T^l_k = T^i_l S^l_k = \delta^i_k \qquad (2.10)$$
$$T^l_i \mathcal{D}^m_l(q) S^k_m = \mathcal{D}^k_i(q) = -\varepsilon_{ji}(q)\varepsilon^{jk}(q)$$
$$S^l_i(\mathcal{D}^{-1})^m_l T^k_m = (\mathcal{D}^{-1})^k_i = -\varepsilon_{ij}(q)\varepsilon^{kj}(q)$$

where the notation $\mathcal{D}$ and $\mathcal{D}^{-1}$ for $SU_q(2)$-metrics is introduced.

The unitarity condition for the matrix T can be formulated with the help of involution [7]

$$T^i_k \to \overline{T^i_k} = S^k_i \qquad (2.11)$$

The condition $D(T) = 1$ corresponds to the case of $SU_q(2)$. Let us define quantum harmonics as matrix elements of the $SU_q(2)$-matrix u^i_a. We shall distinguish the upper $SU_q(2)$ index $i = 1, 2$ and low $U(1)$-index $a = +, -$. $SU^L_q(2) \times U(1)$ co-transformations of the harmonics have the following form:

$$u^i_\pm \to l^i_k u^k_\pm \exp(\pm i\alpha) \qquad (2.12)$$

where α is the $U(1)$ parameter and l is the $SU^L_q(2)$-matrix.

Eqs(2.7) for the matrix elements u^i_a are equivalent to the basic relations

$$\varepsilon_{ki}(q)u^i_\pm u^k_\pm = 0 \qquad (2.13)$$
$$\varepsilon_{ki}(q)u^i_a u^k_b = \varepsilon_{ba}(q)\, ,$$
$$\varepsilon^{ba}(q)u^i_a\, u^k_b = \varepsilon^{ki}(q)$$

We shall use the left-covariant 3-dimensional differential calculus [10], [11] for the quantum harmonics. Consider the q-traceless left-invariant 1-forms satisfying the Maurer-Cartan equations

$$\theta^a_b = \bar{u}^a_i du^i_b \qquad (2.14)$$
$$\mathrm{Tr}_q\theta = q\theta^+_+ + q^{-1}\theta^-_- = 0 \qquad (2.15)$$
$$d\theta^a_b = -\theta^a_c\theta^c_b \qquad (2.16)$$

where $\bar{u}^a_i$ are components of the inverse $SU_q(2)$-harmonics.

Introduce the simple $U(1)$ notation

$$\theta_0 = \theta^+_+\, , \quad \theta_{(+2)} = \theta^-_+\, , \quad \theta_{(-2)} = \theta^+_- \qquad (2.17)$$

Consider the left-covariant bilinear relations between harmonics and θ-forms

$$q^{\pm 2}\theta_0 u^i_\pm = u^i_\pm \theta_0$$
$$q^{\pm 1}\theta_{(p)} u^i_\pm = u^i_\pm \theta_{(p)}, \quad p \neq 0 \tag{2.18}$$

These formulas are consistent with Eqs(2.14)-(2.16). Using the standard Leibniz rules for the operator d one can obtain the relations for the θ-forms

$$\theta^2_{(p)} = 0, \quad \theta_{(+2)}\theta_{(-2)} = -q^2\theta_{(-2)}\theta_{(+2)}$$
$$\theta_{(\pm 2)}\theta_0 = -q^{\pm 4}\theta_0\theta_{(\pm 2)} \tag{2.19}$$

Consider the $SU_q(2) \times U(1)$ invariant decomposition of the harmonic external derivative

$$d_u = \delta_0 + \delta + \bar{\delta}$$
$$\delta_0 = \theta_0 D_0, \quad \delta = \theta_{(-2)}D_{(+2)}, \quad \bar{\delta} = \theta_{(+2)}D_{(-2)} \tag{2.20}$$

where D_0 and $D_{(\pm 2)}$ are left-invariant differential operators. Note that the D-operators are generators of the q-deformed Lie algebra [11]

$$q^2 D_{(+2)}D_{(-2)} - D_{(-2)}D_{(+2)} = D_0$$
$$D_0 D_{(+2)} - q^4 D_{(+2)}D_0 = q^2(1+q^2)D_{(+2)}$$
$$D_{(-2)}D_0 - q^4 D_0 D_{(-2)} = q^2(1+q^2)D_{(-2)} \tag{2.21}$$

The standard basis of the universal enveloping algebra $\mathbf{U}_q[SU(2)]$ [6] can be obtained by the nonlinear substitution [11]

$$D_0 = \tfrac{q^2}{1-q^2}(1 - q^{2H})$$
$$D_{(\pm 2)} = q^{H/2}X^{(\pm)} \tag{2.22}$$

The operators δ_0, δ and $\bar{\delta}$ are nilpotent and obey the additional condition

$$\{\delta_0, \delta\} + \{\delta_0, \bar{\delta}\} + \{\delta, \bar{\delta}\} = 0 \tag{2.23}$$

Define the manifest expressions for the action of these operators on quantum harmonics

$$[\delta_0, u^i_+] = u^i_+\theta_0, \quad [\delta, u^i_+] = 0, \quad \bar{\delta}u^i_+ = u^i_-\theta_{(+2)}$$
$$[\delta_0, u^i_-] = -\theta_0 u^i_-, \quad [\delta, u^i_-] = u^i_+\theta_{(-2)}, \quad [\bar{\delta}, u^i_-] = 0 \tag{2.24}$$

An invariant decomposition of the Maurer-Cartan equations on $SU_q(2)/U(1)$ has the following form:

$$d_u\theta_0 = 2\{\delta, \theta_0\} = 2\{\bar{\delta}, \theta_0\} = -\theta_{(-2)}\theta_{(+2)}$$
$$d_u\theta_{(+2)} = 2\{\delta_0, \theta_{(+2)}\} = 2\{\bar{\delta}, \theta_{(+2)}\} = q^2(1+q^2)\theta_0\theta_{(+2)}$$
$$d_u\theta_{(-2)} = 2\{\delta_0, \theta_{(-2)}\} = 2\{\delta, \theta_{(-2)}\} = q^2(1+q^2)\theta_{(-2)}\theta_0 \tag{2.25}$$

Global functions on the quantum sphere $S^2_q = SU_q(2)/U(1)$ satisfy the invariant condition

$$[\delta_0, f(u)] = \theta_0 D_0 f(u) = 0 \tag{2.26}$$

We shall consider also the $U(1)$-charged functions of the harmonics $f_{(p)}(u)$

$$[H, f_{(p)}(u)] = p f_{(p)}(u) \tag{2.27}$$

where p is an integer number.

We shall treat harmonic functions as formal expansions on irreducible harmonic polynomials. The q-symmetrized product of r harmonics u^i_+ and s harmonics u^i_- is the basis of the irreducible $SU_q(2)$-representation with the $U(1)$-charge $p = r - s$

$$\Phi^{(r,s)}(u) = \Phi^{(i_1 \cdots i_{r+s})}(u) = u_+^{(i_1}u_+^{i_2}\cdots u_+^{i_r}u_-^{i_{r+1}}\cdots u_-^{i_{r+s})} = (u_+)^r(u_-)^s \tag{2.28}$$

where $(r,s) = I$ is the q-symmetrized multiindex

$$P^{(+)}_{k,k+1}\Phi^{(r,s)} = q^{-1}R_{k,k+1}\Phi^{(r,s)} = \Phi^{(r,s)} \tag{2.29}$$

Here the R-matrix and the projectional operator $P^{(+)}$ act on the indices i_k and i_{k+1}.

The monomials $\Phi^{(r,s)}$ obey complicated commutation relations depending on the values r, s , so the polynomials $f_{(p)}(u)$ with complex numerical coefficients have not covariant commutation properties. It is useful to extend the algebra of harmonics by adding the set of noncommuting coefficients $C_{(r,s)}$. These coefficients are the components of the covariant neutral harmonic polynomials (covariant q-harmonic fields)

$$F(u) = \sum C_{(r,r)}\Phi^{(r,r)}(u) = \sum C_I\Phi^I \tag{2.30}$$

The bilinear commutation relations between C_I and u follow from the requirement of harmonic commutativity :

$$[u^i_\pm, F(u)] = 0 \tag{2.31}$$

Relations between different coefficients C_I can be obtained , for instance, from the additional assumption of commutativity for the monomials in Eq(2.30). If one has a matrix harmonic field $F_b^a(u)$ satisfying the bilinear relations, then new relations for the corresponding coefficients arise too.

A construction of the differential calculus on covariant harmonic fields includes the relations for the harmonic external derivatives (2.21) and C_I

$$[\delta_0, C_I] = [\delta, C_I] = [\bar{\delta}, C_I] = 0 \qquad (2.32)$$

3 Quantum Euclidean space and quantum self-duality equation

Quantum deformations of the Minkowski and Euclidean 4-dimensional spaces have been considered in Refs[16]-[19]. We shall use the coordinates x^i_α of q-deformed Euclidean space $E_q(4)$ as generators of a noncommutative algebra covariant under the coaction of the quantum group $G_q(4) = SU^L_q(2) \times SU^R_q(2)$

$$x^i_\alpha \to (lxr)^i_\alpha = l^i_k r^\beta_\alpha \otimes x^k_\beta \qquad (3.1)$$

where l and r are quantum matrices of the left and right $SU_q(2)$ groups:

$$R^{ik}_{lm} x^l_\alpha x^m_\beta = x^i_\gamma x^k_\rho R^{\gamma\rho}_{\alpha\beta} \qquad (3.2)$$

$$R\, r\, r' = r\, r'\, R, \qquad R\, l\, l' = l\, l'\, R \qquad (3.3)$$

$$[r, l'] = [r, x'] = [l, x'] = 0, \qquad \mathrm{Det}_q(l) = 1 = \mathrm{Det}_q(r)$$

We use two identical copies of R-matrices for $SU^L_q(2)$ and $SU^R_q(2)$.

The q-deformed central Euclidean interval τ can be constructed by analogy with the quantum determinant

$$\tau(x) = -\frac{q}{1+q^2}\varepsilon^{\beta\alpha}(q)\varepsilon_{ki}(q) x^i_\alpha x^k_\beta \qquad (3.4)$$

We do not consider the quantum-group structure on $E_q(4)$ but we shall apply, the standard formula (2.10) for a definition of the inverse matrix $S(x)$.

It is convenient to use the following $E_q(4)$-involution:

$$\overline{x^i_\alpha} = \varepsilon_{ik}(q) x^k_\beta \varepsilon^{\beta\alpha}(q) = \tau S^\alpha_i(x) \qquad (3.5)$$

$$\bar{\tau} = \tau\,, \quad \overline{\overline{x^i_\alpha}} = x^i_\alpha$$

Let us consider the bicovariant differential calculus on the quantum group $U_q(2)$ [20]-[23]

$$T dT' = R dT T' R \qquad (3.6)$$

$$D(T) dT = q^2 dT\, D(T) \qquad (3.7)$$

$$\omega R \omega + R\omega R\omega R = 0 \qquad (3.8)$$

$$T\omega' = R\omega R T \qquad (3.9)$$

where $\omega^i_k(T) = dT^i_j\, S(T^j_k)$ are the right-invariant differential forms.

The quantum trace ξ of the form ω plays an important role in this calculus

$$\xi(T) = \mathcal{D}^k_i(q)\omega^i_k(T) \neq 0, \quad \xi^2 = 0, \quad d\xi = 0 \qquad (3.10)$$

$$dT = \omega T = (q^2\lambda)^{-1}[T, \xi], \quad q dD(T) = \xi D(T) \qquad (3.11)$$

$$d\omega = \omega^2 = -(q^2\lambda)^{-1}\{\xi, \omega\} \qquad (3.12)$$

All these formulae can be used for a construction of the $G_q(4)$-covariant differential calculus on $E_q(4)$ via the substitution

$$T \to x, \quad dT \to dx, \quad \omega(T) \to \omega(x) = dx\, S(x) \qquad (3.13)$$

The noncommutative algebra of differential complexes [12]-[14] can be used for a consistent formulation of the $U_q(2)$ gauge theory on the quantum space $E_q(4)$. Consider the $U_q(2)$ gauge matrix T^a_b defined on $E_q(4)$. Suppose that Eqs(2.2,3.7 - 3.12) locally satisfy for each "point" x. Coaction of the gauge group $U_q(2)$ on the connection 1-form A^a_b has the following form [12]-[14]:

$$A \to T(x)\, A\, S(T(x)) + dT(x)\, S(T(x)) = T\, A\, S + \omega(T) \qquad (3.14)$$

$$A^a_b = dx^i_\alpha A^{\alpha a}_{ib}(x)$$

The basic commutation relations for the form A are covariant under the gauge transformation

$$A\, R\, A + R\, A\, R\, A\, R = 0 \qquad (3.15)$$

Note that the general relation for A contains a nontrivial right-hand side [14].

The restriction $\alpha = \mathrm{Tr}_q A = 0$ is inconsistent with Eq(3.15), but we can choose the zero field-strength condition $d\alpha = \mathrm{Tr}_q dA = 0$. This constraint for the $U(1)$-gauge field is gauge invariant.

The curvature 2-form is q-traceless for this model

$$F = dA - A^2 = dx^i_\alpha dx^k_\beta F^{\beta\alpha}_{ki}(x) \tag{3.16}$$

Basic 2-forms on $E_q(4)$ can be decomposed with the help of the projectional operators $P^{(\pm)}$ (2.3)

$$\begin{aligned} dx^i_\alpha dx^k_\beta &= [\, P^{(-)} dx dx' P^{(+)} + P^{(+)} dx dx' P^{(-)} \,]^{ik}_{\alpha\beta} = \\ &= \tfrac{q}{1+q^2}[\varepsilon^{ki}(q) d^2 x_{\alpha\beta} + \varepsilon_{\beta\alpha}(q) d^2 x^{ik}] \end{aligned} \tag{3.17}$$

By analogy with the classical case we can treat these two parts as self-dual and anti-self-dual 2-forms under the action of a duality operator $*$.

Let us consider the deformed anti-self-duality equation

$$* F = -F \tag{3.18}$$

We can obtain a 5-parameter solution for the q-deformed anti-self-dual $U_q(2)$-connection [23]:

$$A^a_b = dx^a_\alpha \, \varepsilon_{bk}(q) \, \hat{x}^k_\beta \, \varepsilon^{\beta\alpha}(q)(c + \hat{\tau})^{-1} \tag{3.19}$$

$$\hat{x}^k_\beta = x^k_\beta - c^k_\beta, \quad d\hat{x} = dx, \quad dc = 0 \tag{3.20}$$

$$\begin{gathered} R\,\hat{x}\,\hat{x}' = \hat{x}\,\hat{x}'\,R, \quad R\,c\,c' = c\,c'\,R, \quad c\,x' = R\,x\,c'\,R \\ c\,dx' = R\,dx\,c'\,R, \quad [\hat{x}, \tau(\hat{x})] = 0 \\ \tau(\hat{x}) dx = q^2 dx \tau(\hat{x}) \end{gathered}$$

where c and c^k_β are some "parameters" and a central function $\hat{\tau} = \tau(\hat{x})$ can be defined by substitution $x \to \hat{x}$ in Eq(3.4).

Note that one can treat c as a central periodical function which define a solution of the first-order finite-difference equation: $c(\tau) = c(q^2\tau)$. This solution is a deformed analogue of Belavin-Polyakov-Schwarz-Tyupkin instanton. The multiparameter q-generalization of the 't Hooft solution can be considered too.

4 Harmonic (twistor) interpretation of quantum-group self-duality equation

The QGSD-equation for the field strength has the following form:

$$F^{\beta\alpha}_{ki} = [P^{(+)} F P^{(-)}]^{\alpha\beta}_{ik} = \varepsilon_{ki}(q) F^{\beta\alpha} \tag{4.1}$$

One can obtain the integrability condition multiplying this equation by the product of q-harmonics $u^i_+ u^k_+$.

Let us discuss the covariant formulation of this integrability condition using the deformed harmonic space. It is convenient to introduce new analytic coordinates $x_{\alpha(\pm)}$ for $E_q(4) \otimes_q S^2_q$. One should use the following commutation relations

$$\partial^\alpha_k x^i_\beta = \delta^\alpha_\beta \delta^i_k + R^{ij}_{kl} R^{\alpha\rho}_{\beta\gamma} x^l_\rho \partial^\gamma_j \tag{4.2}$$

$$q \partial^\alpha_i u^l_a = R^{lm}_{ik} u^k_a \partial^\alpha_m \tag{4.3}$$

$$q u^i_a x^k_\beta = R^{ik}_{lm} x^l_\beta u^m_a \tag{4.4}$$

Define the charged analytical coordinates and derivatives and the corresponding commutation relations

$$x_{\alpha a} = \varepsilon_{ab}(q) x^b_\alpha = \varepsilon_{ik}(q) x^k_\alpha u^i_a = -q^2 \varepsilon_{ki}(q) u^i_a x^k_\alpha \tag{4.5}$$

$$R^{cd}_{ab} x_{\alpha c} x_{\beta d} = R^{\gamma\rho}_{\alpha\beta} x_{\gamma a} x_{\rho b} \tag{4.6}$$

$$\partial^\alpha_a = u^i_a \partial^\alpha_i, \quad R^{cd}_{ab} \partial^\alpha_c \partial^\beta_d = R^{\beta\alpha}_{\gamma\rho} \partial^\rho_a \partial^\gamma_b$$

$$\partial^\alpha_a x^b_\beta = \delta^\alpha_\beta \delta^b_a + q^{-1} R^{\alpha\rho}_{\beta\gamma} R^{fb}_{ga} x^g_\rho \partial^\gamma_f \tag{4.7}$$

Note that upper and low indices $a, b \ldots$ have opposite $U(1)$-charges.

Consider the symmetrical decomposition of the external derivative d_x on $E_q(4)$

$$\begin{gathered} d_x = dx^i_\alpha \partial^\alpha_i = \kappa^a_\alpha \partial^\alpha_a = d^a_a = d_1 + d_2 \\ d^2_x = 0, \quad d^a_b = \kappa^a_\alpha \partial^\alpha_b \end{gathered} \tag{4.8}$$

where $\kappa^a_\alpha = \varepsilon^{ab}(q) \kappa_{\alpha b}$ are the covariant analytic 1-forms:

$$\kappa_{\alpha a} = \varepsilon_{ki}(q) dx^i_\alpha u^k_a = dx_{\alpha a} - x_{\alpha b} \theta^b_a \tag{4.9}$$

$$\{d_x, \kappa^a_\alpha\} = 0, \quad \kappa^a_\alpha \kappa^b_\beta = -R^{ba}_{dc} \kappa^c_\gamma \kappa^d_\rho R^{\gamma\rho}_{\alpha\beta} \tag{4.10}$$

$$\partial^\alpha_a \kappa^b_\beta = R^{\alpha\rho}_{\beta\gamma} R^{ga}_{fb} \kappa^f_\rho \partial^\gamma_g$$

It is not difficult to check the following relations:

$$d^2_1 = 0, \quad d^2_2 + \{d_1, d_2\} = 0 \tag{4.11}$$

Stress that $d^2_2 \to 0$ in the limit $q \to 1$.

An analyticity condition for the functions of $x_{\alpha a}$ and u^i_a has manifest solutions Λ depending on the analytical coordinate $x_{\alpha(+)}$

$$\partial^\alpha_+ \Lambda = 0 \Longleftrightarrow d_1\Lambda = 0 \tag{4.12}$$

It should be remarked that the action of the harmonic derivatives δ_0 and δ (2.25) conserves the analyticity

$$\{\delta_0, d_1\}\Lambda = 0, \qquad \{\delta, d_1\}\Lambda = 0 \tag{4.13}$$

Consider a decomposition of the $U_q(2)$-connection in the central basis (CB) (3.15) $A = a_1 + a_2$ corresponding to the decomposition (4.9) where $a_1 = \kappa^+_\alpha A^\alpha_+(x)$ is a connection for the derivative d_1. The quantum-group self-duality equation (4.3) is equivalent to the zero-curvature equation

$$d_1 a_1 - a_1^2 = 0 \tag{4.14}$$

This equation has the following harmonic solution:

$$a_1 = d_1 h\ S(h) = \omega(h, d_1 h) \tag{4.15}$$

where $h(x,u)$ is a "bridge" $U_q(2)$-matrix function. The matrix elements of $h, d_1 h, d_x h$ and $d_u h$ satisfy the relations analogous to Eqs(3.7-3.9). Additional harmonic conditions are

$$\delta_0 h = 0, \quad d\mathrm{Tr}_q \omega(h, dh) = 0 \tag{4.16}$$

where d is a nilpotent operator (d_1, d_x or d_u).

The bridge solution possesses a nontrivial gauge freedom

$$h \to T(x) h \Lambda(x_{(+)}, u)\ , \quad \delta_0\Lambda = d_1\Lambda = 0 \tag{4.17}$$

where Λ is an analytical $U_q(2)$ gauge matrix.

The matrix h is a transition matrix from the central basis to the analytic basis (AB) where d_1 has no connection. Consider formally the decomposition $d = d_x + d_u$ in the CB equations (3.15-3.16) although the CB-harmonic connection is equal to zero ($d_u T = 0 = d_u A$). The bridge transform is a transition to a new u-dependent basis $\mathcal{A}$ in the algebra of $U_q(2)$ differential complexes

$$\mathcal{A} = S(h)Ah - S(h)dh = \tilde{A}_x + V \tag{4.18}$$

$$\tilde{A}_x = S(h)Ah - S(h)d_x h = \kappa^{(-)}_\alpha A^\alpha_- \tag{4.19}$$

$$V = v + \bar{v} = -S(h)d_u h, \quad v = \theta_{(-2)}V_{(+2)}, \quad \bar{v} = \theta_{(+2)}V_{(-2)} \tag{4.20}$$

where $\tilde{A}_x, V, v$ and $\bar{v}$ are the AB-connection 1-forms for the operators d_x, d_u, δ and $\bar{\delta}$ correspondingly.

A general solution of QGSDE can be obtained as a solution of the basic harmonic gauge equation [4],[5]

$$\delta h + hv = \theta_{(-2)}[D_{(+2)}h + hV_{(+2)}] = 0 \tag{4.21}$$

where the connection v contains the analytic prepotential $V_{(+2)}$.

We can discuss also the harmonic equations for the AB-gauge fields by analogy with Refs[24]

$$\partial^\alpha_+ V_{(+2)} = 0, \quad A^\alpha_- = -q^{-2}\partial^\alpha_+ V_{(-2)} \tag{4.22}$$

$$[D_{(+2)} + V_{(+2)}]V_{(-2)} - q^{-2}[D_{(-2)} + V_{(-2)}]V_{(+2)} = 0 \tag{4.23}$$

where $V_{(-2)}$ is the nonanalytic gauge field for $D_{(-2)}$.

One can obtain explicit or perturbative solutions of these equations by using the noncommutative generalizations of classical harmonic expansions and harmonic Green functions [4],[5],[24].

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