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QUANTUM MECHANICAL OBSERVABLES IN GENERAL RELATIVITY

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Квантовомеханические наблюдаемые в общей теории относительности

Обсуждается общая схема формулировки ковариантной квантовой механики в общей теории относительности. Показано, почему при учете хотя бы первых релятивистских поправок нужно специальным образом, из некоторых общерелятивистских выражений, определять операторы наблюдаемых в каждой данной системе отсчета. При этом операторы импульса, координаты и спина не образуют стандартную алгебру Гейзенберга, хотя принцип соответствия, конечно, соблюдается. Каждая полная пространственноподобная гиперповерхность определяет нормальную геодезическую систему отсчета, которая, в свою очередь, определяет пространство волновых функций, допускающих борновскую вероятностную интерпретацию. Такие пространства, относящиеся к существенно разным системам отсчета, некогерентны, и, таким образом, возникает правило суперотбора, связанное не с каким-либо зарядом, а с самим понятием «частица во внешнем гравитационном поле». Высказывается предположение, что указанная некогерентность разрешает известный парадокс потери информации в черных дырах.

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Quantum Mechanical Observables in General Relativity

A general scheme of consecutive and covariant formulation of quantum mechanics in general relativity is dicussed. It is shown, why operators of observables should be defined in a special way from general relativistic relations in each given frame of reference if relativistic corrections are taken into account. The operators of momentum, (curvilinear) coordinates and spin so defined do not form a standard Heisenberg algebra, though the principle of correspondence, of course, is satisfied. Any complete space-like hypersurface determines normal geodesic system, which, in turn, determines a space of wave functions that have Born's probabilistic interpretation. The spaces determined by the essentially different frames are incoherent and so a superselection rule arises that is connected not with a charge, but with the notion itself of «a particle in an external gravitational field». A conjecture is done that this incoherence solves the known paradox of information loss in black holes.

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1.Introduction

A consecutive formulation of quantum mechanics of a particle and more complicated physical systems in the framework of general relativity seems to be interesting both for investigation of quantum effects in an external gravitational field and for more profound insight onto basic concepts of the theory through modification of its geometric background.

The problem does not seem difficult when it concerns the evolution equation and a structure of the space of wave functions. Quantum mechanics in the Schrödinger picture can be considered as an asymptotic theory in c^{-2} , c being the velocity of light, of the theory of field $\phi(x)$ with a mass m and a spin s , in a *riemannian space–time* V_4 with the metric tensor $g_{\alpha\beta}(x)$ of the signature -2 , $\alpha, \beta, \gamma, ... = 0, 1, 2, 3$. The theory so obtained is non–relativistic in the sense that it is asymptotic but having been correctly formulated it permits to take into account relativistic corrections in a form which is general–covariant and identical in a wide class of frames of reference; in the latter sense the theory is general–relativistic. In the present paper a short description of the approach mentioned above will be outlined with a special stress on determination of operators of observables which is not a direct generalisation of notions of the standard theory in R_4 if the relativistic corrections are taken into account. We refer for more details to papers [1, 2] for the case of $s = 0$ and to [3] for $s = 1/2$.

2. Schrödinger Equation and Born Interpretation of Wave Functions in General Relativity

We start with $\phi(x)$, $x \in V_4$ (a scalar field for $s = 0$ and a bispinor for $s = 1/2$) in the form

$$\phi(x) = const \cdot \exp\Big(-i(mc/\hbar)S(x)\Big)\hat{K}(x)\psi(x), \qquad (1)$$

where ψ is a scalar or a (two–component) spinor field, $\hat{K}$ is a transformation of ψ defined below. (Some unessential changes in notation are done in the present paper with respect to [1–3]). Assume also, that the non–exponential part in Eq.(1) is adiabatic, i.e.

$$\mid c^{-1}\partial_\alpha \arg(\hat{K}(x)\psi(x))\ \partial^\alpha S(x) \mid = o(1), \qquad (2)$$

and that $S(x)$ satisfies a Hamilton–Jacobi equation for geodesics of V_4

$$\partial_\alpha S\, \partial^\alpha S = 1, \tag{3}$$

or a modification of this equation including any additional terms at most of order $O(c^{-1})$.

Then, we substitute ϕ into the corresponding field equation in V_4 (we have considered the Dirac equation in the tetrad formalism in [3] and the minimal or non-minimal Klein–Gordon–Fock equation in [1, 2]), introduce the time-like vector field

$$\tau_\alpha \stackrel{def}{=} c\partial_\alpha S, \tag{4}$$

which might be called a normal (almost) geodesic *frame of reference*, and come to a generally covariant Schrödinger or Pauli equation in the form

$$\begin{aligned} i\hbar T\psi &= \Big(-\frac{\hbar^2}{2m}D^2 + \sum_{n=1}^{N} c^{-2n}H_n + O(c^{-2(N+1)})\Big)\psi, \\ &\stackrel{def}{=} H\psi, \end{aligned} \tag{5}$$

where (we write out the expressions for $s = 1/2$ to give a general idea on the structure of the formulae)

$$T \stackrel{def}{=} \big(\tau^\alpha\partial_\alpha + \frac{1}{2}\nabla_\alpha\tau^\alpha\big)\cdot\mathbf{I} + \frac{i}{2}\Omega_\alpha^{(ij)}\varepsilon_{(ijk)}\sigma^{(k)}\tau^\alpha,$$

$$D \stackrel{def}{=} \sigma^{(i)}\ \big(h^\alpha_{(i)}\tilde{\partial}_\alpha + \frac{1}{2}\nabla_\alpha h^\alpha_{(i)}\big) - \frac{i}{4}\Omega_\alpha^{(ij)}\varepsilon_{(ijk)}h^{(k)\alpha}\cdot\mathbf{I},$$

and the following notation is used: $\sigma_{(i)}$ are the Pauli matrices; ∇_α is the covariant derivative; $\mathbf{I}$ is an identity operator; the indices in parentheses mean projections to tetrad basis $h^\alpha{}_{(\beta)}$ with $h^\alpha{}_{(0)} = \partial^\alpha S(x)$ and $i, j, k, ... = 1, 2, 3$; $\varepsilon_{(ijk)}$ is an antisymmetric symbol; $\Omega_\alpha^{\beta\gamma}$ are the Ricci rotation coefficients.

The asymptotic operator $\hat{K}$ can be calculated simultaneously with operators H_n by iterations as a differential operator series in powers of c^{-2} and both the *hamiltonian* H and $\hat{K}$ *contain only derivatives along the space–like hypersurface*

$$S = \{x \in V_4;\ S(x) = const\}.$$

So, when all the mentioned conditions and relations are fulfilled, $\phi(x)$ is an asymptotic solution of the corresponding general–relativistic field equation of the order c^{-2N} , N being the order of iteration in calculation of operators H and $\hat{K}$. *Note that all previous and subsequent essential relations and quantities are scalars with respect to general transformations of coordinates* x^α .

Further, an important step can be done due to that the relations determining these operators leave a freedom in determination of $\hat{K}(x)$. It may be used to fulfill the following asymptotic relations

$$\{\phi_1,\ \phi_2\}_S = (\psi_1,\ \psi_2)_S + O(c^{-2(N+1)}) \tag{6}$$

where

$$\{\phi_1,\ \phi_2\}_S \stackrel{def}{=} \int_S d\sigma^\alpha(x)\ (\overline{\phi_1}\ \partial_\alpha\phi_2 - \overline{\partial_\alpha\phi_1}\ \phi_2) \quad \text{for } s = 0, \tag{7}$$

$$\stackrel{def}{=} \int_S d\sigma^\alpha(x)\ \overline{\phi_1}\ \gamma_\alpha(x)\phi_2 \quad \text{for } s = \frac{1}{2}; \tag{8}$$

$$(\psi_1, \psi_2)_S \stackrel{def}{=} \int_\Sigma d\sigma\ \psi_1{}^\dagger\ \psi_2; \tag{9}$$

and the following notation is used: $d\sigma^\alpha(x)$ and $d\sigma(x)$ are correspondingly normal and volume elements of S ; $\overline{\phi}$ is the ordinary complex $(s = 0)$ or Dirac $(s = 1/2)$ conjugation of ϕ and $\psi^\dagger$ is respectively the ordinary complex or hermitean conjugation of ψ .

So, the generally non–positive–definite bilinear form $\{\cdot,\cdot\}_S$, value of which actually does not depend on choice of the space-like hypersurface S , becomes positive–definite in the space $\Phi_{S,N}$ of asymptotic solutions of the general–relativistic field equation which are represented by solutions $\psi(x)$ of the Schrödinger – Pauli equation 5. We denote the space of wave functions $\psi(x)$ as $\Psi_{S,N}$. As one sees from Eqs.(6), (7) and (8) the general– relativistic "scalar product" $\{\cdot,\cdot\}_S$ induces in $\Psi_{S,N}$ a natural generalization for V_4 of the pre-Hilbert structure of the space of wave functions in the coordinate representation. Of course, $\Psi_{S,N} \supseteq \Psi_{S,N+1}$.

A very important point is that as a consequence of conservation of $\{\cdot,\cdot\}_S$ the hamiltonian H from Eq.(5) is hermitean with respect to the scalar product $(\cdot,\cdot)_S$ in the usual sense, i.e.

$$H = H^\dagger$$

and the system described by a wave function $\psi(x)$ is stable. This is a reason for *the Born probabilistic interpretation* $\psi(x)$. The operator

$\hat{K}$ is still determined up to the multiplication from the right by a unitary operator so that the unitarity of the theory takes place.

3. Operators of observables

To continue the formulation of quantum mechanics in V_4 we should define operators of observables. The conventional postulate of the standard quantum mechanics is that any hermitean (well defined on the state space) operator is an observable and all operators of observables for spinless particle can be realized in configuration representation as functions of operators of (cartesian) spatial coordinates $\hat{x}^i \stackrel{def}{=} x^i \mathbf{I}$ and of their canonically conjugate **momenta** $\hat{p}_i \stackrel{def}{=} -i\hbar\partial_i$. This scheme could be generalized to V_4 as follows.

The operator in the space $\Psi_{S,N}$ of projection of four–momentum on the given unit four–vector field $V^\alpha(x)$ could be defined as

$$\hat{p}_V \stackrel{def}{=} \left(i\hbar V^\alpha \nabla_\alpha + \frac{1}{2}\nabla_\alpha V^\alpha\right)_{i\hbar T=H}, \qquad (10)$$

where the general index $i\hbar T = H$ signifies that it should be used if V^α has a component along τ^α. Due to the hermiticity of H and to the second term at the right hand side of Eq.(10) one has $\hat{p}_V = \hat{p}_V^\dagger$.

As concerns spatial coordinates, we introduce three arbitrary scalar functions $q^A(x)$, $A, B, ... = 1,2,3,$ satisfying the conditions:

$$d\sigma^\alpha \partial_\alpha q^A = 0, \qquad \text{rank } \|\partial_\alpha q^A\| = 3. \qquad (11)$$

Their values determine a position on S, where they can be considered as intrinsic (curvilinear) coordinates. Then the spatial position operator could be defined as

$$\hat{q}^A(x) = q^A(x)\, \mathbf{I} \qquad (12)$$

and its conjugate momentum will be $\hat{p}_{V_A}$, where

$$V^\alpha{}_A \stackrel{def}{=} \omega_{AB}\partial^\alpha q^B, \qquad \omega^{AB} \stackrel{def}{=} \partial^\alpha q^A \partial_\alpha q^B.$$

A fact is, however, that the scheme so described is plausible at most only in the case of $N = 0$, that corresponds to $c^{-1} = 0$, i.e. in the purely non-relativistic theory of a spinless particle. If we are going to take into account relativistic corrections, i.e. consider $N \geq 1$, and spin, we should

expect the corrections not only in H and ψ, but also in $\hat{p}_V$ and $\hat{q}^A$. This is just the point at which the approach of [4] to formulation of quantum mechanics in general relativity seems to be incorrect.

In fact, if we calculate the mean projection on V^α of momentum of the field $\phi \in \Phi_{S,N}$ through the well–known general–relativistic formula

$$P_V(\phi;\ S) = \int_S d\sigma^\alpha V^\beta T_{\alpha\beta}, \qquad (13)$$

where $T_{\alpha\beta}$ is the (metric) energy–momentum tensor of field ϕ, then we shall obtain

$$P_V(\phi;\ S) - (\psi, \hat{p}_V\psi)_S = O(c^{-2}),$$

for $\hat{p}_V$ defined by Eq.(10). It seems natural to assume that the correct value is given by the fundamental formula (13) although it cannot be represented in the form $\{\phi,\ \hat{p}_V\ \phi\}$ except V^α being a Killing field.

As concerns the coordinate operator, the situation is even worse because there is no universally recognized expression analogous to Eq(13). In [1–3] we use the following simplest general–relativistic expression for "meanvalue of spatial coordinate q^A":

$$Q^A(\phi;\ S) = \{\phi, q^A\, \mathbf{I}\ \phi\}, \qquad (14)$$

which is an immediate generalization of the definition used in [5] for R_4 and a cartesian system $\{x^\alpha\}$.

On contrary, an analogous expression for "mean spin tetrad components" $S_{(i)}(\phi; S)$ is well known, but for brevity we restrict ourselves with consideration of operators of momentum and coordinates.

Thus, we should construct operators $\hat{p}_V, \hat{q}^A$ in $\Psi_{S,N}$ which are consistent with the general–relativistic *real quadratic functionals* $P_V(\phi;\ S)$, $Q^A(\phi;\ S)$ on $\Phi_{S,N}$. The prescription is very simple. A real quadratic functional $Z(\phi)$ uniquely defines a *sesquilinear hermitean functional* $Z(\phi_1, \phi_2; S)$ through the known procedure of *polarisation*, see, e.g., [6] :

$$4Z(\phi_1,\phi_2) = Z(\phi_1+\phi_2) - Z(\phi_1-\phi_2) - iZ(\phi_1+i\phi_2) + iZ(\phi_1-i\phi_2). (15)$$

The hermiticity of $Z(\phi_1, \phi_2)$ means that

$$Z(\phi_1,\phi_2) = \overline{Z(\phi_2,\phi_1)}. \qquad (16)$$

If $Z(\phi; S)$ is a local real quadratic functional in the form of an invariant integral over given S as $P_V(\phi;\ S)$, $Q^A(\phi;\ S)$ and $S_{(i)}(\phi; S)$, then it can be represented asymptotically on $\Phi_{S,N}$ as

$$Z(\phi_1, \phi_2;\ S) = (\psi_1,\ \hat{z}(x)\ \psi_2)_S + O(c^{-(2n+2)}), \tag{17}$$

where $\hat{z}(x)$ is a (matrix for $s = 1/2$) differential operator in terms of derivatives along S , and

$$\hat{z}(x) = \hat{z}(x)^\dagger$$

as a consequence of hermiticity of $Z(\phi_1, \phi_2;\ S)$.

Application of the described procedure to $P_V(\phi;\ S)$, in the particular case of $s = 1/2$, $N = 1$ and $V_{(0)} = 0$ gives

$$\hat{p}_{V,N}(x)\mid_{N=1} = \hat{p}_{V,0}(x) + \frac{\hbar^2}{4m^2c^2}[D\ ,\ [D\ ,\ p_{V,0}(x)]] + O(c^{-4}), \tag{18}$$

where

$$p_{V,0}(x) = i\hbar((V^\alpha\partial_\alpha + \frac{1}{2}\nabla_\alpha V^\alpha)\cdot\mathbf{I} - \frac{i}{4}L^{(j)}(V, h^{(i)})\ \varepsilon_{(ijk)}\sigma^{(k)}), \tag{19}$$

$L^{(j)}(V, W)$ being a spatial tetrad component of the Lie derivative of a vector field W in the direction of V .

We see here, that $\hat{p}_V$ differs from the right hand side of Eq.(10) even in the case of $N = 0$ (owing to spin terms) and commutators of projections of momentum on three independent vector fields do not vanish except the fields commute too, what is possible at most in the spatially flat Robertson – Walker space–times.

Similarly, we come from Eq.(14) to

$$\hat{q}^A\mid_{N=1} = q^A\ \mathbf{I}\ +\ \frac{\hbar^2}{8m^2c^2}[D\ ,\ [D\ ,\ q^A)]]\ +\ O(c^{-4}). \tag{20}$$

Here a difference with the naive definition Eq.(12) arises in the order $O(c^{-2})$ (or even higher if $s = 0$ q^A form a harmonic coordinates on S).

Obviously, $[\hat{q}^A(x)\ ,\ \hat{q}^B(x)] = 0$ at least for $N = 0$ it can be easily shown that this is the case if $s = 0$. So in these cases the coordinate operators (together with spin ones if $s = 1/2$) may be used to form a complete set of quantum mechanical observables in contrast with the momentum operators $\hat{p}_V$.

An important fact is that the asymptotic series for $\hat{q}^A(x)$ may be represented in a closed form if $\partial_\alpha S$ is a Killing vector, i.e. V_4 is static. Then, as it is shown in [2], there exist another representation space, not Born intepretable, in which the corresponding coordinate operator is a generalization to curvilinear coordinates and static external gravitation of the well–known Newton–Wigner operator in the case of $s = 0$. The case of $s = 1/2$ has not been investigated yet in this aspect.

At last, an interesting point is that the algebra (if it is an algebra at all) of the operators of momentum and coordiates differs from the direct generalization of the Heisenberg algebra to V_4 on the level of $N = 1$ and this circumstance needs further investigation.

4. Conclusion

So, for any appropriate solution of the Hamilton–Jacobi equation (3) or its proper modification there exists a space $\Phi_{S,N}$ of asymptotic solutions of order N of the general–relativistic field under consideration and its equivalent representation in the space $\Psi_{S,N}$ of solutions ψ of the Schrödinger or Pauli equation (5) that may be considered as having the Born probabilistic interpretation. A class of general–relativistic real quadratic functionals and their polarisations induce hermitean asymptotic operators of observables in each $\Psi_{S,N}$ in an uniform way (but in the form depending on S) and so the general mashinery of quantum mechanics proves to be produced there. It should be emphasized the appearance of the mentioned functionals as general–relativistic pre–images of quantum mechanical matrix elements of observables. This approach as a whole looks as a new sort of quantization, different from the canonical and geometrical ones because it does not lead to the Heisenberg algebra for coordinates and momenta.

A superposition of functions ϕ from different spaces $\Psi_{S,N}$ is , of course, an asymptotic solution of the field equation, but it cannot be expressed as a superposition of the corresponding wave functions which belong to the different spaces. So, the spaces $\Psi_{S,N}$ are *not coherent* in the quantum–mechanical sense and situation is just similar to that with the *superselection rules*, see, e.g., [7]. The one–particle state in a given frame of reference, or more exactly in a class of equivalent frames, has no quantum–mechanical (Born interpretable) sense in another essentially different frame. This assertion does not contradict to that in inertial frames of reference the spaces of one-particle states are coherent. Here the coherence is supported by the Lorentz symmetry.

However, an approximate basis in $\Psi_{S,N}$ can be used for second quantization of the field ϕ and it seems that then one- particle states from a given frame of reference may be represented as superposition of many-particle states in another but this is only a conjecture .

We have **met** in our construction of the quantum mechanics two sorts of incoherence. The first is incoherence due to abcense of **an exact** complete of observables. This incoherence increases with increase of external fields and of velocities of quantum particle. The second is incoherence connected with different frames of reference. A question may be posed: could these **uncertainties** to solve the known paradox [8] of information loss in the thermodynamics of black holes.

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