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ON ONE-LOOP RENORMALIZATION OF BLACK-HOLE ENTROPY

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Фурсаев Д.В., Солодухин С.Н. E2-94-462
Об однопетлевой перенормировке энтропии черной дыры

Доказано, что однопетлевые расходимости, возникающие в энтропии квантовой черной дыры, полностью устраняются стандартной перенормировкой гравитационной постоянной и других коэффициентов при R^2-членах в эффективном гравитационном действии. Существенным моментом при доказательстве является то, что благодаря слагаемым высшего порядка по кривизне энтропия отличается от энтропии Бекенштейна-Хокинга в эйнштейновской теории гравитации поправками, зависящими от внутренней и внешней геометрии поверхности горизонта.

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Fursaev D.V., Solodukhin S.N. E2-94-462
On One-Loop Renormalization of Black-Hole Entropy

One-loop divergences appearing in the entropy of a quantum black hole are proven to be completely eliminated by the standard renormalization of both the gravitational constant and other coefficients by the R^2-terms in the effective gravitational action. The essential point of the proof is that due to the higher order curvature terms the entropy differs from the Bekenstein-Hawking one in the Einstein gravity by the contributions depending on the internal and external geometry of the horizon surface.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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That there are analogs of the thermodynamical laws for classical black holes is a remarkable and a long-known fact (a recent review of this issue is [1].) The central point of this analogy is the entropy identified with the area of the black-hole horizon. Now it is also well-known that for a quantum black hole its entropy additionally acquires the ultraviolet divergent corrections, S_{div}, concentrated on the horizon surface [2]. The same feature inherent in the "entanglement" entropy appearing in ordinary quantum theory under tracing out a part of the pure state residing inside a region of space [3]. Analogously, one can interpret S_{div} as the entanglement entropy to be related with the loss, for an external observer, of information about field excitations located inside the horizon [4]. The physical reason for the surface ultraviolet divergences to occur in this way is correlations across the horizon between inside and outside quantum fluctuations.

It has shortly been realized that for the Rindler space-time, where the divergences scale simply as the horizon area, they are eliminated from the entropy together with renormalization of the gravitational constant [5], [6]. However, for the Schwarzschild black hole S_{div} is not reduced only to the horizon area [7]. In this case an addition in S_{div} is removed by the renormalization of a gravitational coupling at the R^2-term necessarily generated in the effective action by quantum corrections [7]. The complete form of the divergent terms appearing on the horizon surface has been derived explicitly for arbitrary static black-hole geometry in [8]. This allowed to find out all divergent corrections to black hole entropy [9], [10].

The aim of our Letter is to demonstrate, using these results, that in the general case *all* one-loop divergences S_{div} are removed from the entropy under standard renormalization of the constants in the effective gravitational action, including couplings at the second order curvature terms. To put it in another way, we show that the bare tree-level and "entanglement", S_{div}, pieces of the black hole entropy appear in such a combination to reproduce tree-level entropy expressed through the renormalized constants.

So far as in quantum theory in curved space-time quantum corrections are known to result in higher order curvature contributions to the Einstein action [11], we begin our consideration with the following action functional:

$$W = \int \sqrt{g} d^4x \left(-\frac{1}{16\pi G} R + a_1 R^2 + a_2 R^{\mu\nu} R_{\mu\nu} + a_3 R^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho} \right) \tag{1}$$

where G and a_i are the bare gravitational couplings. Due to topological properties, only two of these couplings are independent in the four-dimensional theory.

We will follow the Gibbons-Hawking path integral approach to the gravitational thermodynamics [12] to give to our analysis a transparent statistical meaning. In this approach the functional (1), being considered on a Euclidean section $\mathcal{M}_\beta$ of the corresponding space-time with period β in time, is associated, in the semiclassical approximation, with the tree-level free energy of the system at temperature $T = \beta^{-1}$

$$F(\beta) = \beta^{-1} W(\beta) \quad . \tag{2}$$

Nothing unusual happens as compared to thermodynamics in the Minkowsky space when space-time possesses a globally defined time-like Killing vector field which is not null anywhere. A non-trivial point appears in the presence of the Killing horizon, as in the case of a black hole geometry. In this case, for arbitrary temperature β^{-1} the Euclidean manifold $\mathcal{M}_\beta$ has conical singularities at the horizon surface Σ, in the vicinity of which it looks topologically as a space product $C_\beta \times \Sigma$ of a two-dimensional cone C_β and the horizon surface Σ. This leads to a specific Hawking temperature $\beta^{-1} = \beta_H^{-1}$ for which the Euclidean manifold is regular. The black-hole thermodynamics is considered at this temperature. However, to get the entropy using the standard definition

$$S(\beta_H) = \left(\beta \frac{\partial}{\partial \beta} - 1 \right) W(\beta)|_{\beta=\beta_H} \tag{3}$$

we must let β be slightly different from β_H. This procedure being applied to the action (1) faces a difficulty due to the terms of higher order in curvature which turn out to be ill-defined on the conical singularities.

There is a method how to avoid this problem [13] when one approximates $\mathcal{M}_\beta$ by a sequence of smooth manifolds $\tilde{\mathcal{M}}_\beta$ converging to $\mathcal{M}_\beta$. For the "regularized" spaces $\tilde{\mathcal{M}}_\beta$ the action (1) is well-defined and in the limit $\tilde{\mathcal{M}}_\beta \to \mathcal{M}_\beta$ we get the following formulas [13]

$$\int_{\mathcal{M}_\beta} R = \alpha \int_{\mathcal{M}_{\beta_H}} R + 4\pi(1-\alpha) \int_\Sigma \quad , \tag{4}$$

$$\int_{\mathcal{M}_\beta} R^2 = \alpha \int_{\mathcal{M}_{\beta_H}} R^2 + 8\pi(1-\alpha) \int_\Sigma R + O((1-\alpha)^2) \quad , \tag{5}$$

$$\int_{\mathcal{M}_\beta} R^{\mu\nu} R_{\mu\nu} = \alpha \int_{\mathcal{M}_{\beta_H}} R^{\mu\nu} R_{\mu\nu} + 4\pi(1-\alpha) \int_\Sigma R_{\mu\nu} n_i^\mu n_i^\nu + O((1-\alpha)^2) \quad , \tag{6}$$

$$\int_{\mathcal{M}_\beta} R^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho} = \alpha \int_{\mathcal{M}_{\beta_H}} R^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho} + 8\pi(1-\alpha) \int_\Sigma R_{\mu\nu\lambda\rho} n_i^\mu n_i^\lambda n_j^\nu n_j^\rho + O((1-\alpha)^2) \quad , \tag{7}$$

where $\alpha = \beta/\beta_H$ and n_i^μ are two orthonormal vectors orthogonal to Σ. The first integrals in right part of (4)-(7) are defined on the smooth space at $\beta = \beta_H$; they are proportional to β and do not affect $S(\beta_H)$. As for the terms $O((1-\alpha)^2)$ in (5)-(7), they depend on the regularization prescription and turn out to be singular in the limit $\tilde{\mathcal{M}}_\beta \to \mathcal{M}_\beta$, but they do not contribute into the entropy and energy at the Hawking temperature ($\alpha = 1$). Indeed, from (4)-(7) one obtains for S the following integral over the horizon surface Σ:

$$S(G, a_i) = \lim_{\tilde{\mathcal{M}}_\beta \to \mathcal{M}_\beta} \left(\beta \frac{\partial}{\partial \beta} - 1 \right) W(\tilde{\mathcal{M}}_\beta)(G, a_i)|_{\beta=\beta_H}$$

$$= \frac{1}{4G} A_\Sigma - \int_\Sigma \left(8\pi a_1 R + 4\pi a_2 R_{\mu\nu} n_i^\mu n_i^\nu + 8\pi a_3 R_{\mu\nu\lambda\rho} n_i^\mu n_i^\lambda n_j^\nu n_j^\rho \right) \tag{8}$$

where A_Σ is the horizon area. Remarkably, this expression differs from the Bekenstein-Hawking entropy $S = A_\Sigma / 4G$ in the Einstein gravity by the contributions depending on both internal and external geometry of the horizon due to the high curvature terms in (1). However, it is easy to see that the effect of internal geometry of Σ is reduced to the integral curvature of this surface which, being a topological invariant, is an irrelevent constant addition to the entropy. It is worth noting that exactly the same expression can be derived by the Noether charge method suggested by Wald [14]. A difference between two approaches is that Wald's method seems to be more general, but it is defined "on-shell", whereas the above derivation of (8) did not operate with the equations of motion.

Consider now quantum theory on the black-hole background. For a massive scalar field the one-loop effective action reads

$$W_{eff} = W + \frac{1}{2} \log \det(-\Box + m^2) \quad . \tag{9}$$

To define this action on the singular manifold $\mathcal{M}_\beta$, we make use of the same procedure going to $\tilde{\mathcal{M}}_\beta$. On the smoothed space W_{eff} consists of a finite W_{fin} and an ultraviolet divergent W_{div} parts. So far as the latter has the same structure as the bare functional

(1), the divergences are taken off by the standard renormalization of the gravitational couplings G and a_i [11]

$$W(\tilde{\mathcal{M}}_\beta)(G,a_i) + W_{div}(\tilde{\mathcal{M}}_\beta)(\epsilon) = W(\tilde{\mathcal{M}}_\beta)(G^{ren}, a_i^{ren}) \tag{10}$$

where G^{ren} and a_i^{ren} are renormalized couplings expressed through the bare ones and an ultraviolet cut-off parameter ϵ.

According to (4)-(7), W_{div} generates additional surface ultraviolet divergent terms in the limit $\tilde{\mathcal{M}}_\beta \to \mathcal{M}_\beta$. However, one should expect that in this limit the finite part W_{fin} also results in the surface divergences, as has been demonstrated in two-dimensional case where W_{fin} is exactly known [7]. In the four-dimensional theory the total exact structure of the surface divergent terms $W_{div}^{exact}(\mathcal{M}_\beta)$ has been found in [8]. For the action (9) comparison of both the results at $\beta \simeq \beta_H$, the details of which we omit here, can be expressed as follows:

$$\lim_{\tilde{\mathcal{M}}_\beta \to \mathcal{M}_\beta} W_{div}(\tilde{\mathcal{M}}_\beta) = W_{div}^{exact}(\mathcal{M}_\beta) + O((1-\alpha)^2) \quad . \tag{11}$$

The last term in the right-hand side of (11) comes from the finite part of $W_{eff}(\tilde{\mathcal{M}}_\beta)$ in the limit $\tilde{\mathcal{M}}_\beta \to \mathcal{M}_\beta$. This shows that the surface divergences of the order $(1-\alpha)$ are completely removed by the standard renormalization of the gravitational constants (10). However, to get rid off the divergent terms of higher order in $(1-\alpha)$ one is forced to introduce the surface counterterms additional to those we have in the regular case [10].

The consequence of equation (11) is that the entropy at the Hawking temperature does not acquire additional divergences apart from the standard ones removed by renormalization of G and a_i. Indeed, from (11) for the divergent part of S, caused by correlations between field fluctuations inside and outside the horizon, one has

$$S_{div}(\epsilon) \equiv \left(\beta\frac{\partial}{\partial\beta} - 1\right) W_{div}^{exact}(\mathcal{M}_\beta)(\epsilon)|_{\beta=\beta_H} =$$

$$\left(\beta\frac{\partial}{\partial\beta} - 1\right) \lim_{\tilde{\mathcal{M}}_\beta \to \mathcal{M}_\beta} W_{div}(\tilde{\mathcal{M}}_\beta)(\epsilon)|_{\beta=\beta_H} \quad . \tag{12}$$

Finally, taking into account (8), (12) and (10) the renormalization of the entropy can be presented as follows:

$$S(G,a_i) + S_{div}(\epsilon) = S(G^{ren}, a_i^{ren}) \quad . \tag{13}$$

Here $S(G^{ren}, a_i^{ren})$ has the form (8) expressed through G^{ren} and a_i^{ren} related with the bare constants by the usual equations originated from the one-loop renormalization (10) in quantum theory on space-times without horizons.

The equation (13) proves the main statement of this Letter. Following from (13) is a consequence that the finite observed entropy of a hole $S(G^{ren}, a_i^{ren})$ always comes out as a combination of the tree-level bare entropy $S(G,a_i)$ and $S_{div}(\epsilon)$ interpreted as quantum "entanglement" entropy. Thus, if the gravitational action is totally induced by quantum effects, then $S(G^{ren}, a_i^{ren})$ is "purebred entanglement entropy" [6]. Besides, we see from (13) that all the dependence of $S_{div}(\epsilon)$ on the number of field species is absorbed into the observable constants G^{ren} and a_i^{ren}.

For simplicity we derived (13) for the scalar model (9), but the effect of higher spins can also be incorporated in our analysis. A special treatment, however, is needed for the case of non-zero curvature coupling $\xi R\phi^2$ in the scalar Lagrangian. It should also be noted that our result concerns the static black holes and the extension on the stationary geometries is of interest as well.

To conclude, the following remarks are in order. Generally speaking, there are two ways to compute the quantum corrections to the entropy of a black hole. Here we considered the statistical-mechanical derivation based on equation (3). The other approach to this problem is to infer the Hawking temperature, entropy, etc. from the metric of a hole which takes into account the back reaction effect caused by the quantum matter [15]. In this case the source of quantum corrections to the black-hole geometry is the renormalized stress tensor $< T_{\mu\nu} >_{ren}$ computed in the Hartle-Hawking vacuum. No ultraviolet divergences appear in such an approach in addition to those removed from $< T_{\mu\nu} >$ by the renormalization of the gravitational couplings G and a_i. One can expect that the "statistical" and "geometrical" methods give similar results. From this point of view the absence of additional divergences in the black-hole entropy S, eq. (13), is not surprising.

In this Letter we concerned the divergent corrections to the entropy. However, it is worth pointing out that also finite quantum corrections to S, that result in its deviation from the tree-level form (8), are of great interest. Some information about these can be extracted from the two-dimensional models where the analysis indicates the terms logarithmically depending on the mass of the hole [7]. The analogous terms in four dimensions might be important for understanding the thermodynamics of quantum holes.

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