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дубна

E2-94-444

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CLASSICAL DYNAMICS OF ROTATING RELATIVISTIC STRING WITH MASSIVE ENDS: THE REGGE TRAJECTORIES AND QUARK MASSES

Submitted to «Nuclear Physics B»

1994

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Классическая динамика вращающейся релятивистской струны с массивными концами: траектории Редже и массы кварков

Динамические уравнения релятивистской струны с точечными массами на концах формулируются в терминах геометрических инвариантов мировых траекторий массивных концов струны (кривизна k_i и кручение $\kappa_i(\tau)$, $i = 1,2$ траекторий). Эти инварианты дают возможность воспроизвести мировую поверхность струны с точностью до ее положения в пространстве Минковского. Кручение $\kappa_i(\tau)$ подчиняется системе дифференциальных уравнений второго порядка, k_i — постоянные. Случай постоянных κ_i детально исследуется. В этом случае мировая поверхность струны есть геликод в пространстве E_2^1. Нелинейные соотношения (траектории Редже) между угловым моментом системы J и квадратом массы M выводятся и обсуждаются. Для данной массы M (масса мезона) и данного J (спин мезона) вычисляются массы кварков.

Работа выполнена в Лаборатории теоретической физики им. Н.Н.Боголюбова ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 1994

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Classical Dynamics of Rotating Relativistic String with Massive Ends: the Regge Trajectories and Quark Masses

Dynamic equations in the theory of a relativistic string with point masses at the ends are formulated in terms of geometric invariants of the world trajectories of the massive ends of the string (curvature k_i and torsion $\kappa_i(\tau)$, $i = 1,2$ of the trajectories). With these characteristics we reproduce the string world surface up to its position in Minkowski space E_2^1. The torsions $\kappa_i(\tau)$, $i = 1,2$ obey a system of second order differential equations with delay arguments describing the retardation effects of the interaction of masses through the string, k_i being constants. The constant torsions are investigated in detail. In this case the string world sheet is a helicoid in E_2^1. A nonlinear relation (the Regge trajectory) between the angular momentum of the system, J, and the mass squared, M^2, is derived. For given meson masses (M) and spin (J), the masses of quarks are calculated.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 1994

1 Introduction

Interest in the string theory in hadronic physics is inspired by a striking analogy between the open relativistic string with masses at the ends and a quark-antiquark pair linked together by a QCD flux tube [1]. Until now much attention has been paid to solve exactly the classical equations describing the dynamics of this system [2–9], but only the well-known particular solution has been found when a straight-line string with spinless massive ends rotates in a given plane, which corresponds to the constant torsions κ_i of the massive ends of the string. In this case the string world sheet is a helicoid in the three-dimensional space-time [8]. In paper [9], the dynamic equations (boundary conditions) in the theory of a relativistic string with point masses at the ends have been formulated in terms of geometric invariants of world lines of point masses m_i $(i = 1, 2)$ at the string ends. It has been shown that the string variables in three-dimensional space-time are completely defined by the constant curvatures $k_i = \frac{\gamma}{m_i}$, where γ is the string tension and the torsions $\kappa_i(\tau)$, $i = 1, 2$ of the endpoint trajectories.

In the present article, an example of the straight-like string (the case of constant torsions $\kappa_i(\tau) = \kappa_{0i}$) is investigated in detail. A nonlinear relation between J (the angular momentum of the system) and $E = M$ (the energy of the system) is obtained. For given meson masses M and spins J, the masses of quarks m_i are calculated.

2 Equations of motion and boundary conditions

Consider the dynamics of a relativistic string with point masses m_i $(i = 1,2)$ at the ends. The world sheet with coordinates $x^\mu(\tau,\sigma)$ $(\mu = 0,1,2)$ swept out by the string in 3-dimensional Minkowski space-time is an extremal of the functional of the action [1].

$$S = -\gamma \int_{\tau_1}^{\tau_2} \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} \sqrt{(\dot{x}\acute{x})^2 - \dot{x}^2\acute{x}^2}\, d\tau\, d\sigma + \sum_{i=1}^{2} m_i \int_{\tau_1}^{\tau_2} \sqrt{\left(\frac{dx^\mu(\tau,\sigma_i(\tau))}{d\tau}\right)^2}\, d\tau. \tag{2.1}$$

where

$$\dot{x}^\mu = \frac{\partial x^\mu(\tau,\sigma)}{\partial \tau}, \quad \acute{x}^\mu = \frac{\partial x^\mu(\tau,\sigma)}{\partial \sigma},$$

$$\frac{dx^\mu(\tau,\sigma_i(\tau))}{d\tau} = \dot{x}^\mu(\tau,\sigma_i(\tau)) + \dot{\sigma}_i(\tau)\,\acute{x}^\mu(\tau,\sigma_i(\tau)), \quad i = 1,2.$$

The motion of the string endpoints in the plane of the parameters $\tau,\sigma = u_k$ $(k = 1,2)$ is described by the functions $\sigma_i(\tau)$, $i = 1,2$. As for a massless string, the action (2.1) is invariant under the changes of variables, $\tilde{\tau} = \tilde{\tau}(\tau,\sigma)$ and $\tilde{\sigma} = \tilde{\sigma}(\tau,\sigma)$, which allows us to eliminate any two of the three independent component of the world sheet metric $g_{kl} = \eta_{\mu\nu}\frac{\partial x^\mu}{\partial u^k}\frac{\partial x^\nu}{\partial u^l}$, where the flat Minkowski metric $\eta_{\mu\nu}$ of the enveloping 3-dimensional space-time E^1_{d-1} has the signature (+,-,-). It is convenient to introduce isothermal coordinates τ and σ in terms of which the metric is diagonal and traceless (orthonormal gauge)

$$g_{00} + g_{11} = 0,\; g_{01} = g_{10} = 0, \quad or \quad \dot{x}^2 + \acute{x}^2 = 0;\; (\dot{x}\acute{x}) = 0. \tag{2.2}$$

Variation of the action (2.1) with respect to $x^\mu(\tau,\sigma)$ gives equations of motion linear in the gauge (2.2)

$$\ddot{x}^\mu(\tau,\sigma) - x''^\mu(\tau,\sigma) = 0 \tag{2.3}$$

and nonlinear boundary conditions at the string ends:

$$m_i \frac{d}{d\tau}\left[\frac{\dot{x}^\mu(\tau,\sigma_i) + \dot{\sigma}_i(\tau)\,\acute{x}^\mu(\tau,\sigma_i)}{\sqrt{(1-\dot{\sigma}_i^2(\tau))\,\dot{x}^2(\tau,\sigma_i)}}\right] = (-)^{i+1}\gamma\left[\acute{x}^\mu(\tau,\sigma_i) + \dot{\sigma}_i(\tau)\,\dot{x}^\mu(\tau,\sigma_i)\right]. \tag{2.4}$$

A general solution to the equations of motion (2.3) and gauge conditions (2.2) has the form [8,9]

$$x^\mu(\tau,\sigma) = \frac{1}{2}\left[\psi^\mu_+(u^+) + \psi^\mu_-(u^-)\right], \quad u^+ = \tau+\sigma, \quad u^- = \tau-\sigma, \tag{2.5}$$

where $\psi'^\mu_\pm(u^\pm)$ are two isotropic vectors: $\psi'^2_\pm(u^\pm) = 0$ and can be represented in the following form:

$$\begin{aligned}\psi'^\mu_+(u^+) &= \frac{A_+(u^+)}{f'(u^+)}\left[a^\mu + \frac{1}{2}c^\mu f^2(u^+) + b^\mu f(u^+)\right],\\ \psi'^\mu_-(u^-) &= \frac{A_-(u^-)}{g'(u^-)}\left[a^\mu + \frac{1}{2}c^\mu g^2(u^-) + b^\mu g(u^-)\right],\end{aligned} \tag{2.6}$$

where the constant basis $\{a^\mu, b^\mu, c^\mu\}$ is formed of two isotropic vectors a^μ, c^μ, $a^2 = 0$, $c^2 = 0$, $(ac) = 1$ and a space-like vector b^μ, $b^2 = -1$, $(ab) = (cb) = 0$. The functions $f(u^+)$ and $g(u^-)$ are an arbitrary ones and must be defined by boundary conditions. The representations (2.6) fully define the world surface of a relativistic string in 3-dimensional Minkowski space-time and allow us to construct its basic quadratic forms. Non-zero components of metric tensor are given by

$$g_{00} = -g_{11} = \dot{x}^2(\tau,\sigma) = \frac{A_+(u^+)A_-(u^-)}{4f'(u^+)g'(u^-)}\left[f(u^+) - g(u^-)\right]^2. \tag{2.7}$$

As is known [1], the Gauss equation for the world surface of the string $x^\mu(\tau,\sigma)$ reduces to the Liouville equation for $g_{00} = \dot{x}^2(\tau,\sigma)$

$$\frac{\partial^2 \ln \dot{x}^2(u^+,u^-)}{\partial u^+ \partial u^-} = \frac{A_+(u^+)A_-(u^-)}{2\dot{x}^2(u^+,u^-)} \tag{2.8}$$

and (2.7) is the general solution to this equation.

Let us procced to the boundary conditions (2.4). For each of value $i = 1,2$, only 2 of the 3 equations (2.4) are independent of each other since the projections of this system onto the tangent vectors $\dot{x}^\mu(\tau,\sigma)$ and $\acute{x}^\mu(\tau,\sigma)$ coincide. Thus, 4 independent equations of the system (2.4) contain, 6 unknown quantities: two functions $\sigma_i(\tau)$ and 4 independent components of the vectors $\psi'^\mu_\pm$ expressed, according to (2.6), through $A_\pm$, f, g. That indefiniteness is a consequence of the invariance of equations (2.4) and (2.8) under the conformal transformation of the parameters $\tilde{\tau} \pm \tilde{\sigma} = u(\tau \pm \sigma)$. So, the definition of this system may be supplemented by imposing two auxiliary conditions:

$$(\ddot{x}(\tau,\sigma) \pm \dot{x}'(\tau,\sigma))^2 = -A^2 = \text{const}. \tag{2.9}$$

In terms of $\psi'^\mu_\pm$ it means that $\psi''^2_\pm(\tau \pm \sigma) = -A^2$, or according to (2.6)

$$A_+(\tau \pm \sigma) = A_-(\tau \pm \sigma) = A = \text{const} \tag{2.10}$$

Eqs. (2.9) are the gauge conditions which completely fix the coordinate system τ,σ on the string world sheet. This choice of gauge has the simple geometric meaning. Indeed, having fixed $A_\pm$ according to (2.10), we obtain the coefficients of the second quadratic form b_{kl} in the following form

$$b_{00} = b_{11} = \frac{A_+ - A_-}{2} = 0, \quad b_{01} = b_{10} = \frac{A_+ + A_-}{2} = A. \tag{2.11}$$

Geometrically [10], this means that the orthonormal coordinates (2.2) are at the same time the asymptotic lines on the string world surface. The gauge (2.9) fixes the functions $\sigma_i(\tau)$ in eqs. (2.4). In fact, projecting (2.4) onto normals n^μ and taking account of $b_{kl} = \left(n\frac{\partial^2 x}{\partial u^k \partial u^l}\right)$ we get the equation:

$$\left(1 + \dot{\sigma}_i^2(\tau)\right) b_{00} + 2\dot{\sigma}_i(\tau)\, b_{01} = 0. \tag{2.12}$$

With (2.11) it turns into $\acute{\sigma}_i(\tau)=0$, $i=1,2$. and we put $\sigma_1=0$ and $\sigma_2=l$. Now taking advantage of the conformal gauge (2.2) and equations of motion (2.3) it is easy to show that the projections (2.4) onto $\dot{x}^\mu(\tau,\sigma_i)$ vanish, and projections onto $\acute{x}^\mu(\tau,\sigma_i)$ give only two equations for boundaries:

$$\frac{\partial}{\partial\sigma}\left(\frac{1}{\sqrt{\dot{x}^2(\tau,\sigma)}}\right)_{|\sigma=\sigma_i} = (-1)^{i+1}\frac{\gamma}{m_i} \qquad (i=1,2). \tag{2.13}$$

Inserting the general solution (2.7) of the Liouville equation (2.8) into (2.13) we obtain the system of two differential equations for the functions $f(\tau)$ and $g(\tau)$

$$\frac{d}{d\tau}\ln\frac{g'(\tau)}{f'(\tau)}+2\frac{f'(\tau)+g'(\tau)}{f(\tau)-g(\tau)} = \frac{\gamma}{m_1}|A|\frac{|f(\tau)-g(\tau)|}{\sqrt{f'(\tau)g'(\tau)}} \tag{2.14}$$

$$\frac{d}{d\tau}\ln\frac{g'(\tau-l)}{f'(\tau+l)}+2\frac{f'(\tau+l)+g'(\tau-l)}{f(\tau+l)-g(\tau-l)} = -\frac{\gamma}{m_2}|A|\frac{|f(\tau+l)-g(\tau-l)|}{\sqrt{f'(\tau+l)g'(\tau-l)}} \tag{2.15}$$

where A is an arbitrary constant.

In the Minkowski space E_2^1 we shall describe the world trajectories of the massive string endpoints in terms of two geometric invariants, curvature k_i and torsion κ_i $(i=1,2)$. As is well known [10], these characteristics uniquely define a curve in a three-dimensional space up to its position. In refs. [8,9] one demonstrated that the world lines $x^\mu(\tau,\sigma_i)$ have the constant curvatures $k_i=\frac{\gamma}{m_i}$ $i=1,2$. and their torsions are given by the expressions

$$\kappa_1(\tau) = \frac{4f'(\tau)g'(\tau)}{A\left[f(\tau)-g(\tau)\right]^2}, \tag{2.16}$$

$$\kappa_2(\tau) = \frac{4f'(\tau+l)g'(\tau-l)}{A\left[f(\tau+l)-g(\tau-l)\right]^2} \tag{2.17}$$

By using these formulas together with eqs. (2.14) and (2.15) the functions $f(\tau)$ and $g(\tau)$ can be expressed [9] in terms of the curvatures k_i and torsions $\kappa_i(\tau)$ as follows

$$D(f(\tau)) = D\left(\int^\tau\sqrt{A\kappa_1(\eta)}\,d\eta\right)+\frac{A\kappa_1(\tau)}{2}\left(1-\frac{k_1^2}{\kappa_1^2(\tau)}\right)-2k_1\frac{d}{d\tau}\sqrt{\frac{A}{\kappa_1(\tau)}} =$$

$$= D\left(\int^{\tau-l}\sqrt{A\kappa_2(\eta)}\,d\eta\right)+\frac{A\kappa_2(\tau-l)}{2}\left(1-\frac{k_2^2}{\kappa_2^2(\tau-l)}\right)+2k_2\frac{d}{d\tau}\sqrt{\frac{A}{\kappa_2(\tau-l)}} \tag{2.18}$$

$$D(g(\tau)) = D\left(\int^\tau\sqrt{A\kappa_1(\eta)}\,d\eta\right)+\frac{A\kappa_1(\tau)}{2}\left(1-\frac{k_1^2}{\kappa_1^2(\tau)}\right)+2k_1\frac{d}{d\tau}\sqrt{\frac{A}{\kappa_1(\tau)}} =$$

$$= D\left(\int^{\tau+l}\sqrt{A\kappa_2(\eta)}\,d\eta\right)+\frac{A\kappa_2(\tau+l)}{2}\left(1-\frac{k_2^2}{\kappa_2^2(\tau+l)}\right)-2k_2\frac{d}{d\tau}\sqrt{\frac{A}{\kappa_2(\tau+l)}} \tag{2.19}$$

Here $D(f(\tau))$ stands for the Schwarz derivative:

$$D(f(\tau)) = \frac{f'''(\tau)}{f'(\tau)}-\frac{3}{2}\left(\frac{f''(\tau)}{f'(\tau)}\right)^2 = -2\sqrt{f'(\tau)}\frac{d^2}{d\tau^2}\left(\frac{1}{\sqrt{f'(\tau)}}\right)$$

According to (2.5) and (2.6), it follows from (2.18) and (2.19) that the string coordinates $x^\mu(\tau\,\sigma)$ are completely defined by k_i and $\kappa_i(\tau)$, $i=1,2$ of the world trajectories $x^\mu(\tau,\sigma_i)$ of massive string endpoints.

3 The case of constant torsions

We have the simplest solution to eqs. (2.18) and (2.19) for the constant torsions $\kappa_i(\tau)=\kappa_{0i}$, when the string massive ends are moving along helices. In this case, the equations (2.18) and (2.19) take the form

$$D(g(\tau)) = D(f(\tau)) = \frac{A\kappa_{01}}{2}\left(1-\frac{k_1^2}{\kappa_{01}^2}\right) = \frac{A\kappa_{02}}{2}\left(1-\frac{k_2^2}{\kappa_{02}^2}\right), \tag{3.1}$$

$$\frac{A\kappa_{0i}}{2}\left(1-\frac{k_i^2}{\kappa_{0i}^2}\right) = \frac{\omega^2}{2},$$

where ω is a constant which, as will be seen below, can be put to equal to A. Then from the second eq. of (3.1) if $A=\omega$ it follows that:

$$\kappa_{0i} = \frac{\gamma}{m_i}\left(\sqrt{\left(\frac{\omega m_i}{2\gamma}\right)^2+1}+\frac{\omega m_i}{2\gamma}\right), \quad k_i=\frac{\gamma}{m_i}. \tag{3.2}$$

The first eq. of (3.1) connects $f(\tau)$ and $g(\tau)$: $D(f(\tau)) = D(g(\tau))$, therefore these two functions are related by the Moebius transformation:

$$g(\tau)=\frac{af(\tau)+b}{cf(\tau)+d}, \quad g'(\tau)=\frac{f'(\tau)}{(cf(\tau)+d)^2}, \tag{3.3}$$

$$ad-cb=1.$$

Inserting this expression for $g(\tau)$ into the first eq. of (2.14) we obtain for $\rho=\frac{a+d}{2}$ that

$$\rho = \frac{k_1}{\kappa_{01}}\varepsilon\left(\frac{2f'(\tau)}{cf^2(\tau)+(d-a)f(\tau)-b}\right),$$

where $\varepsilon(x)$ is a sign function

$$\varepsilon(x)=\begin{cases}+1, & \text{if } x>0\\ -1, & \text{if } x<0\end{cases}$$

We take the constants $c>0$ and $|\frac{a+d}{2}|=|\rho|<1$, therefore, the polynomial in the argument of the $\varepsilon-$ function is always positive. Further, we set $f'(\tau)>0$, which implies, according to (3.3), that $g'(\tau) > 0$ and the argument of the $\varepsilon-$ function is positive. As a result, using (3.2), we have

$$\rho = \frac{a+d}{2} = \frac{k_1}{\kappa_{01}} = \sqrt{\left(\frac{\omega m_1}{2\gamma}\right)^2 + 1} - \frac{\omega m_1}{2\gamma} < 1. \tag{3.4}$$

Setting the function $f(\tau)$ to be equal to

$$cf(\tau) + d = \rho + \sqrt{1-\rho^2}\tan\frac{\omega\tau}{2}, \quad f'(\tau) = \frac{\omega\sqrt{1-\rho^2}}{2c\cos^2\frac{\omega\tau}{2}}, \tag{3.5}$$

obeying eq. (3.1) $D(f(\tau)) = \frac{\omega^2}{2}$ and the condition $f'(\tau) > 0$, we derive, from (3.3), the following expression for $g(\tau)$:

$$cg(\tau) - a = \frac{-1}{\rho + \sqrt{1-\rho^2}\tan\frac{\omega\tau}{2}}, \quad g'(\tau) > 0. \tag{3.6}$$

that also satisfies the equation $D(g(\tau)) = \frac{\omega^2}{2}$. Now we turn to the second boundary condition (2.15). Without loss of generality, we assume that $a = d = \rho < 1$ and, instead of (3.5) and (3.6), we obtain:

$$f(\tau) = \frac{\sqrt{1-\rho^2}}{c}\tan\frac{\omega\tau}{2}; \quad g(\tau) = \frac{\sqrt{1-\rho^2}}{c}\,\frac{\rho\tan\frac{\omega\tau}{2} - \sqrt{1-\rho^2}}{\sqrt{1-\rho^2}\tan\frac{\omega\tau}{2} + \rho}. \tag{3.7}$$

By shifting arguments in the second of eqs. (2.15) by $(-l)$, we arrive at the equation

$$\frac{d}{d\tau}\ln\frac{g'(\tau-2l)}{f'(\tau)} + 2\,\frac{f'(\tau)+g'(\tau-2l)}{f(\tau)-g(\tau-2l)} = -k_2\omega\frac{|f(\tau)-g(\tau-2l)|}{\sqrt{f'(\tau)g'(\tau-2l)}}$$

Inserting the expression for $f(\tau)$ and

$$g(\tau-2l) = \frac{\sqrt{1-\rho^2}}{c}\,\frac{(\rho\cos\omega l - \sqrt{1-\rho^2}\sin\omega l)\tan\frac{\omega\tau}{2} - (\rho\sin\omega l + \sqrt{1-\rho^2}\cos\omega l)}{(\sqrt{1-\rho^2}\cos\omega l + \rho\sin\omega l)\tan\frac{\omega\tau}{2} + (\rho\cos\omega l - \sqrt{1-\rho^2}\sin\omega l)}$$

into it, we obtain the expression, without the time parameter τ, connecting the parameters ρ (3.4), ω, l and $k_2 = \frac{\gamma}{m_2}$

$$\left(\sqrt{1-\rho^2}\sin\omega l - \rho\cos\omega l\right)^2 + \frac{\omega m_2}{\gamma}\left(\sqrt{1-\rho^2}\sin\omega l - \rho\cos\omega l\right) = 1$$

from which we find

$$\sqrt{1-\rho^2}\sin\omega l - \rho\cos\omega l = \sqrt{\left(\frac{\omega m_2}{2\gamma}\right)^2 + 1} - \frac{\omega m_2}{2\gamma}.$$

Setting $\rho = \sin\phi_0$, we have

$$\sin(\omega l - \phi_0) = \sqrt{\left(\frac{\omega m_2}{2\gamma}\right)^2 + 1} - \frac{\omega m_2}{2\gamma}.$$

Since $\phi_0 = \arcsin\left(\sqrt{\left(\frac{\omega m_1}{2\gamma}\right)^2 + 1} - \frac{\omega m_1}{2\gamma}\right)$, according to (3.4), we can then determine the dimensional parameter l:

$$l = \frac{1}{\omega}\sum_{i=1}^{2}\arcsin\left(\sqrt{\left(\frac{\omega m_i}{2\gamma}\right)^2 + 1} - \frac{\omega m_i}{2\gamma}\right) \tag{3.8}$$

Using (2.6) at $A(u^+) = A(u^-) = A = \omega$,

$$f(u^+) = \frac{\sqrt{1-\rho^2}}{c}\tan\frac{\omega u^+}{2}, \quad g(u^-) = -\frac{\sqrt{1-\rho^2}}{c}\cot\left(\frac{\omega u^-}{2} + \phi_0\right)$$

and taking the vectors a^μ, b^μ, c^μ in the form:

$$a^\mu = \frac{\sqrt{1-\rho^2}}{2c}\{1,0,1\}, \quad b^\mu = \{0,1,0\}, \quad c^\mu = \frac{c}{\sqrt{1-\rho^2}}\{1,0,-1\},$$

we obtain

$$\psi'^\mu_+(\tau+\sigma) = \{1, \sin\omega(\tau+\sigma), \cos\omega(\tau+\sigma)\},$$
$$\psi'^\mu_-(\tau-\sigma) = \{1, -\sin\omega(\tau-\sigma+2\phi_0), -\cos\omega(\tau-\sigma+2\phi_0)\}.$$

Hence, it follows that if we put $A = \omega$, then $\tau = t$ because

$$\dot{x}^\mu(\tau,\sigma) = \frac{\psi'^\mu_+(\tau+\sigma) + \psi'^\mu_-(\tau-\sigma)}{2}, \quad \acute{x}^\mu(\tau,\sigma) = \frac{\psi'^\mu_+(\tau+\sigma) - \psi'^\mu_-(\tau-\sigma)}{2}$$

take the form

$$\dot{x}^\mu(\tau,\sigma) = \{1, \sin(\omega\sigma - \phi_0)[\cos(\omega\tau+\phi_0), -\sin(\omega\tau+\phi_0)]\},$$
$$\acute{x}^\mu(\tau,\sigma) = \{0, \cos(\omega\sigma - \phi_0)[\sin(\omega\tau+\phi_0), \cos(\omega\tau+\phi_0)]\}.$$

and finally we get

$$x^\mu(\tau,\sigma) = \left\{\tau, \frac{\sin(\omega\sigma-\phi_0)}{\omega}[\sin(\omega\tau+\phi_0), \cos(\omega\tau+\phi_0)]\right\}, \tag{3.9}$$

Then from (3.9) it follows that $\tau = t$. The physical picture given by this solution is clear: the string as a piece of a straight line rotates with the angular velocity ω. The linear velocities of the massive string ends can be determined from (3.9) as follows:

$$\vec{v_1} = \dot{\vec{x}}(t,0) = \sin\phi_0[-\cos(\omega t+\phi_0), \sin(\omega t+\phi_0)],$$

$$\vec{v_2} = \dot{\vec{x}}(t,l) = \sin(\omega l - \phi_0)\,[\cos(\omega t + \phi_0), -\sin(\omega t + \phi_0)]\,.$$

The absolute value of $\vec{v}_i$ is equal to

$$v_1 = |\sin\phi_0| = \sqrt{\left(\frac{\omega m_1}{2\gamma}\right)^2 + 1} - \frac{\omega m_1}{2\gamma}\,,$$
$$v_2 = |\sin(\omega l - \phi_0)| = \sqrt{\left(\frac{\omega m_2}{2\gamma}\right)^2 + 1} - \frac{\omega m_2}{2\gamma}\,. \qquad (3.9a)$$

Now we can determine R_i, the distances from the center of rotation of the string to its ends

$$R_i = \frac{v_i}{\omega} = \frac{1}{\omega}\left[\sqrt{\left(\frac{\omega m_i}{2\gamma}\right)^2 + 1} - \frac{\omega m_i}{2\gamma}\right]. \qquad (3.10)$$

Equation (3.10) gives the connection between R_1 and R_2 in the following form

$$\gamma R_1\,(m_1 + \gamma R_1) = \gamma R_2\,(m_2 + \gamma R_2) = \frac{\gamma^2}{\omega^2}. \qquad (3.11)$$

This equations in terms of v_i takes the form:

$$\frac{v_1^2}{R_1}(m_1 + \gamma R_1) = \frac{v_2^2}{R_2}(m_2 + \gamma R_2) = \gamma\,. \qquad (3.12)$$

It has a simple physical meaning: the string tension γ balances the contrifugal forces of rotating masses $\frac{v_i^2}{R_i}m_i$ and the masses γR_i of rotating string pieces $\frac{v_i^2}{R_i}\gamma R_i$, linking m_i with the center of string rotation.

4 Energy and angular momentum of a rotating string

In the gauge, we have chosen $\tau = t$, the Lagrangian of the system is given by the expression:

$$L = -\gamma\int_0^l \sqrt{(\vec{x}_t\vec{x}_\sigma)^2 + \vec{x}_\sigma^2(1-\vec{x}_t^2)}\,d\sigma - \sum_{i=1}^2 m_i\sqrt{1-\vec{x}_t^2(t,\sigma_i)}\,. \qquad (4.1)$$

whereas the orthonormal gauge conditions (2.2) in terms of momenta become [1]

$$\vec{p}^2(t,\sigma) + \gamma^2\vec{x}_\sigma^2(t,\sigma) = \gamma^2;\quad \vec{p}(t,\sigma) = \gamma\vec{x}_t(t,\sigma)\,. \qquad (4.2)$$

The string Hamiltonian is equal to

$$H_{st} = \int_0^l \sqrt{(\vec{p}^2(t,\sigma) + \gamma^2\vec{x}_\sigma^2(t,\sigma)}\,d\sigma = \gamma l \qquad (4.3)$$

and the Hamiltonian of the masses m_i at the ends of the string is given by

$$H_m = \sum_{i=1}^2 \frac{m_i}{\sqrt{1-\vec{x}_t^2(t,\sigma_i)}} = \sum_{i=1}^2 \frac{m_i}{\sqrt{1-v_i^2}}\,. \qquad (4.4)$$

The energy of the system can be expressed [1–5] as the value of the total Hamiltonian on the solution (3.9)

$$E = \gamma l + \sum_{i=1}^2 \frac{m_i}{\sqrt{1-v_i^2}}\,. \qquad (4.5)$$

Using (3.8) and (3.9a) we can express the energy only in terms of linear velocities of the ends of the string v_i and masses m_i or in terms of the angular velocity of the string ω and m_i:

$$E = \sum_{i=1}^2\left[\frac{\gamma}{\omega}\arcsin v_i + \frac{m_i}{\sqrt{1-v_i^2}}\right] = \qquad (4.6)$$
$$= \sum_{i=1}^2\left[\frac{\gamma}{\omega}\arcsin\left(\sqrt{\left(\frac{\omega m_i}{2\gamma}\right)^2 + 1} - \frac{\omega m_i}{2\gamma}\right) + \frac{m_i}{\sqrt{\frac{\omega m_i}{\gamma}\left(\sqrt{\left(\frac{\omega m_i}{2\gamma}\right)^2 + 1} - \frac{\omega m_i}{2\gamma}\right)}}\right].$$

In the nonrelativistic case, $v_i \ll 1$, from the first equation (4.6) we get

$$E \simeq \sum_{i=1}^2\left(\frac{\gamma}{\omega}v_i + m_i + \frac{m_i v_i^2}{2}\right)$$

and taking into account that $v_i = \omega R_i$ we arrive at the linearly growing potential

$$E = \gamma(R_1 + R_2) + \sum_{i=1}^2\left(m_i + \frac{m_i v_i^2}{2}\right), \qquad (4.7)$$

where $R = R_1 + R_2$ is the length of the straight-line string.

For the total angular momentum we obtain the following expression

$$J = \int_0^l (xp_y - yp_x)\,d\sigma + \sum_{i=1}^2 (x_i p_{y_i} - y_i p_{x_i}) =$$
$$= \frac{1}{2\omega}\sum_{i=1}^2\left[\frac{\gamma}{\omega}\arcsin v_i + \frac{m_i v_i^2}{\sqrt{1-v_i^2}}\right]. \qquad (4.8)$$

Comparing the expressions for E (4.6) and for J (4.8) we find first that their difference is equal to

$$E - 2\omega J = \sum_{i=1}^2 m_i\sqrt{1-v_i^2} = \sum_{i=1}^2 m_i\sqrt{\frac{\omega m_i}{\gamma}\left(\sqrt{\left(\frac{\omega m_i}{2\gamma}\right)^2 + 1} - \frac{\omega m_i}{2\gamma}\right)}, \qquad (4.9)$$

tends to zero both when $m_i \to 0$ and when $\omega \to 0$ in both the cases $v_i \to 1$. Second, these expressions result in the inequality:

$$J < \alpha' E^2, \text{ if } m_i \neq 0, \tag{4.10}$$

where $\alpha' = \frac{1}{2\pi\gamma}$ is the Nambu slope of Regge trajectories. Indeed, using expressions (4.6) and (4.8), we determine the difference:

$$E^2 - 2\gamma J \left(\sum_{i=1}^{2} \arcsin v_i\right) =$$

$$= \frac{\gamma}{\omega}\left(\sum_{i=1}^{2} \arcsin v_i\right) \sum_{i=1}^{2} m_i \left(\frac{1}{\sqrt{1-v_i^2}} + \sqrt{1-v_i^2}\right) + \left(\sum_{i=1}^{2} \frac{m_i}{\sqrt{1-v_i^2}}\right)^2 > 0,$$

when $m_i \neq 0$, from which it follows that

$$J < \frac{E^2}{2\pi\gamma}\left(\frac{\pi}{\sum_{i=1}^{2} \arcsin v_i}\right), \text{ if } m_i \neq 0;$$

and since

$$\frac{\pi}{\sum_{i=1}^{2} \arcsin v_i} \leq 1,$$

inequality (4.10) holds true. At $m_i = 0$ we have $v_i = 1$ and the equality $J = \alpha' E^2$ is valid exactly.

If consider the case of small quark masses $m_i \ll \frac{2\gamma}{\omega}$, then from (3.9a) we obtain for the velocities $v_i \simeq 1 - \frac{\omega m_i}{2\gamma}$, which means that v_i is close to the velocity of light. Upon simple calculations we obtain

$$J = \alpha' E^2 \left(1 - \sqrt{\pi}\frac{m_1^{3/2} + m_2^{3/2}}{E^{3/2}}\right), \tag{4.11}$$

which is consistent with inequality (4.10).

This formula allows us to calculate the masses of quarks m_i by using $E = M$ (meson mass) and J (meson spin) from the Particle Data for mesons with equal or close quark masses $m_1 \approx m_2$. From (4.11) we have

$$m_1^{3/2} + m_2^{3/2} = \frac{1}{\pi}\left[E^{3/2} - \frac{J}{\alpha' E^{1/2}}\right]. \tag{4.11a}$$

The corresponding results are listed in Table 1. A good agreement is observed with the mass values of current quarks m_c and m_b, however, for the π^+ meson, the mass $m_{u,d}$ is very small. For vector mesons ρ^+ and K^{**}, the calculated quark masses turn out to be negative since for them $E^{3/2} < \frac{J}{\alpha' E^{1/2}}$. Thus, for heavy mesons we obtain satisfactory values of m_q.

In the other limiting case when the masses of qurks are large $m_i \gg \frac{2\gamma}{\omega}$, from (3.9a) we get $v_i \simeq \frac{\gamma}{m_i\omega} \ll 1$ and

$$J = \alpha'(E - m_1 - m_2)^{3/2} \pi \sqrt{\frac{m_1 m_2}{m_1 + m_2}}. \tag{4.12}$$

In this case the contribution of masses to E and J is much large than that from the string $E - m_1 - m_2 \ll m_1 + m_2$.

One interesting case is that when $m_1 = m_2 = \gamma R$, and therefore from (3.10) $R_1 = R_2 = \frac{1}{\sqrt{2}\omega}$, $v = \omega R = \frac{1}{\sqrt{2}} = \frac{m\omega}{\gamma}$. From (4.6) and (4.8) we obtain:

$$E = \frac{\pi + 4}{\sqrt{2}} m \quad \gamma\pi J = \frac{2\pi + \pi^2}{2} m^2, \text{ then } 2\gamma\pi J = \frac{(2\pi + \pi^2)^2}{(\pi + 4)^2} E^2,$$

and finally:

$$J = \alpha'(0.6)E^2 \tag{4.13}$$

with inequality (4.10) being valid.

Another interesting class of mesons (heavy–light mesons) is when one of the quarks (m_1) becomes very massive, specifically $m_1 \gg m_2$ and $m_1 \gg \frac{2\gamma}{\omega}$, $m_2 \ll \frac{2\gamma}{\omega}$. Then from (3.9a) we have $v_1 \simeq \frac{\gamma}{\omega m_1} \ll 1$, $v_2 = 1 - \frac{\omega m_2}{\gamma}$ is very close to the velocity of light. Let us analyse the calculation in this case in more detail. Using (4.9) we can express $\frac{\omega}{\gamma}$ through E, J, m_i, γ. Indeed,

$$E - 2\gamma J\frac{\omega}{\gamma} \simeq m_1 + m_2\sqrt{\frac{m_2\omega}{\gamma}},$$

then $\frac{\omega}{\gamma} \simeq \frac{E - m_1}{2\gamma J}$. From (4.6) E can be determined as follows

$$E = m_1 + \frac{\pi\gamma}{2\omega} + \sqrt{\frac{m_2\gamma}{\omega}} \simeq m_1 + \frac{\pi\gamma J}{E - m_1} + m_2^{1/2}\sqrt{\frac{2\gamma J}{E - m_1}}.$$

Finally:

$$\pi\gamma J = (E - m_1)^2 - \sqrt{\frac{2m_2}{\pi}}(E - m_1)^{3/2}. \tag{4.14}$$

Thus, the first term of this formula describes the dependence of J on E at a small mass m_2 (4.11), whereas the second term with opposite sign corresponds to formula (4.12) for large masses m_i. However, in the case under consideration, as it has been mentioned in ref. [12], the slope for heavy–light mesons is double the Nambu slope $2\alpha'$. Indeed, from (4.14) it can be obtained as follows

$$J = 2\alpha'(E - m_1)^2 \left(1 - \sqrt{\frac{2m_2}{\pi(E - m_1)}}\right). \tag{4.15}$$

To conclude this consideration, we present the expressions for E (4.6) and J (4.8) as functions of R_i and v_i^2. Expanding the arcsin v_i in a power series of $v_i = \sqrt{\frac{\gamma R_i}{m_i+\gamma R_i}}$, we obtain

$$E = \gamma(R_1 + R_2) + \sum_{i=1}^{2}\left[\gamma R_i \sum_{k=1}^{\infty} \frac{(2k)!v_i^{2k}}{2^{2k}(k!)^2(2k+1)} + \frac{m_i}{\sqrt{1-v_i^2}}\right]. \qquad (4.16)$$

The first term is a linearly growing potential and the second one is a correction to it in powers of v_i^2. For J we have

$$J = \frac{\gamma}{2\omega}(R_1 + R_2) + \frac{1}{2\omega}\sum_{i=1}^{2}\left[\gamma R_i \sum_{k=1}^{\infty} \frac{(2k)!v_i^{2k}}{2^{2k}(k!)^2(2k+1)} + \frac{m_i v_i^2}{\sqrt{1-v_i^2}}\right]. \qquad (4.17)$$

Here also the infinite sum over degrees of v_i^2 gives a correction to the leading first term. In nonrelativistic limit we get (4.7) and

$$J = \frac{\gamma}{2\omega}(R_1 + R_2) + \frac{1}{2\omega}\sum_{i=1}^{2}\left(\gamma R_i \frac{v_i^2}{3} + m_i v_i^2\right).$$

In Fig. 1, the numerical computation is plotted for the dependence of J on E^2 for various ratios of masses $\frac{m_2}{m_1} = a$. It is clear that the curves in Fig. 2 have the asymptotic behaviour $J = \alpha' E^2$.

5 Conclusion

The geometrical method propose here for solving the boundary problem in the theory of the relativistic string with massive ends is essentially based on the geometrical notion of torsions κ_i of world trajectories of the massive string ends. The world surface of a string with massive ends is completely defined by trajectories of the massive ends. These trajectories are characterized by geometrical invariants k_i and $\kappa_i(\tau)$. In our case $\kappa_i(\tau) = \kappa_0 -$ const. and the boundary curves are helices, the string world surface is a helicoid. The solution derived here for a rotating straight–line string reproduces all the results found in the flux tube model [12], the so–called yrast solution that here follows automatically from (3.10)–(3.12) without the auxiliary condition

$$\gamma R_i = \frac{m_i v_i^2}{1 - v_i^2},$$

resulting from (3.12) and covering the cases of equal and different masses m_i. Quasi-classical quantization of this system according to N. Bohr: $J^2 = \hbar(j+1)j$ gives the energy spectrum of the system E_j, however, we should know the analytic dependence of E on J determined by an equation of the fourth order in E for which it is very difficult to find physically reasonable roots but this equation can be solved numerically with the use of Fig. 2. A drawback of the latter solution is the absence of quark radial

Table 1

$$m_1^{3/2} + m_2^{3/2} = \frac{1}{\sqrt{\pi}}\left[E^{3/2} - \frac{J}{\alpha'} E^{-1/2}\right]$$

$m_{1,2}$ — mass quarks ($m_{1,2} < E$)
$E = M$ — mass mesons J — spin mesons
$\alpha' = 0.18$

mesons	J	M GeV	m_q GeV	m_q Particle Data
$\pi^+ = u\bar{d}$	0	0.14	0.06	$m_{ud} \sim 0.35$
$\rho^+ = u\bar{d}$	1	0.768	< 0	
$J/\psi' = c\bar{c}$	1	3.097	1.24	$1.3 < m_c < 1.7$
$\psi'' = c\bar{c}$	1	3.69	1.5	
$\psi''' = c\bar{c}$	1	3.77	1.53	
$\psi = c\bar{c}$	1	4.03	1.64	
$\chi_0 = c\bar{c}$	0	3.45	1.73	
$\chi = c\bar{c}$	1	3.51	1.41	

	J	M	m_q	Particle Data
$B^- = b\bar{u}$	0	5.23	$m_b = 6.84$	$4.7 \leq m_b \leq 5.3$
$B^{-*} = b\bar{u}$	1	5.32	$m_b = 6.66$	
$D_s^+ = c\bar{s}$	0	1.97	$m_c = 1.56$	$1.3 \leq m_c \leq 1.7$
$D_s^{+*} = c\bar{s}$	1	2.11	$m_c = 1.29$	
$K^+ = u\bar{s}$	0	0.49	$m_s = 0.19$	$0.1 < m_s < 0.3$
$K^{+*} = u\bar{s}$	1	0.89	$m_s = < 0$	

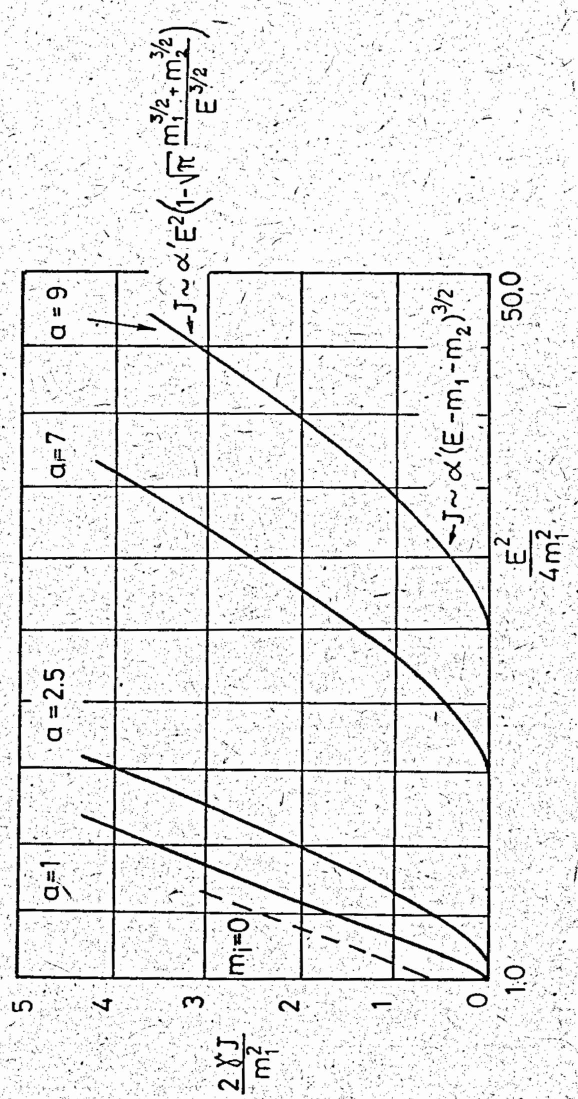

Fig.1

motions since at a given angular velocity ω the masses m_i move along circles of radius R_i. This solution cannot pretend to be a meson realistic model, however, from Table 1 it is seen that the masses m_i calculated by formula (4.11a) are in good agreement with experimental data on current masses of heavy quarks.

Acknowledgments.

The author expresses his gratitude to V.V. Nesterenko, A.M. Chervyakov and A.L. Koshkarov for useful discussions and help in numerical computations. The work is supported by the Russian Science Foundation under grant ℵ93 – 02 – 03972.

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Received by Publishinf Department
on November 18, 1994.