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Некоторые новые интегрируемые уравнения из уравнений автодуальности модели Янга–Миллса

Используя редукцию по подгруппам группы симметрии уравнений автодуальности модели Янга–Миллса в (2 + 2) измерениях, мы вводим новые интегрируемые уравнения, являющиеся «деформацией» уравнений следующих моделей: киральной модели в (2 + 1) измерениях, обобщенной нелинейной модели Шредингера, Кортевега–де Фриза, Эйлера–Арнольда, обобщенной модели Калоджеро–Мозера и Эйлера–Калоджеро–Мозера. Пары Лакса для всех этих уравнений получены редукцией по симметриям пары Лакса для уравнений автодуальности модели Янга–Миллса.

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Some New Integrable Equations from the Self-Dual Yang-Mills Equations

Using the symmetry reductions of the self-dual Yang-Mills (SDYM) equations in (2 + 2) dimensions, we introduce new integrable equations which are «deformations» of the chiral model in (2 + 1) dimensions, generalized nonlinear Schrödinger, Korteweg-de Vriez, Toda lattice, Garnier, Euler-Arnold, generalized Calogero-Moser and Euler-Calogero-Moser equations. The Lax pairs for all of these equations are derived by the symmetry reductions of the Lax pair for the SDYM equations.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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1. Introduction

The purpose of this paper is to describe eight new systems of differential equations and to write out the Lax pairs for them. We derive equations for all these integrable systems using the method of symmetry reduction (see [1, 2] and references therein) applied to the self-dual Yang-Mills (SDYM) equations in the space $R^{2,2}$ with the metric of the signature $(++--)$. For derivation of the Lax pairs for these equations we use the algorithm of reduction of the Lax pair for the SDYM equations described in [3].

We use the SDYM equations in $R^{2,2}$ and the symmetry reduction method only as a tool for obtaining new integrable systems in lower dimensions, but there are at least three reasons in view of which the connection between these integrable systems and the SDYM equations is important. Firstly, the importance of the SDYM equations in $R^{2,2}$ is motivated by the conjecture [4] that the SDYM equations may be a universal integrable system, i.e. that all integrable equations in $1 \leq d \leq 3$ dimensions can be obtained from it by suitable reductions. In fact, it has recently been shown that many integrable equations can be embedded into the SDYM equations [4–14]. It is obvious that besides the known equations, the symmetry reductions of the SDYM equations give the opportunity to obtain some new integrable equations valuable for applications. In the following, we illustrate this by deriving "deformations" of the equations mentioned in the abstract. Under the deformations of some equations we mean equations which coincide with the initial ones up to some additional terms. Secondly, to the equations derived from the SDYM equations, one may apply the twistor techniques for solving equations and for analysing properties of solutions (see, e.g., [15, 16, 10, 11]). Thirdly, the SDYM equations are known to arise in the $N=2$ supersymmetric string theory [17, 18] which is considered now as the universal string theory including the conventional $N=0$ and $N=1$ strings as particular vacua [19, 20]. Therefore, the soliton-type solutions of the SDYM equations and their reductions are important for the analysis of nonperturbative effects in string theories.

2. Definitions and notation

We consider the space $R^{2,2}$ with the metric $(g_{\mu\nu}) := diag(+1,+1,-1,-1)$ and the potentials A_μ of the Yang-Mills (YM) fields $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, where $\mu, \nu, ... = 1,...,4$, $\partial_\mu = \partial/\partial x^\mu$. Fields A_μ and $F_{\mu\nu}$ take values in the Lie algebra $gl(n, C)$.

In $R^{2,2}$ we introduce null coordinates $t = \frac{1}{2}(x^2 - x^4), u = \frac{1}{2}(x^2 + x^4), y = \frac{1}{2}(x^1 - x^3), z = \frac{1}{2}(x^1 + x^3)$ and set $A_t = A_2 - A_4,\ A_u = A_2 + A_4,\ A_y = A_1 - A_3,\ A_z = A_1 + A_3$. The

SDYM equations in the null coordinates have the following form:

$$F_{tz} = 0, \quad F_{uy} = 0, \quad F_{tu} + F_{zy} = 0. \tag{1}$$

Equations (1) can be obtained as compatibility conditions of the following linear system of equations (cf. ref. [21, 22]):

$$(\partial_t - \lambda\partial_y + A_t - \lambda A_y)\Psi = 0, \tag{2a}$$

$$(\partial_z + \lambda\partial_u + A_z + \lambda A_u)\Psi = 0, \tag{2b}$$

$$\partial_{\bar\lambda}\Psi = 0, \tag{2c}$$

where $\bar\lambda$ is a complex conjugate to λ. Here Ψ is a column vector depending on the coordinates of $R^{2,2}$ and the "coordinates" $\lambda, \bar\lambda$, parametrizing the upper sheet of the hyperboloid $H^2 = SO(2,1)/SO(2)$. Notice that Ψ is defined on the twistor space $\mathcal{Z} = R^{2,2} \times H^2$ for the space $R^{2,2}$, and eqs.(2) mean the holomorphicity of the vector-function Ψ (Ward theorem [22]).

3. Symmetry reduction

We consider the inhomogeneous group of rotations $ISO(2,2)$ (rotations and translations) and an arbitrary subgroup G of the group $ISO(2,2)$. We would like to impose the conditions of G-invariance on the YM potentials A_μ and on the vector-function Ψ. For that, we have to define the generators of the group $ISO(2,2)$ as vector fields on $R^{2,2}$ when considering the action of G on A_μ, and as vector fields on the twistor space $R^{2,2} \times H^2$ when considering the action of G on Ψ [3].

Let us introduce the following constant tensors:

$$f^a_{\mu\nu} = \{f^a_{bc},\ \mu = a, \nu = b;\ \delta^a_\mu, \nu = 4;\ -\delta^a_\nu, \mu = 4\}, \quad I_a{}^\mu{}_\nu = -\frac{1}{2} g_{ab} g^{\mu\lambda} f^b_{\lambda\nu}, \tag{3a}$$

$$\bar f^a_{\mu\nu} = \{f^a_{bc},\ \mu = a, \nu = b;\ -\delta^a_\mu, \nu = 4;\ \delta^a_\nu, \mu = 4\}, \quad J_a{}^\mu{}_\nu = -\frac{1}{2} g_{ab} g^{\mu\lambda} \bar f^b_{\lambda\nu}, \tag{3b}$$

where $a, b, ... = 1, 2, 3$, $g_{11} = g_{22} = -g_{33} = 1$ and $f^1_{23} = f^2_{31} = -f^3_{12} = 1$ are the structure constants of the group $SO(2,1)$. Then, the generators of the group $ISO(2,2)$ can be realized in terms of the following vector fields on $R^{2,2}$:

$$X_a = I_a{}^\mu{}_\nu x^\nu \partial_\mu, \quad Y_a = J_a{}^\mu{}_\nu x^\nu \partial_\mu, \quad P_\mu = \partial_\mu. \tag{4}$$

The vector fields on $\mathcal{Z} = R^{2,2} \times H^2$, which also form the generators of $ISO(2,2)$, are given by

$$\tilde X_a = X_a, \quad \tilde Y_a = Y_a + Z_a, \quad \tilde P_\mu = P_\mu, \tag{5a}$$

with the following expression of the generators Z_a of the $SO(2,1)$-rotations on H^2:

$$Z_1 = \frac{1}{2}[(1-\lambda^2)\partial_\lambda + (1-\bar\lambda^2)\partial_{\bar\lambda}],\ Z_2 = -[\lambda\partial_\lambda + \bar\lambda\partial_{\bar\lambda}],\ Z_3 = -\frac{1}{2}[(1+\lambda^2)\partial_\lambda + (1+\bar\lambda^2)\partial_{\bar\lambda}]. \tag{5b}$$

It can be easily shown that $[X_a, X_b] = f^c_{ab} X_c$, $[Z_a, Z_b] = f^c_{ab} Z_c$, $[\tilde Y_a, \tilde Y_b] = f^c_{ab} \tilde Y_c$ and so on.

In order to reduce the SDYM equations (1) and the linear system (2) under a subgroup G of the group $ISO(2,2)$, it is necessary to impose the following conditions of G-invariance on the gauge potentials A_μ and on the vector-function Ψ [23]:

$$W_\xi A_\mu + A_\sigma W^\sigma_{\xi,\mu} = 0, \quad \forall \xi \in \mathcal{G}, \tag{6a}$$

$$\tilde W_\xi \Psi = 0, \quad \forall \xi \in \mathcal{G}, \tag{6b}$$

where $\mathcal{G}$ is a Lie algebra of the group G, $W_\xi = W^\sigma_\xi \partial_\sigma$ are vector fields on $R^{2,2}$ and $\tilde W_\xi = \tilde W^\sigma_\xi \partial_\sigma + \tilde W^a_\xi Z_a$ are vector fields on $R^{2,2} \times H^2$. Both W_ξ and $\tilde W_\xi$ form a realization of the Lie algebra $\mathcal{G}$.

In accordance with the general method of symmetry reduction (see [1] and references therein), as new coordinates on $R^{2,2} \times H^2$, one should choose the coordinates θ_ξ on the orbits Q of the group G in $R^{2,2} \times H^2$, and the invariant coordinates θ_A ($A = 1, ..., 4 - dim Q$) and ζ which parametrize the space of orbits and satisfy

$$\tilde W_\xi \theta_A = 0, \quad \tilde W_\xi \zeta = 0, \quad \partial_{\bar\lambda}\zeta = 0, \quad \forall \xi \in \mathcal{G}. \tag{7}$$

Here, the invariant complex coordinate ζ represents the new "spectral parameter". Then, substituting solutions of eqs.(6) and (7) into eqs.(1), (2), we obtain the reduced SDYM equations and their Lax pairs in terms of functions of the invariant coordinates [1, 3].

Now we consider examples of reduction of the SDYM equations to the integrable equations in $1 \le d \le 3$. In what follows, we shall firstly write out some known integrable equations which arise as reduction of the SDYM equations under translations. After that we shall describe new nonautonomous versions of these equations (i.e. their deformations), derived via reduction with respect to the action of the subgroups containing rotations.

4. Reductions to integrable systems in $(2+1)$ dimensions

Chiral model equation in $R^{2,1}$ [24, 16]. Let us consider the one-dimensional Abelian group with generator $P_y - P_z$. Then, $\varphi = y - z$ will be the coordinate on the orbit and the invariant coordinates are $x = y + z$, t, u and λ. The YM potentials A_μ, satisfying (6a), and the vector-function Ψ, satisfying (6b) and (2c), are given by

$$A_t = T_t(t,u,x), A_u = T_u(t,u,x), A_y = T_y(t,u,x), A_z = T_z(t,u,x), \Psi = \psi(t,u,x,\lambda). \tag{8}$$

Substituting (8) into the linear system (2), we obtain the following reduced Lax pair:

$$(\partial_t - \lambda\partial_x + T_t - \lambda T_y)\psi = 0, \quad (\partial_x + \lambda\partial_u + T_z + \lambda T_u)\psi = 0. \tag{9}$$

Accordingly, the SDYM equations (1) are reduced to the compatibility conditions of the Lax pair (9):

$$\partial_t T_z - \partial_x T_t + [T_t, T_z] = 0, \quad \partial_x T_u - \partial_u T_y + [T_y, T_u] = 0, \tag{10a}$$

$$\partial_x(T_y - T_z) + \partial_t T_u - \partial_u T_t + [T_t, T_u] + [T_z, T_y] = 0. \tag{10b}$$

Now let us impose the algebraic constraints $T_z = T_t = 0$. Then, from eqs.(10a) we obtain $T_u = g^{-1}\partial_u g$, $T_y = g^{-1}\partial_x g$, where g is an arbitrary function of t, u, x with values in the group $GL(n, C)$, and eqs. (10b) coincide with the equation of the chiral field model considered in the papers [24, 16]:

$$\partial_x(g^{-1}\partial_x g) + \partial_t(g^{-1}\partial_u g) = 0. \tag{11}$$

Nonautonomous chiral model equation in $R^{2,1}$. Now we consider the one-dimensional Abelian group of rotations generated by the vector field $X_2 + Y_2$. From (4) and (5), we obtain $\tilde{X}_2 + \tilde{Y}_2 = X_2 + Y_2 + Z_2 = z\partial_z - y\partial_y - \lambda\partial_\lambda - \bar{\lambda}\partial_{\bar{\lambda}}$. Let us introduce the coordinates ρ, θ, η, ξ by formulae $y = \frac{1}{2}\rho\, e^{-\theta}$, $z = \frac{1}{2}\rho\, e^{\theta}$, $\lambda = \eta\, e^{i\xi}$, then $X_2 + Y_2 = \partial_\theta$ and $\tilde{X}_2 + \tilde{Y}_2 = \partial_\theta - \eta\partial_\eta$. Therefore, $\varphi = \frac{1}{2}(\theta - \ln\eta)$ will be the coordinate on the orbit and $t, u, \rho, \zeta = \lambda e^{\theta}$ will be the invariant coordinates.

The invariant YM potentials A_μ, satisfying eqs.(6a), have the form

$$A_t = T_t(t,u,\rho),\ A_u = T_u(t,u,\rho),\ A_y = T_y(t,u,\rho)e^{\theta},\ A_z = T_z(t,u,\rho)e^{-\theta}. \tag{12a}$$

The vector-function

$$\Psi = \psi(t, u, \rho, \zeta) \tag{12b}$$

is the solution of equations (6b) and (2c).

Substituting (12) into (2), we obtain the following reduced Lax pair:

$$\nabla_{V_1}\psi \equiv [\partial_t - \zeta\partial_\rho + \frac{1}{\rho}\zeta^2\partial_\zeta + T_t - \zeta T_y]\psi = 0,\ \nabla_{V_2}\psi \equiv [\partial_\rho + \zeta\partial_u + \frac{1}{\rho}\zeta\partial_\zeta + T_z + \zeta T_u]\psi = 0, \tag{13}$$

where $V_1 = \partial_t - \zeta\partial_\rho + \frac{1}{\rho}\zeta^2\partial_\zeta$, $V_2 = \partial_\rho + \zeta\partial_u + \frac{1}{\rho}\zeta\partial_\zeta$. Remind that in the general case $[V_1, V_2] \neq 0$ and then for linear systems like (13) the compatibility condition is

$$[\nabla_{V_1}, \nabla_{V_2}] - \nabla_{[V_1, V_2]} = 0. \tag{14}$$

Correspondingly, the SDYM equations (1) are reduced to

$$\partial_t T_z - \partial_\rho T_t + [T_t, T_z] = 0, \quad \partial_\rho T_u - \partial_u T_y + [T_y, T_u] = 0, \tag{15a}$$

$$\partial_\rho(T_y - T_z) + \frac{1}{\rho}(T_y - T_z) + \partial_t T_u - \partial_u T_t + [T_t, T_u] + [T_z, T_y] = 0 \tag{15b}$$

which agree with the compatibility condition (14) of the Lax pair (13).

Choosing the same algebraic constraints $T_z = T_t = 0$ as in (9), (10), from eq.(15a) we obtain $T_u = g^{-1}\partial_u g, T_y = g^{-1}\partial_\rho g$. Then, eq.(15b) is reduced to the nonautonomous chiral model equation in $R^{2,1}$:

$$\partial_\rho(g^{-1}\partial_\rho g) + \frac{1}{\rho}g^{-1}\partial_\rho g + \partial_t(g^{-1}\partial_u g) = 0 \Longleftrightarrow \frac{1}{\rho}\partial_\rho(\rho g^{-1}\partial_\rho g) + \partial_t(g^{-1}\partial_u g) = 0. \tag{16}$$

The Lax pair for this equation has the form (13) with $T_z = T_t = 0$.

Remark. Notice that if one uses an additional condition of invariance under $P_t + P_u$: $(\partial_t + \partial_u)\psi = (\partial_t + \partial_u)g = 0$ in the Lax pair (9) and in eqs.(11), then one obtains the equation of the principal chiral model in $R^{1,1}$. But if we impose the same condition on ψ and g in (13) and (16), then we obtain the nonautonomous equation of the principal chiral model in $R^{1,1}$ [25, 26], which in a particular case of the gauge group $GL(2, R)$ is equivalent to the Ernst equations [25, 10].

5. Reductions to integrable systems in $(1+1)$ dimensions

Generalized nonlinear Schrödinger equation (NLS) [9]. Let us consider the two-dimensional Abelian group with the generators $\{P_y - P_z, P_u\}$. Then, solutions A_μ and Ψ of eqs.(6) and (2c) are given by

$$A_t = T_t(t,x),\ A_u = T_u(t,x),\ A_y = T_y(t,x),\ A_z = T_z(t,x),\ \Psi = \psi(t,x,\lambda). \tag{17}$$

The linear system (2) is reduced to the following one:

$$\begin{cases} (\partial_t - \lambda\partial_x + T_t - \lambda T_y)\psi = 0, \\ (\partial_x + T_z + \lambda T_u)\psi = 0 \end{cases} \Rightarrow \begin{cases} [\partial_t + T_t + \lambda(T_z - T_y) + \lambda^2 T_u]\psi = 0, \\ (\partial_x + T_z + \lambda T_u)\psi = 0 \end{cases} \tag{18}$$

and the SDYM equations (1) are reduced to the compatibility condition of the Lax pair (18):

$$\partial_t T_z - \partial_x T_t + [T_t, T_z] = 0, \quad \partial_x T_u - [T_u, T_y] = 0, \tag{19a}$$

$$\partial_t T_u - \partial_x(T_z - T_y) + [T_t, T_u] + [T_z, T_y] = 0. \tag{19b}$$

To reduce eqs.(19) to the generalized NLS equations, introduced by Fordy and Kulish [27], one should impose the algebraic constraints on the elements of matrices in (19).

Let us choose in $GL(n,C)$ the subgroups N and H so that N/H be a compact Hermitian symmetric space. Let $\mathcal{N}$ and $\mathcal{H}$ be the Lie algebras of the Lie groups N and H. Then $\mathcal{N} = \mathcal{H}\oplus\mathcal{P}$ and $[\mathcal{H},\mathcal{H}] \subset \mathcal{H}, [\mathcal{H},\mathcal{P}] \subset \mathcal{P}, [\mathcal{P},\mathcal{P}] \subset \mathcal{H}$. A special feature of Hermitian symmetric spaces is the existence of an element $A \in \mathcal{H}$ such that $\mathcal{H} = \{B \in \mathcal{N} : [A,B] = 0\}$. The matrix ad_A has only three distinct eigenvalues $0, \pm i$ and $[A,\mathcal{H}] = 0, [A, X^{\pm}] = \pm i X^{\pm}$ for all $X^{\pm} \in \mathcal{P}^{\pm}, \mathcal{P} = \mathcal{P}^{+} \oplus \mathcal{P}^{-}$. Let $e_{\pm\alpha}$ be a basis of the space $\mathcal{P}^{\pm}$. Then

$$[A, e_{\pm\alpha}] = \pm i e_{\pm\alpha}, \quad [e_\mu, [e_\nu, e_{-\sigma}]] = R^{\alpha}_{\mu,\nu,-\sigma} e_\alpha,$$

$$[e_{-\mu}, [e_{-\nu}, e_\sigma]] = R^{-\alpha}_{-\mu,-\nu,\sigma} e_{-\alpha}, \quad R^{-\alpha}_{-\mu,-\nu,\sigma} = \bar{R}^{\alpha}_{\mu,\nu,-\sigma}, \tag{20}$$

where $R^{\alpha}_{\mu,\nu,-\sigma}$ are components of the curvature tensor defined at the initial point of the symmetric space N/H, and $R^{-\alpha}_{-\mu,-\nu,\sigma}$ are complex conjugate to the $R^{\alpha}_{\mu,\nu,-\sigma}$ components.

For the matrices from (17) we choose the following ansatz:

$$T_t = \sum_\alpha (\phi^\alpha e_\alpha + \bar{\phi}^\alpha e_{-\alpha}) + \sum_{\alpha,\beta} \Omega^{\alpha,-\beta}[e_\alpha, e_{-\beta}], \quad T_u = A,$$

$$T_y = 0, \quad T_z = \sum_\alpha (\psi^\alpha e_\alpha + \bar{\psi}^\alpha e_{-\alpha}), \tag{21}$$

where ϕ^α, ψ^α and $\Omega^{\alpha,-\beta}$ are arbitrary complex-valued functions of t, x and the bar over the letter means complex conjugation. Substituting (21) into eqs.(19), we obtain that

$$\phi^\alpha = i\partial_x \psi^\alpha, \; \Omega^{\alpha,-\beta} = i(\psi^\alpha \bar{\psi}^\beta + \Omega^{\alpha,-\beta}_o), \; \Omega^{\beta,-\alpha}_o = \bar{\Omega}^{\alpha,-\beta}_o = const \tag{22}$$

and eqs.(19) are reduced to the generalized NLS equations on the functions ψ^α:

$$i\partial_t \psi^\alpha + \partial_x^2 \psi^\alpha + \sum_{\mu,\nu,\sigma} R^{\alpha}_{\mu,\nu,-\sigma} \psi^\mu \psi^\nu \bar{\psi}^\sigma + \sum_{\mu,\nu,\sigma} R^{\alpha}_{\mu,\nu,-\sigma} \Omega^{\nu,-\sigma}_o \psi^\mu = 0. \tag{23}$$

Notice that the constant components $\Omega^{\nu,-\sigma}_o$ can always be chosen so that $\sum_{\nu,\sigma} R^{\alpha}_{\mu,\nu,-\sigma}\Omega^{\nu,-\sigma}_o = \omega_\alpha \delta^\alpha_\mu$, where ω_α are real constants [27]. The Lax pair for eqs.(23) can be deduced via substitution of (21) and (22) in (18).

Nonautonomous generalized NLS equation. Now let us consider the two-dimensional Abelian group with the generators $\{X_2 + Y_2, P_u\}$. Then, invariant A_μ and Ψ are given by formulae (12) where T_μ and ψ do not depend on u. The reduced Lax pair and SDYM equations have the form

$$[\partial_t - \zeta\partial_\rho + \frac{1}{\rho}\zeta^2\partial_\zeta + T_t - \zeta T_y]\psi = 0, \quad [\partial_\rho + \frac{1}{\rho}\zeta\partial_\zeta + T_z + \zeta T_u]\psi = 0, \tag{24}$$

$$\partial_t T_z - \partial_\rho T_t + [T_t, T_z] = 0, \qquad \partial_\rho T_u + [T_y, T_u] = 0, \tag{25a}$$

$$\partial_\rho(T_y - T_z) + \frac{1}{\rho}(T_y - T_z) + \partial_t T_u + [T_t, T_u] + [T_z, T_y] = 0. \tag{25b}$$

For matrices from (24), (25) we choose the ansatz (21) again. Substituting (21) into (25), we obtain that

$$\phi^\alpha = i(\partial_\rho \psi^\alpha + \frac{1}{\rho}\psi^\alpha),$$

$$\Omega^{\alpha,-\beta} = i(\psi^\alpha \bar{\psi}^\beta + \Omega^{\alpha,-\beta}_o + 2\int \frac{d\rho}{\rho}\psi^\alpha\bar{\psi}^\beta), \; \Omega^{\beta,-\alpha}_o = \bar{\Omega}^{\alpha,-\beta}_o = const \tag{26}$$

and the functions ψ^α have to satisfy the nonautonomous generalized NLS equations

$$i\partial_t\psi^\alpha + \partial_\rho^2\psi^\alpha + \sum_{\mu,\nu,\sigma} R^{\alpha}_{\mu,\nu,-\sigma}\psi^\mu\psi^\nu\bar{\psi}^\sigma + \sum_{\mu,\nu,\sigma} R^{\alpha}_{\mu,\nu,-\sigma}\Omega^{\nu,-\sigma}_o\psi^\mu =$$

$$= -\partial_\rho(\frac{1}{\rho}\psi^\alpha) - 2\sum_{\mu,\nu,\sigma} R^{\alpha}_{\mu,\nu,-\sigma}\psi^\mu \int \frac{d\rho}{\rho}\psi^\nu\bar{\psi}^\sigma. \tag{27}$$

The Lax pair for eqs.(27) can be obtained by substitution of (21) and (26) into (24).

Remark. In the case of $N = SU(2)$ and $H = U(1)$, ansatz (21) has the form

$$T_t = \begin{pmatrix} \Omega & \bar{\phi} \\ \phi & -\Omega \end{pmatrix}, \; T_u = \frac{1}{2i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; T_y = 0, \; T_z = \sqrt{\kappa}\begin{pmatrix} 0 & \bar{\psi} \\ \psi & 0 \end{pmatrix}. \tag{28}$$

where Ω, ϕ and ψ are arbitrary complex-valued functions of t and ρ, and κ is an arbitrary real constant parameter. Substituting (28) into (25), we obtain that

$$\Omega = -i\kappa(\bar{\psi}\psi - \gamma^2) - 2i\kappa\int\frac{d\rho}{\rho}\bar{\psi}\psi, \qquad \phi = i\sqrt{\kappa}(\partial_\rho\psi + \frac{1}{\rho}\psi), \quad \gamma = const. \tag{29}$$

and the function ψ has to satisfy the equation

$$i\partial_t\psi + \partial_\rho^2\psi - 2\kappa(\bar{\psi}\psi - \gamma^2)\psi = -\partial_\rho(\frac{1}{\rho}\psi) + 4\kappa\psi\int\frac{d\rho}{\rho}\bar{\psi}\psi. \tag{30}$$

The Lax pair for eqs.(30) can be obtained by substitution of (28) and (29) into (24).

The nonautonomous NLS equation (30) has been considered in the paper [26]. When $\kappa = -1$ and $\gamma^2 = 0$, this equation is gauge equivalent to the equation of the Heisenberg ferromagnet in axial geometry. By change of variables t, ρ and ψ, eq.(30) can be transformed to the equation, which has been introduced and integrated in [28]. Thus, the nonautonomous NLS equation is shown to be the reduction of the SDYM equations.

Korteweg-de Vries equation [5,6]. Now, considering the generators $\{P_y - P_z, P_u\}$, the Lax pair (18) and the compatibility conditions (19), we choose the matrices from (19) in the form of the following 2×2 matrices

$$T_t = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \; T_u = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \; T_y = \begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix}, \; T_z = \begin{pmatrix} 0 & g \\ f & 0 \end{pmatrix}, \tag{31a}$$

where a, b, c, f, g and h are arbitrary real-valued functions.

Substituting (31a) in (19), we obtain that

$$a = \frac{1}{4}\partial_x f,\ b = -\frac{1}{2}f,\ c = \frac{1}{2}f^2 + \frac{1}{4}\partial_x^2 f,\ h = \frac{1}{2}f,\ g = -1, \tag{31b}$$

and the function f has to satisfy the KdV equation

$$\partial_t f - \frac{3}{2}f\partial_x f - \frac{1}{4}\partial_x^3 f = 0. \tag{32}$$

The Lax pair for eqs.(32) is obtained after substitution of (31a) and (31b) into (18).

Nonautonomous KdV equation. Now we consider the generators $\{X_2 + Y_2, P_u\}$, Lax pair (24) and its compatibility conditions (25). For matrices from (25) let us choose the ansatz (31a). Substituting (31a) in (25), we obtain that

$$a = \frac{1}{4}\partial_\rho f - \frac{1}{4\rho}\int\frac{d\rho}{\rho}f,\quad b = -\frac{1}{2\rho}f - \frac{1}{2\rho}\int\frac{d\rho}{\rho}f,\quad h = \frac{1}{2}f + \frac{1}{2}\int\frac{d\rho}{\rho}f,$$

$$c = \frac{1}{4}\rho\partial_\rho^2 f - \frac{1}{4\rho}f + \frac{1}{2}f^2 + (\frac{1}{4\rho} + \frac{f}{2})\int\frac{d\rho}{\rho}f,\quad g = -\frac{1}{\rho}, \tag{33}$$

and the function f satisfies the equation

$$\partial_t f - \frac{3}{2}f\partial_\rho f - \frac{1}{4}\partial_\rho^3(\rho f) =$$

$$= \frac{1}{2\rho^2}f + \frac{1}{2\rho}f^2 - \frac{1}{4\rho}\partial_\rho f - \frac{1}{2}\partial_\rho^2 f + (\frac{1}{2}\partial_\rho f - \frac{f}{2\rho} - \frac{1}{4\rho^2})\int\frac{d\rho}{\rho}f. \tag{34}$$

The Lax pair for eq.(34) is obtained after substitution of (31a) and (33) into the Lax pair (24).

Remark. Nonautonomous KdV equations have been considered in the papers [26, 28]. Equation (34) differs from ones, considered in [26, 28], and it is a new deformation of the KdV equation.

6. Reductions to integrable dynamical systems

Periodic Toda lattice with damping. Let us consider the three-dimensional non-Abelian subgroups of $ISO(2,2)$ generated by the vector fields $X_2+\beta Y_2, P_y, P_z$, where $\beta \in R, \beta \neq 1$. Notice that the SDYM equations, reduced with respect to the symmetry group with the generators X_2+Y_2, P_y and P_z, lead to the zero curvature condition $F_{\mu\nu} = 0$ and, therefore, they are not interesting. That is why we shall investigate the case $\beta \neq 1$.

Let us introduce the coordinates τ, θ by formulae $\tau = \frac{1}{4}\ln(4tu)^2$, $\theta = \frac{1}{4}\ln(\frac{u}{t})^2$. Then, the orbit coordinates are $\chi = \frac{2(1+\beta^2)}{(1-\beta)}\theta + \frac{1}{2}\beta\ln(\bar\lambda\lambda)$, y, z and the invariant coordinates

are τ, $\zeta = \lambda e^{\gamma\theta}$, where $\gamma = 2\beta/(1-\beta)$. The invariant YM potentials and Ψ satisfying eqs.(2c) and (6) are given by:

$$A_t = T_t(\tau)e^{\theta-\tau},\ A_u = T_u(\tau)e^{-\theta-\tau},\ A_y = T_y(\tau)e^{(1+\gamma)(\theta-\tau)},\ A_z = T_z(\tau)e^{-(1+\gamma)(\theta+\tau)}$$

$$\Psi = \psi(\tau,\zeta). \tag{35}$$

Substituting (35) into the linear system (2), changing the variables and using (6), we obtain the following reduced Lax pair:

$$[\partial_\tau - \gamma\zeta\partial_\zeta + T_t - \zeta e^{-\gamma\tau}T_y]\psi = 0,\quad [\zeta\partial_\tau + \gamma\zeta^2\partial_\zeta + e^{-\gamma\tau}T_z + \zeta T_u]\psi = 0, \tag{36}$$

Using the compatibility condition (14) for the Lax pair (36), we obtain the following reduced SDYM equations:

$$\frac{d}{d\tau}T_y + [T_u, T_y] = 0,\quad \frac{d}{d\tau}T_z + [T_t, T_z] = 0, \tag{37a}$$

$$\frac{d}{d\tau}(T_u - T_t) + [T_t, T_u] + e^{-2\gamma\tau}[T_z, T_y] = 0. \tag{37b}$$

The equations of the periodic Toda lattice with damping are derived via the algebraic reduction of eqs.(37). Let us choose for $T_t, T_u, T_y, T_z \in gl(n, C)$ the following (algebraic) ansatz:

$$T_t = -T_u = \begin{pmatrix} p_1 & 0 & \dots & 0 \\ 0 & p_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & p_n \end{pmatrix},\quad T_y = T_z^T = 2\begin{pmatrix} 0 & a_1 & 0 & \dots & 0 \\ 0 & 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & a_{n-1} \\ a_n & 0 & \dots & 0 & 0 \end{pmatrix}, \tag{38}$$

where $a_\alpha = \exp(q_\alpha - q_{\alpha+1})$ and the superscript ^{T} means matrix transpose. Then, after substitution of (38) into eqs.(37), we obtain

$$\frac{d}{d\tau}q_\alpha = p_\alpha,\quad \frac{d}{d\tau}p_\alpha = 2\exp(-2\gamma\tau)\{\exp[2(q_{\alpha-1} - q_\alpha)] - \exp[2(q_\alpha - q_{\alpha+1})]\}. \tag{39}$$

When $\gamma = 0$, the latter equations coincide with the standard periodic Toda lattice equations. If $\gamma \neq 0$, then using the variable $\varphi = \exp(-\gamma\tau)$, we obtain the equations of the Toda lattice with damping:

$$\frac{d^2}{d\varphi^2}q_\alpha + \frac{1}{\varphi}\frac{d}{d\varphi}q_\alpha = \frac{2}{\gamma^2}\{\exp[2(q_{\alpha-1} - q_\alpha)] - \exp[2(q_\alpha - q_{\alpha+1})]\}. \tag{40}$$

The corresponding Lax pair is obtained by inserting (38) in (36).

Integrable Hamiltonian systems with quartic potentials. We are still considering the generators $X_2 + \beta Y_2, P_y, P_z$, the Lax pair (36) and the compatibility conditions (37). Let us choose in $GL(n, C)$ the subgroups N and H in such a way that N/H be the Hermitian symmetric space (for definitions and notation see Sec.5).

For the matrices from (37) we choose the following ansatz:

$$N_t = 0, \quad N_u = i \sum_\alpha q^\alpha (e_\alpha + e_{-\alpha}),$$

$$N_y = \sum_\alpha r^\alpha (e_\alpha - e_{-\alpha}) + \sum_{\alpha,\sigma} \Omega^{\alpha,-\sigma}[e_\alpha, e_{-\sigma}], \quad N_z = A, \tag{41}$$

where q^α, r^α and $\Omega^{\alpha,-\sigma}$ are arbitrary real-valued functions of τ.

Substituting (41) in (37) we obtain that

$$r^\alpha = -e^{2\gamma\tau}\frac{dq^\alpha}{d\tau}, \; \Omega^{\alpha,-\sigma} = i\Omega_o^{\alpha,-\sigma} - ie^{2\gamma\tau}q^\alpha q^\sigma + 2i\,\gamma \int d\tau\; e^{2\gamma\tau}q^\alpha q^\sigma, \; \Omega_o^{\alpha,-\sigma} = const, \tag{42}$$

and eqs.(37) are reduced to the equations

$$\frac{d^2}{d\tau^2}q^\alpha - \sum_{\mu,\nu,\sigma} R^\alpha_{\mu,\nu,-\sigma} q^\mu q^\nu q^\sigma + \sum_{\mu,\nu,\sigma} R^\alpha_{\mu,\nu,-\sigma}\Omega_o^{\nu,-\sigma} q^\mu =$$

$$= -2\gamma \left(\frac{dq^\alpha}{d\tau} + e^{-2\gamma\tau}\sum_{\mu,\nu,\sigma} R^\alpha_{\mu,\nu,-\sigma} q^\mu \int d\tau\; e^{2\gamma\tau} q^\nu q^\sigma \right) + (1 - e^{-2\gamma\tau}) \sum_{\mu,\nu,\sigma} R^\alpha_{\mu,\nu,-\sigma}\Omega_o^{\nu,-\sigma} q^\mu. \tag{43}$$

Notice that $\Omega_o^{\alpha,-\sigma}$ may always be chosen so that $\sum_{\nu,\sigma} R^\alpha_{\mu,\nu,-\sigma}\Omega_o^{\nu,-\sigma} = \omega_\mu \delta^\alpha_\mu$, where $\omega_\alpha = const$.

When $\gamma = 0$, eqs.(43) coincide with the equations of motion in quartic potentials, considered in [29]. Equations of the Garnier system are the particular case of eqs.(43), corresponding to $\gamma = 0, N = SU(n), H = S(U(1) \times U(n-1))$. The Lax pair for eqs.(43) can be obtained by inserting (41) and (42) into (36).

Euler-Arnold equations and their deformations. Now let us consider the three-dimensional non-Abelian symmetry group with the generators $\alpha X_2 + \beta Y_2$, P_u, P_y, where $\alpha, \beta \in R$, $\alpha^2 - \beta^2 = 1$. Let us introduce the coordinates $\tau = \frac{1}{2}(\alpha - \beta)\ln z^2 + \frac{1}{2}(\alpha + \beta)\ln t^2$, $\theta = \frac{1}{2}(\alpha - \beta)\ln z^2 - \frac{1}{2}(\alpha + \beta)\ln t^2$. In this case the orbits are parametrized by the coordinates $\chi = \theta - \frac{1}{2}\beta \ln(\bar\lambda\lambda)$, u, y and the invariant coordinates are τ, $\zeta = \lambda e^{\beta\theta}$. Solving eqs. (6) and (2c), we obtain the following formulae for the invariant YM potentials and for the vector-function Ψ

$$A_t = (\alpha + \beta)\exp\left(\frac{\theta - \tau}{2(\alpha + \beta)}\right) T_t(\tau), \; A_u = (\alpha - \beta)\exp\left(-\frac{\tau}{2(\alpha - \beta)} - \frac{\theta}{2(\alpha + \beta)}\right) T_u(\tau),$$

$$A_y = (\alpha + \beta)\exp\left(-\frac{\tau}{2(\alpha + \beta)} + \frac{\theta}{2(\alpha - \beta)}\right) T_y(\tau), \; A_z = (\alpha - \beta)\exp\left(\frac{-\theta - \tau}{2(\alpha - \beta)}\right) T_z(\tau).$$

$$\Psi = \psi(\tau, \zeta). \tag{44}$$

Substitute (44) into the linear system (2) and express the derivatives in (2) via the new coordinates. Then, after using the conditions of invariance of ψ, the linear system (2) is reduced to the following one:

$$(\partial_\tau - \beta\zeta\partial_\zeta + T_t - \zeta T_y)\psi = 0, \quad (\partial_\tau + \beta\zeta\partial_\zeta + T_z + \zeta T_u)\psi = 0. \tag{45}$$

If we put $N_t = \frac{1}{2}(T_z + T_t)$, $N_u = \frac{1}{2}(T_y + T_u)$, $N_y = \frac{1}{2}(T_y - T_u)$, $N_z = \frac{1}{2}(T_z - T_t)$, then we can rewrite the Lax pair (45) in the form

$$(\partial_\tau + N_\tau - \zeta N_y)\psi = 0, \quad (\beta\zeta\partial_\zeta + N_z + \zeta N_u)\psi = 0. \tag{46}$$

The compatibility conditions of the Lax pair (46) are

$$\partial_\tau N_u + \beta N_y + [N_t, N_u] + [N_z, N_y] = 0, \quad [N_u, N_y] = 0 \tag{47a}$$

$$\partial_\tau N_z + [N_t, N_z] = 0. \tag{47b}$$

Let us choose N_t, N_z to be antisymmetric $n \times n$ matrices and N_u, N_y to be diagonal matrices satisfying the equation $\partial_\tau N_u + \beta N_y = 0$. We choose the solution of this equation in the following form: $N_u = A_u - \beta\tau N_y$, where $A_u = diag(a_1, \ldots, a_n)$ and $N_y = diag(b_1, \ldots, b_n)$ are constant diagonal matrices with $a_i \neq a_j$, $b_i \neq b_j$ and $a_i \neq b_j$ when $i \neq j$. Solution of eqs.(47a) may be written in the form

$$N_u = A_u - \beta\tau N_y, \quad A_u = const, \quad N_y = const, \quad (N_t)_{ij} = \lambda_{ij} M_{ij},$$

$$(N_z)_{ij} = (\beta\tau\lambda_{ij} - 1)M_{ij}, \tag{48}$$

where $\lambda_{ij} = (b_i - b_j)/(a_i - a_j)$, $M = (M_{ij}) = (-M_{ji})$ is an arbitrary antisymmetric matrix and $(N_t)_{ij}$, $(N_z)_{ij}$ are components of the matrices N_t, N_z.

After substitution of (48), eqs.(47b) can be written in the form

$$\frac{d}{d\tau}[(1 - \beta\tau\lambda_{ij})M_{ij}] = \sum_k (\lambda_{kj} - \lambda_{ik})M_{ik}M_{kj}. \tag{49}$$

When $\beta = 0$, these equations coincide with the Euler-Arnold equations describing the rotation of the n-dimensional rigid body [30]. One may also rewrite eqs.(49) in the form

$$\frac{d}{d\tau}M_{ij} - \frac{\beta\lambda_{ij}}{1 - \beta\tau\lambda_{ij}}M_{ij} = \frac{1}{1 - \beta\tau\lambda_{ij}}\sum_k (\lambda_{kj} - \lambda_{ik})M_{ik}M_{kj}. \tag{50}$$

Deformed generalized Calogero-Moser system. Here we consider the same symmetry group with the generators $\alpha X_2+\beta Y_2$, P_u, P_y ($\alpha,\beta\in R$, $\alpha^2-\beta^2=1$), the Lax pair (46) and the compatibility conditions (47). Now for the matrices $N_t, N_u, N_y, N_z \in gl(n,C)$ we choose the following algebraic ansatz:

$$N_t=-i\sum_{\substack{j,k=1\\ j\neq k}}^{n}\frac{f_j^+f_k}{(q_j-q_k)^2}e_{jk}-2N_z,\quad N_u=-i\sum_{j,k=1}^{n}f_j^+f_ke_{jk},\quad N_y=2N_u,$$

$$N_z=h\left\{\sum_{k=1}^{n}p_ke_{kk}+i\sum_{\substack{j,k=1\\ j\neq k}}^{n}\frac{f_j^+f_k}{q_j-q_k}e_{jk}\right\}. \tag{51}$$

Here p_j, q_j and h are the real-valued functions of τ, the vector-functions $f_j(\tau)$ belong to N-dimensional vector space C^N, f_j^+ are canonical conjugate to f_j: $f_j^+f_j=\sum_{a=1}^{N}\bar{f}_j^af_j^a=1$, and $(e_{jk})_{mn}=\delta_{jm}\delta_{kn}$ are the generators of the group $SL(n,C)$: $[e_{jk},e_{lm}]=\delta_{kl}e_{jm}-\delta_{mj}e_{lk}$.

Substituting (51) into (47), we obtain that $h=\exp(2\beta\tau)$ and eqs.(47) are reduced to the following system of equations:

$$\frac{d}{d\tau}q_j=p_j,\quad \frac{d}{d\tau}p_j+2\beta p_j=2\sum_{\substack{k=1\\ k\neq j}}^{n}\frac{(f_j^+f_k)(f_k^+f_j)}{(q_j-q_k)^3},$$

$$\frac{d}{d\tau}f_j+\beta f_j=-i\sum_{\substack{k=1\\ k\neq j}}^{n}\frac{f_k(f_k^+f_j)}{(q_j-q_k)^2},\quad \frac{d}{d\tau}f_j^++\beta f_j^+=i\sum_{\substack{k=1\\ k\neq j}}^{n}\frac{(f_j^+f_k)f_k^+}{(q_j-q_k)^2}. \tag{52}$$

The Lax pair for eqs.(52) can be obtained by substituting (51) into (46). When $\beta=0$, eqs.(52) coincide with those of the generalized Calogero-Moser system introduced in [31] (see also [32] and references therein).

Deformed Euler-Calogero-Moser system. Considering the same symmetry group with the generators $\alpha X_2+\beta Y_2$, P_u, P_y ($\alpha,\beta\in R$, $\alpha^2-\beta^2=1$), the Lax pair (46) and its compatibility conditions (47), for matrices in (46), (47) we choose now the following ansatz:

$$N_t=\sum_{\substack{j,k=1\\ j\neq k}}^{n}\frac{h_{jk}}{(q_j-q_k)^2}e_{jk}-2N_z,\quad N_u=-\sum_{j,k=1}^{n}h_{jk}e_{jk},\quad N_y=2N_u,$$

$$N_z=\exp(2\beta\tau)\left\{\sum_{k}p_ke_{kk}-\sum_{\substack{j,k=1\\ j\neq k}}^{n}\frac{h_{jk}}{q_j-q_k}e_{jk}\right\}, \tag{53}$$

where p_j, q_j and $h_{ij}=-h_{ji}$ are the real-valued functions of τ, and e_{jk} are the generators of the group $SL(n,R)$.

Substituting (53) into eqs.(47) yields

$$\frac{d}{d\tau}q_j=p_j,\quad \frac{d}{d\tau}p_j+2\beta p_j=2\sum_{\substack{k=1\\ k\neq j}}^{n}\frac{h_{jk}h_{jk}}{(q_j-q_k)^3},$$

$$\frac{d}{d\tau}h_{jk}+2\beta h_{jk}=\sum_{\substack{m=1\\ m\neq j,k}}^{n}h_{jm}h_{mk}\left[\frac{1}{(q_k-q_m)^2}-\frac{1}{(q_j-q_m)^2}\right]. \tag{54}$$

The Lax pair for eqs.(54) has the form (46) with the matrices N_t, N_u, N_y, N_z from (53). When $\beta=0$, eqs.(54) coincide with those of the Euler-Calogero-Moser system introduced in the paper [33] (see also the discussion of this integrable system in [32]).

7. Conclusion

In this paper, we have introduced eight new systems of nonlinear integrable differential equations. The Lax pairs for all of these systems contain derivatives of the form $\zeta\frac{\partial}{\partial\zeta}$ with respect to the spectral parameter ζ. The differential operator $\zeta\frac{\partial}{\partial\zeta}$ corresponds to the extention of the loop algebra associated with the Lie algebra of the gauge group, and this extention is important in the standard approach to the integrable equations in $(1+1)$ dimensions [34]. The dressing method for the Lax pairs containing the additional term $\zeta\partial_\zeta$ has been developed by Belinsky and Zakharov [25]. Namely, if one chooses a "seed solution" of an integrable system and constructs the corresponding solution ψ_0 of the linear system, then the ansatz for iteration is $\psi_n=(I+\frac{R_n}{\zeta-\mu_n})\psi_{n-1}$, where the matrices R_n are independent of ζ, and μ_n are functions of the coordinates and do not depend on ζ (*moving* poles). For more detailed discussions see, e.g., [25, 26, 34].

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