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THE EULERIAN BOUND STATES: 8D QUANTUM OSCILLATOR

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Эйлеровы связанные состояния:
8-мерный квантовый осциллятор

Предпринят анализ задачи о 8-мерном осцилляторе в эйлеровых координатах. Сконструирован сферический и цилиндрический базис, доказано два представления для коэффициентов сфероцилиндрического и цилиндросферического межбазисного разложения и установлены трехчленные рекуррентные соотношения, порождающие сфероидальный базис 8-мерного осциллятора.

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Karayan Kh.H. et al. E2-94-439

The Eulerian Bound States: 8D Quantum Oscillator

The Eulerian coordinates are applied for analyzing the 8D oscillator bound states problem. We construct the spherical and cylindrical bases, derive two types of representations for the coefficients of the spheric-cylindrical and cylindric-spherical interbasis expansions, and establish the trinominal recurrent relations for the amplitudes generating the spheroidal basis of the 8D oscillator.

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1 Introduction

The 5D Coulomb and 8D oscillator problems are connected with each other by Hurvitz transformation [1]. This Coulomb–Oscillator duality is expressed most transporant by the Eulerian coordinates [2].

In previous paper we undertook the Eulerian analysis of the 5D Coulomb problem. Here we concentrate our attention on the 8D quantum oscillator problem and calculate the Eulerian spherical and cylindrical bound states of 8D oscillator, derive the expressions for the amplitudes of interbasis expansions, investigate the spheroidal states as the superposition of the spherical and cylindrical states.

2 Spherical 8D Coordinates

Eulerian spherical coordinates for 8D space are defined by

$$
\begin{aligned}
u_0 &= r\cos\theta/2\cos\beta_1/2\cos(\alpha_1+\gamma_1)/2\\
u_1 &= r\cos\theta/2\cos\beta_1/2\sin(\alpha_1+\gamma_1)/2\\
u_2 &= r\cos\theta/2\sin\beta_1/2\cos(\alpha_1-\gamma_1)/2\\
u_3 &= r\cos\theta/2\sin\beta_1/2\sin(\alpha_1-\gamma_1)/2\\
u_4 &= r\sin\theta/2\cos\beta_2/2\cos(\alpha_2+\gamma_2)/2\\
u_5 &= r\sin\theta/2\cos\beta_2/2\sin(\alpha_2+\gamma_2)/2\\
u_6 &= r\sin\theta/2\sin\beta_2/2\cos(\alpha_2-\gamma_2)/2\\
u_7 &= r\sin\theta/2\sin\beta_2/2\sin(\alpha_2-\gamma_2)/2\,.
\end{aligned}
\tag{1}
$$

These coordinates may be varied in the ranges

$$0\le u<\infty, \qquad 0\le\theta\le\pi, \qquad 0\le\beta_1,\beta_2\le\pi, \qquad 0\le\alpha_1,\alpha_2<4\pi, \qquad 0\le\gamma_1,\gamma_2<2\pi\,.$$

Oscillator potential, differential elements of the length and volume and the Laplace operator have the next forms

$$V=\frac{\mu_0\omega^2u^2}{2}$$

$$dl_8^2=du^2+\frac{u^2}{4}(d\theta^2+\cos^2\theta/2dl_1^2+\sin^2\theta/2dl_2^2)$$

$$dV_8=\frac{1}{16}u^7du\sin^3\theta d\theta d\Omega_1 d\Omega_2$$

$$\Delta_8=\frac{1}{u^7}\frac{\partial}{\partial u}\left(u^7\frac{\partial}{\partial u}\right)+\frac{4}{u^2\sin^3\theta}\frac{\partial}{\partial\theta}\left(\sin^3\theta\frac{\partial}{\partial\theta}\right)-\frac{4}{u^2\cos^2\theta/2}\hat{J}_1^2-\frac{4}{u^2\sin^2\theta/2}\hat{J}_2^2\,,$$

where

$$dl_a^2 = d\alpha_a^2 + d\beta_a^2 + d\gamma_a^2 + 2\cos\beta_a d\alpha_a d\gamma_a$$

$$d\Omega_a = \frac{1}{8}\sin\beta_a d\beta_a d\alpha_a d\gamma_a$$

$$\widehat{J}_a^2 = -\left[\frac{\partial^2}{\partial\beta_a^2} + \cot\beta_a\frac{\partial}{\partial\beta_a} + \frac{1}{\sin^2\beta_a}\left(\frac{\partial^2}{\partial\alpha_a^2} - 2\cos\beta_a\frac{\partial^2}{\partial\alpha_a\partial\beta_a} + \frac{\partial^2}{\partial\gamma_a^2}\right)\right].$$

3 Spherical Bound States

In the coordinates (1) the scheme of separation of the variables corresponds to the factorization

$$\Psi^{sph} = R(u)\Phi(\theta)\mathcal{D}^{j_1}_{m_1 m_1'}(\alpha_1,\beta_1,\gamma_1)\mathcal{D}^{j_2}_{m_2 m_2'}(\alpha_2,\beta_2,\gamma_2), \tag{2}$$

where $\mathcal{D}^{j}_{mm'}$ is Wigner function [4]. Taking into account that

$$\widehat{J}_1^2\mathcal{D}^{j_1}_{m_1 m_1'}(\alpha_1,\beta_1,\gamma_1) = j_1(j_1+1)\mathcal{D}^{j_1}_{m_1 m_1'}(\alpha_1,\beta_1,\gamma_1)$$

$$\widehat{J}_2^2\mathcal{D}^{j_2}_{m_2 m_2'}(\alpha_2,\beta_2,\gamma_2) = j_2(j_2+1)\mathcal{D}^{j_2}_{m_2 m_2'}(\alpha_2,\beta_2,\gamma_2)$$

we arrive to the two coupled differential equations

$$\left[\frac{1}{\sin^3\theta}\frac{d}{d\theta}\left(\sin^3\theta\frac{d}{d\theta}\right) - \frac{4j_1(j_1+1)}{\cos^2\theta/2} - \frac{4j_2(j_2+1)}{\sin^2\theta/2} + \frac{1}{4}\Lambda(\Lambda+6)\right]\Phi(\theta) = 0. \tag{3}$$

$$\left[\frac{1}{u^7}\frac{d}{dr}\left(u^7\frac{d}{dr}\right) - \frac{\Lambda(\Lambda+6)}{u^2} - a^4u^2 + 2a^2(N+4)\right]R = 0. \tag{4}$$

Here $a = (\mu_0\omega/\hbar)^{1/2}$ and $\Lambda(\Lambda+6)$ is a separation constant. We choose the normalization condition

$$\frac{1}{16}\int_0^\pi |\Phi_{\Lambda j_1 j_2}(\theta)|^2 \sin^3\theta d\theta = 1$$

and find:

$$\Phi_{\Lambda j_1 j_2}(\theta) = \sqrt{\frac{2(\Lambda+3)(\Lambda/2-j_1-j_2)!(\Lambda/2+j_1+j_2+2)!}{(\Lambda/2-j_1+j_2+1)!(\Lambda/2+j_1-j_2+1)!}}$$

$$(\cos\theta/2)^{2j_1}(sin\theta/2)^{2j_2}P^{2j_2+1,2j_1+1}_{\Lambda/2-j_1-j_2}(\cos\theta). \tag{5}$$

The function $P_n^{(a,b)}$ is Jocobi polynomial (for definition see [5]). The radial function R, under the normalization condition

$$\int_0^\infty R^2_{N\Lambda}(u)u^7du = 1$$

can be expressed via confluent hypergeometrical function

$$R_{N\Lambda}(u) = \frac{a^4}{(\Lambda+3)!}\sqrt{\frac{2(\frac{N+\Lambda}{2}+3)!}{(\frac{N-\Lambda}{2})!}}(au)^\Lambda exp\left(-\frac{a^2u^2}{2}\right)F\left(-\frac{N-\Lambda}{2};\Lambda+4;a^2u^2\right). \tag{6}$$

Thus,the normalized spherical wave function for the 8D oscillator has the next form

$$\Psi^{sph} = \sqrt{\frac{(2j_1+1)(2j_2+1)}{2\pi^4}}R_{N\Lambda}(U)\Phi_{\Lambda j_1 j_2}(\theta)\mathcal{D}^{j_1}_{m_1 m_1'}(\alpha_1,\beta_1,\gamma_1)\mathcal{D}^{j_2}_{m_2 m_2'}(\alpha_2,\beta_2,\gamma_2). \tag{7}$$

The spherical basis (7) is the eigenfunction of the following operators $\{\widehat{H},\widehat{\Lambda}^2,\widehat{J}_1^2,\widehat{J}_2^2,\widehat{J}_{1z},\widehat{J}_{1z'},\widehat{J}_{2z},\widehat{J}_{2z'}\}$,where

$$\widehat{\Lambda}^2 = -\frac{1}{\sin^3\theta}\frac{\partial}{\partial\theta}\left(\sin^3\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\cos^2\theta/2}\widehat{J}_1^2 + \frac{1}{\sin^2\theta/2}\widehat{J}_2^2. \tag{8}$$

In Cartesian coordinates:

$$\widehat{\Lambda}^2 = -\frac{1}{4}u^2\Delta_8 + \frac{1}{4}u_iu_j\frac{\partial^2}{\partial u_i\partial u_j} + \frac{7}{4}u_i\frac{\partial}{\partial u_i}, \tag{9}$$

where $i,j = 0,1,2,\cdots,7$

4 Cylindrical 8D Coordinates

Let us denote the Eulerian 8D cylindrical coordinates by $\rho_1,\rho_2,\alpha_1,\beta_1,\gamma_1,\alpha_2,\beta_2,\gamma_2$ and assume that $\rho_1,\rho_2 \in [0,\infty)$ and $\alpha_1,\beta_1,\gamma_1,\alpha_2,\beta_2,\gamma_2$ have the same meaning as for Eulerian spherical coordinates. By definition

$$\begin{aligned}
u_0 &= \rho_1\cos\beta_1/2\cos(\alpha_1+\gamma_1)/2\\
u_1 &= \rho_1\cos\beta_1/2\sin(\alpha_1+\gamma_1)/2\\
u_2 &= \rho_1\sin\beta_1/2\cos(\alpha_1-\gamma_1)/2\\
u_3 &= \rho_1\sin\beta_1/2\sin(\alpha_1-\gamma_1)/2\\
u_4 &= \rho_2\cos\beta_2/2\cos(\alpha_2+\gamma_2)/2\\
u_5 &= \rho_2\cos\beta_2/2\sin(\alpha_2+\gamma_2)/2\\
u_6 &= \rho_2\sin\beta_2/2\cos(\alpha_2-\gamma_2)/2\\
u_7 &= \rho_2\sin\beta_2/2\sin(\alpha_2-\gamma_2)/2.
\end{aligned} \tag{10}$$

In the coordinates (10)

$$V = \frac{\mu_0\omega^2}{2}({\rho_1}^2+{\rho_2}^2).$$

As a consequence of the (10)

$$dl_8^2 = d{\rho_1}^2 + d{\rho_2}^2 + \frac{{\rho_1}^2}{4}d{l_1}^2 + \frac{{\rho_2}^2}{4}d{l_2}^2.$$

$$dV_8 = \rho_1{}^3\rho_2{}^3 d\rho_1 d\rho_2 d\Omega_1 d\Omega_2,$$

$$\Delta_8 = \frac{1}{\rho_1{}^3}\frac{\partial}{\partial\rho_1}\left(\rho_1{}^3\frac{\partial}{\partial\rho_1}\right) + \frac{1}{\rho_2{}^3}\frac{\partial}{\partial\rho_2}\left(\rho_2{}^3\frac{\partial}{\partial\rho_2}\right) - \frac{4}{\rho_1^2}\hat{J}_1^2 - \frac{4}{\rho_2^2}\hat{J}_2^2.$$

The differentials $dl_a{}^2, d\Omega_a$ and operator $\hat{J}_a^2$ have been introduced above.

5 Cylindrical Bound States

Starting from the representation

$$\Psi^{cyl} = \Phi_1(\rho_1)\Phi_2(\rho_2)\mathcal{D}^{j_1}_{m_1 m_1'}(\alpha_1,\beta_1,\gamma_1)\mathcal{D}^{j_2}_{m_2 m_2'}(\alpha_2,\beta_2,\gamma_2) \tag{11}$$

we can simply derive the two equations

$$\left[\frac{1}{\rho_1^3}\frac{d}{d\rho_1}\left(\rho_1^3\frac{d}{d\rho_1}\right) - \frac{j_1(j_1+1)}{\rho_1^2} - a^4\rho_1{}^2 + C_1\right]\Phi_1 = 0 \tag{12}$$

$$\left[\frac{1}{\rho_2^3}\frac{d}{d\rho_2}\left(\rho_2^3\frac{d}{d\rho_2}\right) - \frac{j_2(j_2+1)}{\rho_2^2} - a^4\rho_2{}^2 + C_2\right]\Phi_2 = 0, \tag{13}$$

where C_1 and C_2 are the separation constants and $C_1 + C_2 = 2\mu_0 E/\hbar^2$. Let us introduce the cylindrical quantum numbers

$$N_i = -j_i - 1 - C_i/4a^2 \tag{14}$$

with $i = 1, 2$ and $N = 2N_1 + 2N_2 + 2j_1 + 2j_2$ as the principal quantum number. Then, it is easily to show that cylindrical basis must be given by following expressions

$$\Psi^{cyl} = \sqrt{\frac{(2j_1+1)(2j_2+1)}{4\pi^4}}\Phi_{N_1 j_1}(\rho_1)\Phi_{N_2 j_2}(\rho_2)\mathcal{D}^{j_1}_{m_1 m_1'}(\alpha_1,\beta_1,\gamma_1)\mathcal{D}^{j_2}_{m_2 m_2'}(\alpha_2,\beta_2,\gamma_2), \tag{15}$$

where

$$\Phi_{N_i j_i}(\rho_i) = \frac{a^2}{(2j_i+1)!}\sqrt{\frac{2(N_i+2j_i+1)!}{(N_i)!}}(a\rho_i)^{2j_i} exp\left(-\frac{a^2\rho_i^2}{2}\right)F(-N_i; 2j_i+2; a^2\rho_i^2). \tag{16}$$

The additional integral of motion

$$\widehat{M} = -\frac{1}{\rho_2^3}\frac{\partial}{\partial\rho_2}\left(\rho_2^3\frac{\partial}{\partial\rho_2}\right) + \frac{4}{\rho_2^2}\hat{J}_2^2 + a^4\rho_2{}^2 \tag{17}$$

has eigenvalues

$$C_2 = 4a^2(N_2 + j_2 + 1). \tag{18}$$

In Cartesian coordinates, the operator $\widehat{M}$ can be rewritten as

$$\widehat{M} = -\frac{\partial^2}{\partial u_\sigma \partial u_\sigma} + a^4 u_\sigma u_\sigma \tag{19}$$

with $\sigma = 4, 5, 6, 7$.

6 Interbasis Amplitudes 1

Now, we can write, for fixed value of energy, the parabolic bound states (11) as a coherent quantum mixture of the spherical bound states:

$$\Psi^{cyl} = \sum_{\Lambda=2j_1+2j_2}^{N} W^{\Lambda}_{N_1 N_2 j_1 j_2}\Psi^{sph}. \tag{20}$$

The aim of this section will be to obtain the explicit form of the amplitudes $W^{\Lambda}_{N_1 N_2 j_1 j_2}$. At first, note that

$$\rho_1 = u\cos\theta/2, \qquad \rho_2 = u\sin\theta/2. \tag{21}$$

Then, substituting $\theta = 0$, taking into account that

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+\beta)_n}{n!} \tag{22}$$

and using the orthogonality relation [6]

$$\int_0^\infty u^5 R_{N\Lambda'}(u)R_{N\Lambda}(u)du = \frac{a^2}{\Lambda+3}\delta_{\Lambda\Lambda'} \tag{23}$$

we get for amplitudes W the following integral representation

$$W^{\Lambda}_{N_1 N_2 j_1 j_2} = \frac{\sqrt{(\Lambda+3)\left(\frac{\Lambda}{2} - j_1 - j_2\right)!}}{(2j_1+1)(\Lambda+3)!}E^{\Lambda j_1 j_2}_{N N_1 N_2}K^{N N_1}_{\Lambda j_1 j_2}. \tag{24}$$

Here

$$E^{\Lambda j_1 j_2}_{N N_1 N_2} = \left[\frac{(\frac{\Lambda}{2} + j_1 - j_2 + 1)!(N_1 + 2j_1 + 1)!(N_2 + 2j_2 + 1)!(\frac{N+\Lambda}{2} + 3)!}{(N_1)!(N_2)!(\frac{\Lambda}{2} - j_1 + j_2 + 1)!(\frac{\Lambda}{2} + j_1 + j_2 + 2)!(\frac{N-\Lambda}{2})!}\right]^{1/2} \tag{25}$$

$$K^{N N_1}_{\Lambda j_1 j_2} = \int_0^\infty x^{\frac{\Lambda}{2}+j_1+j_2+2}e^{-x}{}_1F_1(-N_1, 2j_1+1; x)\,{}_1F_1(-\frac{N-\Lambda}{2}, \Lambda+4; x)\,dx. \tag{26}$$

and

$$x = a^2u^2.$$

After writing the $F(-N_1; 2j_1+1; x)$ as a series, integrating according to [7]

$$\int_0^\infty e^{-\lambda x}x^\nu{}_1F_1(\alpha,\gamma; kx)\,dx = \frac{\Gamma(\nu+1)}{\lambda^{\nu+1}}{}_2F_1\left(\alpha, \nu+1, \gamma; \frac{k}{\lambda}\right) \tag{27}$$

and using the formula

$${}_2F_1(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{28}$$

for the summation of the Gauss hypergeometric function, we obtain

$$K^{NN_1}_{\Lambda j_1 j_2} = \frac{(\Lambda+3)!(\frac{N}{2}-j_1-j_2)(\frac{\Lambda}{2}+j_1+j_2+2)!}{(\frac{\Lambda}{2}-j_1-j_2)!(\frac{N+\Lambda}{2}+3)!}$$

$$ {}_3F_2\left\{ \begin{matrix} -N_1, -\frac{\Lambda}{2}+j_1+j_2, \frac{\Lambda}{2}+j_1+j_2+3 \\ 2j_1+1, -\frac{N}{2}+j_1+j_2 \end{matrix} \middle| 1 \right\} \qquad (29)$$

and thus come to representation

$$W^{\Lambda}_{N_1 N_2 j_1 j_2} = \left[\frac{(\Lambda+3)(\frac{\Lambda}{2}+j_1+j_2+2)!(\frac{\Lambda}{2}+j_1-j_2+1)!(N_1+2j_1+1)!(N_2+2j_2+1)!}{(N_1)!(N_2)!(\frac{\Lambda}{2}-j_1-j_2)!(\frac{\Lambda}{2}-j_1+j_2+1)!(\frac{N-\Lambda}{2})!(\frac{N+\Lambda}{2}+3)!}\right]^{1/2}$$

$$\frac{\left(\frac{N}{2}-j_1-j_2\right)!}{(2j_1+1)!} {}_3F_2\left\{ \begin{matrix} -N_1, -\frac{\Lambda}{2}+j_1+j_2, \frac{\Lambda}{2}+j_1+j_2+3 \\ 2j_1+1, -\frac{N}{2}+j_1+j_2 \end{matrix} \middle| 1 \right\}. \qquad (30)$$

7 Interbasis Amplitudes 2

The next step is to derive the alternative representation for $W^{\Lambda}_{N_1 N_2 j_1 j_2}$. It is sufficient, for this purpose, to write the Clebsch-Gordan coefficients for the group $SU(2)$ in terms of ${}_3F_2$ - function

$$C^{c\gamma}_{a\alpha;b\beta} = (-1)^{a-\alpha}\delta_{\gamma,\alpha+\beta}(a+b-\gamma)!(b+c-\alpha)!$$

$$\left[\frac{(2c+1)(a+\alpha)!(c+\gamma)!}{(a-\alpha)!(b-\beta)!(b+\beta)!(c-\gamma)!(a+b-c)!(a-b+c)!(b-a+c)!(a+b+c+1)!}\right]^{1/2}$$

$$ {}_3F_2\left\{ \begin{matrix} -a-b-c-1, -a+\alpha, -c+\gamma \\ -a-b+\gamma, -b-c+\alpha \end{matrix} \middle| 1 \right\}. \qquad (31)$$

To use formula [8]

$$ {}_3F_2\left\{ \begin{matrix} s, s', -N \\ t', 1-N-t \end{matrix} \middle| 1 \right\} = \frac{(t+s)_N}{(t)_N} {}_3F_2\left\{ \begin{matrix} s, t'-s', -N \\ t', t+s \end{matrix} \middle| 1 \right\} \qquad (32)$$

equation (31) can be rewritten in the form

$$C^{c\gamma}_{a\alpha;b\beta} = (-1)^{a-\alpha}\delta_{\gamma,\alpha+\beta}\left[\frac{(2c+1)(b-a+c)!(a+\alpha)!(b+\beta)!(c+\gamma)!}{(a-\alpha)!(b-\beta)!(c-\gamma)!(a+b-c)!(a-b+c)!(a+b+c+1)!}\right]^{1/2}$$

$$\frac{(a+b-\gamma)!}{(b-a+\gamma)!} {}_3F_2\left\{ \begin{matrix} -a+\alpha, c+\gamma+1, -c+\gamma \\ \gamma-a-b, b-a+\gamma+1 \end{matrix} \middle| 1 \right\}. \qquad (33)$$

By comparing (33) and (30),we finally obtain the desired representation

$$W^{\Lambda}_{N_1 N_2 j_1 j_2} = (-1)^{N_1} C^{\frac{\Lambda}{2}+1, j_1+j_2+1}_{\frac{N-2j_1+2j_2+2}{4}, \frac{2j_2+N_2-N_1+1}{2}; \frac{N+2j_1-2j_2+2}{4}, \frac{2j_1+N_1-N_2+1}{2}} . \qquad (34)$$

The transformation inverse to (21), namely

$$\Psi^{sph} = \sum_{N_1=0}^{\frac{N}{2}-j_1-j_2} \widetilde{W}^{N_1}_{N\Lambda j_1 j_2}\Psi^{cyl}, \qquad (35)$$

is an immediate consequence of the orthonormality propriety of the $SU(2)$ Clebsch–Gordan coefficients. The expansion coefficients in (35) are thus given by

$$\widetilde{W}^{N_1}_{N\Lambda j_1 j_2} = (-1)^{N_1} C^{\frac{\Lambda}{2}+1, j_1+j_2+1}_{\frac{N-2j_1+2j_2+2}{4}, \frac{N-2j_1+2j_2+2}{4}-N_1; \frac{N+2j_1-2j_2+2}{4}, N_1+2j_1-\frac{N+2j_1-2j_2-2}{4}} \qquad (36)$$

and may be expressed in terms of the ${}_3F_2$ function through (31) or (33).

8 Spheroidal 8D Coordinates

The spheroidal coordinates to be defined as following

$$\begin{aligned} u_0 &= \tfrac{d}{2}\sqrt{(\xi+1)(1+\eta)}\cos\beta_1/2\cos(\alpha_1+\gamma_1)/2 \\ u_1 &= \tfrac{d}{2}\sqrt{(\xi+1)(1+\eta)}\cos\beta_1/2\sin(\alpha_1+\gamma_1)/2 \\ u_2 &= \tfrac{d}{2}\sqrt{(\xi+1)(1+\eta)}\sin\beta_1/2\cos(\alpha_1-\gamma_1)/2 \\ u_3 &= \tfrac{d}{2}\sqrt{(\xi+1)(1+\eta)}\sin\beta_1/2\sin(\alpha_1-\gamma_1)/2 \\ u_4 &= \tfrac{d}{2}\sqrt{(\xi+1)(1+\eta)}\cos\beta_2/2\cos(\alpha_2+\gamma_2)/2 \\ u_5 &= \tfrac{d}{2}\sqrt{(\xi+1)(1+\eta)}\cos\beta_2/2\sin(\alpha_2+\gamma_2)/2 \\ u_6 &= \tfrac{d}{2}\sqrt{(\xi+1)(1+\eta)}\sin\beta_2/2\cos(\alpha_2-\gamma_2)/2 \\ u_7 &= \tfrac{d}{2}\sqrt{(\xi+1)(1+\eta)}\sin\beta_2/2\sin(\alpha_2-\gamma_2)/2 \\ & 1\le\xi<\infty, -1\le\eta\le 1, \end{aligned} \qquad (37)$$

where d is the interfocus distance. In the system of spheroidal coordinates the Coulomb potential can be written as

$$V = \frac{\mu_0 d^2\omega^2}{4}(\xi+\eta).$$

In the coordinates (37) differential elements of the length, volume and the Laplace operator have the next forms

$$dl_8^2 = \frac{d^2}{8}(\xi-\eta)\left(\frac{d\xi^2}{\xi^2-1}+\frac{d\eta^2}{1-\eta^2}\right)+\frac{d^2}{16}(\xi+1)(1+\eta)dl_1^2+\frac{d^2}{16}(\xi-1)(1-\eta)dl_2^2$$

$$dV_8 = \frac{d^8}{2^9}(\xi-\eta)(\xi^2-1)(1-\eta^2)d\xi d\eta d\Omega_1 d\Omega_2$$

$$\Delta_8 = \frac{8}{d^2(\xi-\eta)}\left\{\frac{1}{\xi^2-1}\frac{\partial}{\partial\xi}\left[(\xi^2-1)^2\frac{\partial}{\partial\xi}\right]+\frac{1}{1-\eta^2}\frac{\partial}{\partial\eta}\left[(1-\eta^2)^2\frac{\partial}{\partial\eta}\right]\right\}$$

$$-\frac{16\widehat{J}_1^2}{d^2(\xi+1)(1+\eta)}-\frac{16\widehat{J}_2^2}{d^2(\xi-1)(1-\eta)}. \qquad (38)$$

9 Separation of variables in Spheroidal 8D Coordinates

The Schrödinger equation for the potential is separable in spheroidal coordinates. As a point of fact, by looking for a solution of this equation in the form

$$\Psi^{spheroidal} = f_1(\xi) f_2(\eta) \mathcal{D}^{j_1}_{m_1 m_2'}(\alpha_1, \beta_1, \gamma_1) \mathcal{D}^{j_2}_{m_2 m_2'}(\alpha_2, \beta_2, \gamma_2) \tag{39}$$

we obtain the two ordinary differential equations

$$\left[\frac{1}{\xi^2-1}\frac{d}{d\xi}(\xi^2-1)^2\frac{d}{d\xi} + \frac{2j_1(j_1+1)}{\xi+1} - \frac{2j_2(j_2+1)}{\xi-1} + \frac{\mu_0 d^2 E}{4\hbar^2}(\xi-1) - \frac{a^4 d^4}{16}(\xi^2-1)\right] f_1 =$$

$$= X(d) f_1 , \tag{40}$$

$$\left[\frac{1}{1-\eta^2}\frac{d}{d\eta}(1-\eta^2)^2\frac{d}{d\eta} - \frac{2j_1(j_1+1)}{1+\eta} - \frac{2j_2(j_2+1)}{1-\eta} + \frac{\mu_0 d^2 E}{4\hbar^2}(1-\eta) - \frac{a^4 d^4}{16}(1-\eta^2)\right] f_2 =$$

$$= -X(d) f_2 , \tag{41}$$

where $X(d)$ is a separation constant in spheroidal coordinates. By eliminating the energy E from (40) and (41), we produce the operator

$$\widehat{X} = \frac{1}{\xi-\eta}\left\{\frac{1-\eta}{\xi-1}\frac{\partial}{\partial\xi}(\xi^2-1)^2\frac{\partial}{\partial\xi} - \frac{\xi-1}{1-\eta}\frac{\partial}{\partial\eta}(1-\eta^2)^2\frac{\partial}{\partial\eta}\right\} +$$

$$\frac{2(\xi+\eta)}{(\xi+1)(1+\eta)}\hat{J}_1^2 + \frac{2(\xi+\eta-2)}{(\xi-1)(1-\eta)}\hat{J}_2^2 - \frac{a^4 d^4 (\xi-1)(1-\eta)}{16} \tag{42}$$

eigenvalues of which are $X(d)$ and eigenfunctions of which are $\Psi^{spheroidal}$. The significance of the operator $\widehat{X}$ can be found by switching to Cartesian coordinates. A long calculation gives

$$\widehat{X} = \hat{\Lambda}^2 + \frac{d}{4}\widehat{M} \cdot \tag{43}$$

10 Spheroidal Superpositions

From what precedes, we have three sets of commuting operators, viz., $\{\widehat{H}, \hat{\Lambda}^2, \hat{J}_1^2, \hat{J}_{1z}, \hat{J}_{1z'}, \hat{J}_2^2, \hat{J}_{2z}, \hat{J}_{2z'}\}$, $\{\widehat{H}, \widehat{M}, \hat{J}_1^2, \hat{J}_{1z}, \hat{J}_{1z'}, \hat{J}_2^2, \hat{J}_{2z}, \hat{J}_{2z'}\}$ and $\{\widehat{H}, \widehat{X}, \hat{J}_1^2, \hat{J}_{1z}, \hat{J}_{1z'}, \hat{J}_2^2, \hat{J}_{2z}, \hat{J}_{2z'}\}$ corresponding to the spherical, cylindrical and spheroidal coordinates. In particular, we have

$$\hat{\Lambda}^2 \Psi^{sph} = \frac{1}{4}\Lambda(\lambda+6)\Psi^{sph} \tag{44}$$

$$\widehat{M}\Psi^{cyl} = 4a^2(N_2 + j_2 + 1)\Psi^{cyl} = 2a^2(N - N_1 - j_1 + 2)\Psi^{cyl} \tag{45}$$

and

$$\widehat{X}\Psi^{spheroidal} = X_q(d)\Psi^{spheroidal} \tag{46}$$

for the spherical, cylindrical and spheroidal basises. In (46), the index q labels the eigenvalues of the operator $\widehat{X}$ and varies in the range $0 \le q \le N/2 - j_1 - j_2 - 1$. We are now ready to deal with the interbasis expansions

$$\Psi^{spheroidal} = \sum_{\Lambda=2j_1+2j_2}^{N} T^{\Lambda}_{Nqj_1j_2}(d)\Psi^{sph} \tag{47}$$

$$\Psi^{spheroidal} = \sum_{N_1=0}^{\frac{N}{2}-j_1-j_2} U^{N_1}_{Nqj_1j_2}(d)\Psi^{cyl} \tag{48}$$

for the spheroidal basis in terms of the spherical and cylindrical basises.

First, we consider Eq. (47). Let the operator $\widehat{X}$ act on both sides of (47). Then, by the standard techniques we find that

$$[X_q(d) - \frac{1}{4}\Lambda(\Lambda+6)]T^{\Lambda}_{Nqj_1j_2}(d) = \frac{d^2}{4}\sum_{\Lambda'=2j_1+2j_2}^{N} T^{\Lambda'}_{Nqj_1j_2}(d)(\widehat{M})_{\Lambda\Lambda'} \tag{49}$$

where

$$(\widehat{M})_{\Lambda\Lambda'} = \int \Psi^{*sph}_{\Lambda}\widehat{M}\Psi^{sph}_{\Lambda'}\, dV . \tag{50}$$

The calculation of the matrix element $(\widehat{M})_{\Lambda\Lambda'}$ can be done by expanding the spherical wave functions in (51) in terms of cylindrical wave functions. This leads to

$$(\widehat{M})_{\Lambda\Lambda'} = 2a^2 \sum_{N_1=0}^{\frac{N}{2}-j_1-j_2} (N - 2N_1 - 2j_1 + 2)\widetilde{W}^{N_1}_{N\Lambda j_1 j_2}\widetilde{W}^{N_1}_{N\Lambda' j_1 j_2} \tag{51}$$

Now, by using (26) together with the recursion relation [4]

$$C^{c\gamma}_{a\alpha;b\beta} = -\left[\frac{4c^2(2c-1)(2c+1)}{(c-\gamma)(c+\gamma)(-a+b+c)(a-b+c)(a+b-c+1)(a+b+c+1)}\right]^{1/2}$$

$$\left\{\left[\frac{(c-\gamma-1)(c+\gamma-1)(-a+b+c-1)(a-b+c-1)(a+b-c+2)(a+b+c)}{4(c-1)^2(2c-3)(2c-1)}\right]^{1/2} C^{c-2,\gamma}_{a\alpha;b\beta}\right.$$

$$- \frac{(\alpha-\beta)c(c-1)-\gamma a(a+1)+\gamma b(b+1)}{2c(c-1)} C^{c-1,\gamma}_{a\alpha;b\beta} \Big\} \qquad (52)$$

and the orthonormality condition

$$\sum_{\alpha,\beta} C^{c\gamma}_{a\alpha;b\beta} C^{c'\gamma'}_{a\alpha;b\beta} = \delta_{c'c}\delta_{\gamma'\gamma}, \qquad (53)$$

we find that $(\widehat{M})_{\Lambda\Lambda'}$ is given by

$$(\widehat{M})_{\Lambda\Lambda'} = a^2(N+4)\left[1+\frac{4(j_2-j_1)(j_1+j_2+1)}{(\Lambda+2)(\Lambda+4)}\right]\delta_{\Lambda',\Lambda} + 2a^2\left(A^{\Lambda+2}_{Nj_1j_2}\delta_{\Lambda',\Lambda+2} + A^{\Lambda}_{Nj_1j_2}\delta_{\Lambda',\Lambda-2}\right), \qquad (54)$$

where

$$A^{\Lambda}_{Nj_1j_2} = \sqrt{(N-\Lambda+2)(N+\Lambda+6)}$$
$$\left[\frac{(\Lambda-2j_1-2j_2)(\Lambda+2j_1+2j_2+4)(\Lambda+2j_1-2j_2+2)(\Lambda-2j_1+2j_2+2)}{16(\Lambda+2)^2(\Lambda+1)(\Lambda+3)}\right]^{1/2}. \qquad (55)$$

So, we get the following three-term recursion relation for the coefficient $T^{\Lambda}_{Nqj_1j_2}(d)$

$$\left[X_q(d) - \frac{1}{4}\Lambda(\Lambda+6) - \frac{a^2d^2}{4}(N+4)\left[1+\frac{4(j_2-j_1)(j_1+j_2+1)}{(\Lambda+2)(\Lambda+4)}\right]\right]T^{\Lambda}_{Nqj_1j_2}(d) =$$
$$\frac{a^2d^2}{2}\left[A^{\Lambda+2}_{Nj_1j_2}T^{\Lambda+2}_{Nqj_1j_2}(d) + A^{\Lambda}_{Nj-1j_2}T^{\Lambda-2}_{Nqj_1j_2}(d)\right]. \qquad (56)$$

The recursion relation (56) provides us with a system of $N-2j_1-2j_2$ linear homogeneous equations which can be solved by taking into account the normalization condition

$$\sum_{\Lambda=2j_1+2j_2}^{N} |T^{\Lambda}_{Nqj_1j_2}(d)|^2 = 1. \qquad (57)$$

The eigenvalues $X_q(d)$ of the operator $\widehat{X}$ then follow from the vanishing of the determinant for the latter system.

Second, let us concentrate on the expansion (48) of the spheroidal basis in terms of the cylindrical basis. By employing a technique similar to the one used for deriving (49), we get

$$\left[X_q(d) - \frac{a^2d^2}{2}(N-2N_1-2j_1+2)\right]U^{N_1}_{Nqj_1j_2}(d) = \sum_{N_1'=0}^{\frac{N}{2}-j_1-j_2} U^{N_1'}_{Nqj_1j_2}(d)(\widehat{\Lambda}^2)_{N_1'N_1'}, \qquad (58)$$

where

$$(\widehat{\Lambda}^2)_{N_1N_1'} = \int \Psi^{*cyl}_{N_1}\widehat{\Lambda}^2\Psi^{cyl}_{N_1'}\,dV. \qquad (59)$$

The matrix elements $(\widehat{\Lambda}^2)_{N_1N_1'}$ can be calculated in the same way as $(\widehat{M})_{\Lambda\Lambda'}$ except that now we must use, instead of (52) and (53), the relation [4]

$$[(b-a+c)(a-b+c+1)]^{1/2}C^{c\gamma}_{a\alpha;b\beta} = [(a-\alpha+1)(b-\beta)]^{1/2}C^{c\gamma}_{a+1/2,\alpha-1/2;b-1/2,\beta+1/2}$$
$$+\,[(a+\alpha+1)(b+\beta)]^{1/2}C^{c\gamma}_{a+1/2,\alpha+1/2;b-1/2,\beta-1/2} \qquad (60)$$

and the orthonormality condition

$$\sum_{c,\gamma} C^{c\gamma}_{a\alpha;b\beta}C^{c\gamma}_{a\alpha';b\beta'} = \delta_{\alpha\alpha'}\delta_{\beta\beta'}. \qquad (61)$$

This yields the matrix element

$$(\widehat{\Lambda}^2)_{N_1N_1'} = \left[(N_1+1)\left(\frac{N}{2}-N_1-j_1-j_2\right)+(j_1-j_2)(j_1-j_2-1)-2+\right.$$
$$\left.(N_1+2j_1+1)\left(\frac{N}{2}-N_1-j_1+j_2+2\right)\right]\delta_{N_1'N_1}$$
$$-\left[N_1(\frac{N}{2}-N_1-j_1-j_2+1)(\frac{N}{2}-N_1-j_1+j_2+2)(N_1+2j_1+1)\right]^{1/2}\delta_{N_1',N_1-1} \qquad (62)$$
$$-\left[(N_1+1)(\frac{N}{2}-N_1-j_1-j_2)(\frac{N}{2}-N_1-j_1+j_2+1)(N_1+2j_1+2)\right]^{1/2}\delta_{N_1',N_1+1}.$$

Finally, the introduction of (62) into (58) leads to the three-term recursion relation

$$\left[(N_1+1)\Big(\frac{N}{2}-N_1-j_1-j_2\Big)+(N_1+2j_1+1)\Big(\frac{N}{2}-N_1-j_1+j_2+2\Big)\right.$$
$$\left.+(j_1-j_2)(j_1-j_2-1)-2+\frac{a^2d^2}{2}(N-2N_1-2j_1+2)-X_q(d)\right]U^{N_1}_{Nqj_1j_2}(d) = \qquad (63)$$
$$\left[N_1\Big(\frac{N}{2}-N_1-j_1-j_2+1\Big)\Big(\frac{N}{2}-N_1-j_1+j_2+2\Big)(N_1+2j_1+1)\right]^{1/2}U^{N_1-1}_{Nqj_1j_2}(d)+$$
$$\left[(N_1+1)\Big(\frac{N}{2}-N_1-j_1-j_2\Big)\Big(\frac{N}{2}-N_1-j_1+j_2+1\Big)(N_1+2j_1+2)\right]^{1/2}U^{N_1+1}_{Nqj_1j_2}(d)$$

for the expansion coefficients $U^{N_1}_{Nqj_1j_2}(d)$. This relation can be iterated by taking account of the normalization condition

$$\sum_{N_1=0}^{\frac{N}{2}-j_1-j_2}|U^{N_1}_{Nqj_1j_2}(d)|^2 = 1. \qquad (64)$$

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