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G.N.Afanasiev*, Yu.P.Stepanovsky[1]

THE ELECTROMAGNETIC FIELD OF ELEMENTARY TIME-DEPENDENT TOROIDAL SOURCES

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[1]The Institute of Physics and Technology, Kharkov, Ukraine
*e-mail: afanasev@theor.jinrc.dubna.su

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Афанасьев Г.Н., Степановский Ю.П. E2-94-425

Электромагнитное поле зависящих от времени элементарных тороидальных источников

Изучены поля излучения зависящих от времени тороидальных токовых распределений. Найдены зависящие от времени распределения токов и зарядов, вне которых исчезают электромагнитные напряженности, но не потенциалы. Это может быть использовано для постановки экспериментов типа Ааронова—Бома, в которых потенциалы зависят от времени, а также для передачи информации. Используя параметризацию Неймана—Гельмгольца токовой плотности, мы нашли удобное для применений представление зависящего от времени электромагнитного поля.

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Afanasiev G.N., Stepanovsky Yu.P. E2-94-425

The Electromagnetic Field of Elementary Time-Dependent Toroidal Sources

The radiation field of toroidal-like time-dependent current configurations is investigated. Time-dependent charge-current sources are found outside which the electromagnetic strengths disappear but the potentials survive. This can be used to carry out time-dependent Aharonov—Bohm-like experiments and the information transfer. Using the Neumann—Helmholtz parametrization of the current density we present the time-dependent electromagnetic field in a form convenient for applications.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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1. Introduction

Interest in the time-dependent currents flowing in the toroidal coils is due to the following remark made by James Clerk Maxwell in his memoir " On physical lines of force " / 1 / :

" Let B, fig.3, be a circular ring of uniform section, lapped uniformly with covered wire. It may be shewn that if an electric current is passed through this wire, a magnet placed within the coil of wire will be strongly affected, but no magnetic effect will be produced on any external point. The effect will be that of magnet bent round till its two poles are in contact.

If the coil is properly made, no effect on a magnet placed outside it can be discovered, whether the current is kept constant or made to vary in strength; but if a conducting wire C be made to embrace the ring any number of times, an electromotive force will act on this wire whenever the current in the coil is made to vary; and if the circuit be closed, there will be an actual current in the wire C."

Fig.3 mentioned in this passage shows the torus with a poloidal winding on its surface. At the present time, it is known that in general this Maxwell assertion is not correct. It turns out that for the time-dependent current in the toroidal coil the electromagnetic field strengths appear outside it. Qualitatively this was shown by Mitkevich / 2 / and Page / 3 /. The corresponding experiments were performed by Mitkevich / 2 /, Ryazanov / 4 /, Bartlett and Ward / 5 / and many others. The quantitative results were obtained in ref./ 6 / where the electromagnetic fields were evaluated for a number of time dependences of the current flowing in the toroidal coil. After all, experimentalists widely use the toroidal transformers for their own purposes without philosophizing on this subject. The sole exception for which Maxwell's claim holds is the current linearly rising in time which flows in the toroidal coil. In this case H=0 and E is independent of time outside the torus (see, e.g., Miller / 7 /). The question of the energy transfer into the the wire C embracing the torus was considered by Heald / 8 / (the difficulty is that the Poynting vector equals zero for the treated case).

In the previous paper / 9 /, we have studied the electromagnetic field (EMF) of the static toroidal-like configurations, their interactions with the external EMF and possible physical applications. It is the goal of the present consideration to study nonstatic current configurations. Probably, it would be appropriate to explain the meaning of the words " elementary toroidal sources " in the title of this paper. The words " toroidal source " mean the poloidal current flowing in the winding of the toroidal solenoid (TS), which in turn may be an element of a more complex configuration. When the dimensions of this configuration tend to zero, we obtain an "elementary toroidal source". The TS with finite dimensions has a number of nontrivial topological properties (see, e.g., review papers / 10 /). Suppose that these properties survive when the TS dimensions tend to zero. The reason for the treatment of an elementary toroidal source is due to the considerable simplification of the theoretical consideration. Thus, if we find some interesting property for the elementary toroidal source, there is a chance for it to be survived for the finite toroidal configuration. This is confirmed for the simplest toroidal configurations for which the analytical solutions

can be found. As an example, mention the configuration consisting of the TS with a linearly growing current flowing in its winding and the double charged layer lying at the hole of TS. Outside this configuration, electromagnetic strengths disappear but the nontrivial (i.e., unremovable by the gauge transformation) time- dependent vector potential survives. Thus, the possibility arises to perform a time--dependent Aharonov-Bohm-like experiment. However, the linear time-- dependence of the current is unrealistic. It is the aim of this paper to find elementary charge-current configurations possessing radiationless properties mentioned above but with a rather arbitrary time dependence. The plan of our exposition is as follows.
The radiation of elementary time-dependent toroidal-like configurations, in the winding of which the time-dependent current flows, is studied in sect. 2. It turns out that two different branches of these configurations generate essentially different EMF. On the other hand, the current sources of the same family generate the same EMF if their time dependences are properly adjusted. We give an example of the radiationless charge-current source having the property that electromagnetic field strengths disappear outside it but the nontrivial time-dependent potentials survive there. The extended toroidal-like current sources are considered in sect. 3.
By using the Neumann-Helmholtz parametrization for the current density the convenient formulas for the time-dependent EMF are obtained. Basing on them, the radiationless charge-current sources of higher multipolarities are constructed and their possible applications are considered. Particularly, it is shown that the elementary time-dependent charge distribution and poloidal current produce the

same electromagnetic strengths in the surrounding them space provided their multipolarities and time dependences are properly adjusted. Thus, the multipole expansion of these strengths is the same and this in turn justifies the use of double names (known in physical literature as electric and toroidal multipoles) for the particular terms occurring in this expansion.

2. The radiation of the elementary toroidal sources.

A pedagogical example: time-dependent circular current.

According to the Ampere hypothesis the distribution of the magnetic dipoles $\vec{M}(\vec{r})$ is equivalent to the current distribution J(r)= = rot M(r). For example, the circular current flowing in the Z=0 plane

$$\vec{j} = I \cdot \vec{n}_\varphi \cdot \delta(\rho - d)\cdot\delta(z) \qquad (2.1)$$

is equivalent to the the magnetized sheet

$$\vec{M} = I \cdot \vec{n}_z \cdot \Theta(d-\rho)\,\delta(z) \qquad (2.2)$$

lying in the same plane ($\Theta(x)$ is a step function). When the radius d of the circumference along which the current flows tends to zero, the current J becomes ill-defined (it is not clear what does the vector $\vec{n}_\varphi$ mean at the origin). On the other hand, the vector $\vec{M}$ is still well-defined. In this limit the elementary current (2.1) turns out to be equivalent to the magnetic dipole oriented normally to the plane of this current. It is convenient to introduce $I/\pi d^2$ instead of I in Eqs.(2.1),(2.2). Then, in the limit d $\to$0 one gets

$$\vec{M} = I\cdot\vec{n}\cdot\delta^3(\vec{r}) \qquad (\delta^3(\vec{r}) = \delta(\rho)\cdot\delta(z)/2\pi\rho) \qquad (2.3)$$

and

$$\vec{j} = I\cdot \mathrm{rot}\,\vec{n}\,\delta^3(\vec{r}) \qquad (2.4)$$

It turns out that Eqs.(2.3) and (2.4) give the magnetization and current density corresponding to the elementary magnetic dipole. These questions were considered in detail in ref./ 9 /. Now let the intensity of the elementary current change with time

$$\vec{j}_0 = f_0(t)\cdot \mathrm{rot}\,\vec{n}\,\delta^3(\vec{r}) \qquad (2.5)$$

(the factor I is absorbed into $f_0(t)$). The vector potential (VP) corresponding to this current is elementarily obtained:

$$\vec{A}_0 = -D_0\,\frac{1}{c^2 r^2}(\vec{r}\times\vec{n}), \qquad D_K = \dot{f}_K + \frac{c}{r} f_K \qquad (2.6)$$

Here and hereafter the time derivative will be denoted either by the point above the letter or (especially for higher derivatives) by the superscripts.For example, $f^{(3)} \equiv \dddot{f} = d^3 f/dt^3$ .The argument of the f functions, if not indicated, means $t - r/c$ everywhere in this section. The electromagnetic field strengths (FS) are

$$\vec{E}_0 = \frac{1}{c^3 r^2}\dot{D}_0\cdot(\vec{r}\times\vec{n}), \qquad \vec{H}_0 = \frac{(\vec{r}\vec{n})}{c^3 r^3}\cdot F_0\cdot\vec{r} - \frac{1}{c^3 r} G_0\cdot\vec{n} \qquad (2.7)$$

where for brevity we put

$$F_K = \ddot{f}_K + 3\frac{c}{r}\dot{f}_K + 3\frac{c^2}{r^2} f_K, \qquad G_K = \ddot{f}_K + \frac{c}{r}\dot{f}_K + \frac{c^2}{r^2} f_K$$

The flux of the electromagnetic energy through the sphere of the radius r is

$$S = \int P_r\, r^2 d\Omega = \frac{2}{3c^5}\dot{D}_0\cdot G_0, \qquad \vec{P} = \frac{c}{4\pi}(\vec{E}_0\times\vec{H}_0) \qquad (2.8)$$

This flux is positive for large distances and determined by the second derivative of f_0 $\left(S \approx \frac{2}{3c^5} \ddot{f}_0^2 \right)$ .However, for small distances it may be negative. These results are well known and may be found in many text-books (see,e.g., Stratton / 11 /).

The elementary radiating toroidal solenoid.

The next in complexity case is the radiation of the current flowing in the winding of elementary (i.e.infinitely) small toroidal solenoid (TS). According to / 9 /, this elementary current is given by

$$\vec{j}_1 = f_1(t) \cdot \mathrm{rot}^{(2)} \vec{n}\, \delta^3(\vec{r}) \tag{2.9}$$

where $\mathrm{rot}^{(2)} = \mathrm{rot}\,\mathrm{rot}$ and $\vec{n}$ means the normal to the equatorial plane of TS. The VP and FS are equal to

$$\Phi_1 = 0\,, \quad \vec{A}_1 = -\vec{n} \frac{1}{c^3 r} G_1 + \vec{r} \frac{1}{c^3 r^3} (\vec{r}\vec{n}) F_1\,,$$

$$\vec{E}_1 = \vec{n} \frac{1}{c^4 r} \dot{G}_1 - \vec{r} \frac{1}{c^4 r^3} (\vec{r}\vec{n}) \dot{F}_1\,, \quad \vec{H}_1 = \frac{1}{c^4 r^2} (\vec{r} \times \vec{n}) \ddot{D}_1 \tag{2.10}$$

In this and the following equations occurring in this section we omit the δ function terms giving the field values at the origin (to which the current is confined). Thus, Eqs. (2.10) are valid everywhere except for the origin.

Digression on the radiationless sources of electromagnetic fields.

Consider the electric dipole oriented in the n direction.Its charge density is

$$\rho_d = e\,[\,\delta^3(\vec{r} + a\vec{n}) - \delta^3(\vec{r} - a\vec{n})\,]$$

For small separation a this reduces to

$$\rho_d = 2ea\,(\vec{n}\vec{\nabla})\,\delta^3(\vec{r})$$

Let the intensity of this dipole change with time

$$\rho_d = f_d(t)\,(\vec{n}\vec{\nabla})\,\delta^3(\vec{r})$$

(the factor $2ae$ is absorbed into f_d). The corresponding current density is given by

$$j_d = -\dot{f}_d(t)\,\vec{n}\,\delta^3(\vec{r})$$

These densities generate the following potentials and FS (see, e.g., Weinstein / 12 /):

$$\Phi_d = -\frac{1}{c r^2}(\vec{n}\vec{r})\left(\dot{f}_d + \frac{c}{r} f_d\right), \quad \vec{A}_d = -\vec{n}\,\dot{f}_d / rc\,,$$

$$\vec{H}_d = \frac{1}{c^2 r^2}(\vec{r} \times \vec{n})\left(\ddot{f}_d + \frac{c}{r}\dot{f}_d\right), \tag{2.11}$$

$$\vec{E}_d = \frac{1}{c^2 r}\vec{n}\left(\ddot{f}_d + \frac{c}{r}\dot{f}_d + \frac{c^2}{r^2} f_d\right) - \frac{1}{c^2 r^3}(\vec{r}\vec{n})\vec{r}\cdot\left(\ddot{f}_d + 3\frac{c}{r}\dot{f}_d + 3\frac{c^2}{r^2} f_d\right)$$

From the comparison of Eqs. (2.10) and (2.11) we conclude that FS of the time-dependent current flowing in the winding of the infinitely small TS coincide with that of the electric dipole if their time dependences are properly adjusted

$$f_d = \dot{f}_1 / c^2 \tag{2.12}$$

This means that the electromagnetic strengths of the time-dependent electric dipole and TS can be mutually compensated if $f_d = -\dot{f}_1 / c^2$. Then, in the surrounding space E = H = 0 and

$$\Phi = \frac{1}{c^3 r^2}(\vec{n}\vec{r})\,\dot{D}_1\,, \quad \vec{A} = -\frac{1}{c^2 r^2}\vec{n} D_1 + \frac{1}{c^3 r^3}(\vec{r}\vec{n})\vec{r}\cdot F_1$$

It is remarkable that outside this composite object (electric dipole and TS placed at the same point) there are nonvanishing time dependent electric and vector potentials despite disappearance of the field strengths. This can be used to carry out time-dependent Aharonov--Bohm like experiments. The simplest example corresponds to f_1 = const. Then,

$$\Phi = 0, \quad \vec{A} = \frac{3\vec{r}(\vec{r}\vec{n}) - r^2\vec{n}}{c r^5}$$

that coincides with VP of the elementary (i.e., infinitely small) static TS. The next in complexity case is the composite object consisting of the static electric dipole ($f_d = f$ = const) and the linearly changing with time current flowing in the winding of TS

$$\rho = f(\vec{n}\vec{\nabla})\delta^3(\vec{r}), \quad \vec{j} = -c^2 f t\, \mathrm{rot}^{(2)}\vec{n}\,\delta^3(\vec{r}),$$

$$\Phi = -f\cdot(\vec{n}\vec{r})/r^3,$$

$$\vec{A} = -ctf\,[3\vec{r}(\vec{n}\vec{r}) - r^2\vec{n}]/r^5 \qquad (2.13)$$

The TS of finite dimensions was considered in ref. / 9 /. Another interesting case is the compensation of the EMF generated by the oscillating electric dipole by that of the periodical current flowing in the winding of the TS:

$$\rho = \rho_d = f\cdot\cos\omega t\cdot\delta^3(\vec{r}), \quad \vec{j} = \vec{j}_d + \vec{j}_1, \quad \vec{E} = \vec{H} = 0,$$

$$\vec{j}_d = f\omega\sin\omega t\cdot\delta^3(\vec{r}), \quad \vec{j}_1 = -\frac{fc^2}{\omega}\sin\omega t\cdot\mathrm{rot}^{(2)}\vec{n}\,\delta^3(\vec{r}),$$

$$\Phi = \frac{1}{cr^2}(\vec{r}\vec{n})\cdot f\left(\omega\cdot\sin\Omega - \frac{c}{r}\cos\Omega\right), \quad \Omega = \omega\left(t - \frac{r}{c}\right),$$

$$\vec{A} = \frac{1}{r^2} f\vec{n}\left(\cos\Omega + \frac{c}{\omega r}\sin\Omega\right) +$$

$$+\frac{1}{cr^3}(\vec{r}\vec{n})\vec{r}\, f\left(\omega\cdot\sin\Omega - 3\frac{c}{r}\cos\Omega - 3\frac{c^2}{r^2\omega}\cos\Omega\right) \qquad (2.14)$$

It turns out that the FS are compensated if the charge density of the electric dipole oscillates in the counter-phase with the TS current. Nonvanishing electromagnetic potentials can be used as a new channel for the information transfer (by modulating the phase of the charge particle wave function).

More complicated elementary toroidal sources.

We consider the hierarchy of TS each turn of which is again TS. The simplest example is the usual TS (which is obtained by the installing of the infinitely thin TS into the single turn with current (2.5) in it). We denote this TS by T_1 (the initial current source (2.5) will be denoted by T_0). The next in complexity case is obtained when each turn of T_1 is replaced by the infinitely thin TS with alternating current in its winding. Thus obtained current configuration is denoted by T_2 . When its dimensions tend to zero, we get / 9 /

$$\vec{j}_2 = f_2(t)\cdot\mathrm{rot}^{(3)}\vec{n}\cdot\delta^3(\vec{r}) \qquad (2.15)$$

The corresponding VP and FS are given by

$$\vec{A}_2 = \frac{1}{r^2c^4}(\vec{r}\times\vec{n})\mathcal{D}_2^{(2)}, \quad \vec{E}_2 = -\frac{1}{r^2c^5}(\vec{r}\times\vec{n})\mathcal{D}_2^{(3)},$$

$$\vec{H}_2 = \frac{1}{c^5 r}\vec{n}\,G_2^{(2)} - \frac{1}{r^3c^5}\vec{r}(\vec{n}\vec{r})\,F_2^{(2)} \qquad (2.16)$$

By comparing Eqs.(2.6),(2.7) with (2.16) we conclude that the EMF coincide for the current configurations T_0 and T_2 (everywhere except for the origin) if the following relation between time--dependent intensities is fulfilled $f_2^{(2)} = -f_0/c^2$. This means, particularly, that the EMF of the static magnetic dipole (f_0 = const) coincides with the EMF of the current configuration T if the current in it quadratically varies with time ($f_2 = -f_0 c^2 t^2/2$). It follows from this that the magnetic field of the usual magnetic dipole can be compensated everywhere (except for the origin) by the time-dependent current flowing in T_2. Consider now the periodical currents $f_0 = f_{00} \cdot \cos\omega t$ and $f_2 = f_{20} \cdot \cos\omega t$. Clearly, the EMF of T_0 and T_2 coincide if $f_{20} = f_{00} \cdot c^2/\omega^2$. Now we are able to write out the EMF for the point-like toroidal configuration of the arbitrary order. Let

$$\vec{j}_m = f_m \cdot \mathrm{rot}^{(m+1)}\, \vec{n}\, \delta^3(\vec{r})$$

Then, for m even (m=2k, k ≥ 0)

$$\vec{A}_{2k} = \frac{(-1)^{k+1}}{c^{2k+2}} \frac{\vec{r}\times\vec{n}}{r^2} D_{2k}^{(2k)},$$

$$\vec{E}_{2k} = \frac{(-1)^{k}}{c^{2k+3}} \frac{\vec{r}\times\vec{n}}{r^2} D_{2k}^{(2k+1)}, \qquad (2.17)$$

$$\vec{H}_{2k} = \frac{(-1)^{k}}{c^{2k+3}} \left[\vec{r}\, \frac{(\vec{r}\vec{n})}{r^3} F_{2k}^{(2k)} - \vec{n}\, \frac{1}{r} G_{2k}^{(2k)} \right]$$

The flux of the electromagnetic energy through the sphere of the radius r is equal to

$$S = \frac{2}{3} \frac{1}{c^{4k+5}} \left[f_{2k}^{(2k+2)} + \frac{c}{r} f_{2k}^{(2k+1)} + \frac{c^2}{r^2} f_{2k}^{(2k)} \right] \cdot \left[f_{2k}^{(2k+2)} + \frac{c}{r} f_{2k}^{(2k+1)} \right]$$

On the other hand, for m odd (m = 2k+1, $m \geq 0$)

$$\vec{A}_{2k+1} = \frac{(-1)^{k}}{c^{2k+3}} \left[\frac{1}{r^3} \vec{r}\,(\vec{r}\vec{n}) \cdot F_{2k+1}^{(2k)} - \frac{1}{r} \vec{n}\, G_{2k+1}^{(2k)} \right],$$

$$\vec{E}_{2k+1} = \frac{(-1)^{k+1}}{c^{2k+4}} \left[\frac{1}{r^3} \vec{r}\,(\vec{r}\vec{n}) \cdot F_{2k+1}^{(2k+1)} - \frac{1}{r} \vec{n}\, G_{2k+1}^{(2k+1)} \right],$$

$$\vec{H}_{2k+1} = \frac{(-1)^{k}}{c^{2k+4}} \frac{1}{r^2} (\vec{r}\times\vec{n})\, D_{2k+1}^{(2k+2)}, \qquad (2.18)$$

$$S = \frac{2}{3} \frac{1}{c^{4k+5}} \left(f_{2k+1}^{(2k+3)} + \frac{c}{r} f_{2k+1}^{(2k+2)} + \frac{c^2}{r^2} f_{2k+1}^{(2k+1)} \right) \cdot \left(f_{2k+1}^{(2k+3)} + \frac{c}{r} f_{2k+1}^{(2k+2)} \right)$$

We see that there are two branches of toroidal point-like currents generating essentially different EMF. A representative of the first branch is the usual magnetic dipole. The EMF of the k-th member of this family reduces to the EMF of the circular current if the time dependences of these currents are properly adjusted:

$$f_{2k}^{(2k)} = (-1)^k f_0(t)/c^{2k} \qquad (k \geq 0) \qquad (2.19)$$

We remember that the lower index of the f functions selects a particular member of the first branch, while the upper one means the time derivative. The representative of the second branch is the elementary TS. Again, the EMF of this family are the same if the time dependences of currents are properly adjusted:

$$f_{2k+1}^{(2k)} = (-1)^k f_1(t)/c^{2k} \qquad (k \geq 0) \qquad (2.20)$$

From the equations defining the energy flux it follows that for high frequencies the toroidal emitters of the higher order are more effective (as the time derivatives of higher orders contribute to the energy flux).

So far we have used the usual TS as a corner-stone for the construction of more complicated current configurations. Under the term "usual" we mean the torus $(\rho-d)^2+z^2=R^2$ with the poloidal current flowing on its surface. The VP corresponding to this current falls as r^{-3} at large distances

$$\vec{A}_1 \sim [3\vec{r}(\vec{r}\vec{n})-\vec{n}r^2]/r^5$$

for

Here $\vec{n}$ is the unit vector normal to the equatorial plane. It has been shown in ref. / 13 / that it is possible to distribute the currents inside the torus in such a way as to cancel the leading term ($\sim r^{-3}$) in the expansion of the VP. Then the first non-vanishing term in the expansion of the VP has the form

$$A_i \sim \sum n_j n_k n_e Q^{(4)}_{ijke}/r^9 \tag{2.21}$$

where $Q^{(4)}_{ijke}$ is the following symmetric traceless form

$$Q^{(4)}_{ijke} = x_i x_j x_k x_e - \frac{1}{7}(\delta_{ij} x_k x_e + \delta_{ik} x_j x_e +$$

$$+ \delta_{ie} x_j x_k + \delta_{jk} x_i x_e + \delta_{je} x_i x_k + \delta_{ke} x_i x_j) r^2 +$$

$$+ \frac{1}{35}(\delta_{ij}\delta_{ke} + \delta_{ik}\delta_{je} + \delta_{ie}\delta_{jk}) r^4$$

This VP falls like r^{-5} for $r \to \infty$. With this TS taken as a corner-stone and using the procedure described above we can construct a new hierarchy of TS. This game may be continued further. More complicated current configuration may be found inside the torus for which the VP falls like r^{-7} This current configuration may be in turn used as a corner-stone for the construction of the TS installed in each other. These corner-stone current configurations correspond to higher order toroidal multipoles / 13 /.

At large distances they may be expressed in the form

$$A_i^{(l)} = \sum Q^{(l)}_{i i_2 i_3 \ldots i_l} \cdot n_{i_2} n_{i_3} \ldots n_{i_l} / r^{2l+1} \tag{2.22}$$

where $Q^{(l)}_{i_1 \ldots i_l}$ is the symmetric traceless form of the order l . Clearly, $A_i^{(l)}$ fall as r^{-l-1} for $r \to \infty$.
Only even values of l correspond to the toroidal multipoles. As div A = 0, rot A = 0 for any value of l , so the question arises on the existence of finite toroidal current configurations corresponding to odd l . So far we did not identify them.

3. The finite toroidal-like configurations.

The Neumann - Helmholtz parametrization for the electromagnetic potentials and strengths. Consider now the time-dependent current distribution confined to the finite region of space

$$\vec{j}(\vec{r},t) = f(t)\cdot\vec{j}(\vec{r}) \tag{3.1}$$

An arbitrary vector function and, paricularly, the current distribution can be presented in the form (Neumann-Helmholtz parametrization)

$$\vec{j}(\vec{r}) = \mathrm{grad}\,\psi_1 + \mathrm{rot}(\vec{r}\psi_2) + \mathrm{rot}\,\mathrm{rot}(\vec{r}\psi_3) \tag{3.2}$$

The VP corresponding to the current density (3.1) is given by

$$\vec{A} = \mathrm{grad}\,a_1 + \mathrm{rot}(\vec{r}a_2) + \mathrm{rot}\,\mathrm{rot}(\vec{r}a_3) \tag{3.3}$$

Clearly, Eq.(3.3) is the Neumann-Helmholtz parametrization for VP. The functions entering into it are

$$a_k = \frac{1}{c} I_k \ , \quad I_k = \int \frac{1}{R} f(t-\frac{R}{c}) \psi_k(\vec{r}') dV' \tag{3.4}$$

Here $R = |\vec{r}-\vec{r}'|$. To be complete, we write out the corresponding scalar electric potential

$$\Phi = -\frac{1}{c}\dot{I}_1 + 4\pi F(t)\Psi_1(\vec{r}) + \Phi_{stat}, \quad F(t) = \int^t f(t)dt \quad (3.5)$$

Here the point above I_K means the time derivative, and Φ_{stat} is the scalar potential originating from the time independent part of the charge density ($\Phi_{stat} = \int \frac{1}{R} \rho_{stat}(\vec{r}\,')\,dV'$).

It is convenient to represent the FS in the same form as j and A:

$$\vec{E} = \mathrm{grad}\, e_1 + \mathrm{rot}(\vec{r}\, e_2) + \mathrm{rot}\,\mathrm{rot}(\vec{r}\, e_3),$$

$$\vec{H} = \mathrm{grad}\, h_1 + \mathrm{rot}(\vec{r}\, h_2) + \mathrm{rot}\,\mathrm{rot}(\vec{r}\, h_3) \quad (3.6)$$

It turns out that

$$e_1 = -\Phi_{stat} - 4\pi F(t)\Psi_1(\vec{r}), \quad e_2 = -\frac{1}{c^2}\dot{I}_2, \quad e_3 = -\frac{1}{c^2}\dot{I}_3,$$

$$h_1 = 0, \quad h_2 = -\frac{1}{c^3}\ddot{I}_3 + \frac{4\pi}{c} f(t)\Psi_3(\vec{r}), \quad h_3 = \frac{1}{c}\dot{I}_2 \quad (3.7)$$

These representations are convenient because the potentials and FS are obtained from the relatively simple integrals, their time and space derivatives. It steams from Eq.(3.4) that if

$$\Psi(\vec{r}) = \Delta\tilde{\Psi}(\vec{r}) \quad \text{then}$$

$$I(\Psi) = \frac{1}{c^2}\ddot{I}(\tilde{\Psi}) - 4\pi f(t)\,\tilde{\Psi}(r) \quad (3.8)$$

It follows from this that two different sources

$$\Psi(\vec{r})\, f(t) \quad \text{and} \quad \tilde{\Psi}(\vec{r})\cdot \ddot{f}(t)/c^2$$

give the same EMF everywhere except for the space region where $\Psi \neq 0$. In general, for the given current density $\vec{j}$ confined to the finite space region S, the functions Ψ_K entering into Eq.(3.2) are defined with some ambiguity and may be different outside S

(see,e.g., / 9 /). On the other hand, J certainly vanishes in those space regions where $\Psi_K = 0$. It is known / 9 / that the functions Ψ_2 and Ψ_3 carry information on the magnetic and toroidal (electric) moments, resp. Thus, putting

$$\Psi_2(\vec{r}) = \Psi_2(r)\, Y_{lm}(\theta,\varphi) \quad \text{and} \quad \Psi_3(\vec{r}) = \Psi_3(r)\cdot Y_{lm}(\theta,\varphi)$$

we obtain the formulas describing the radiation of particular magnetic and toroidal (electric) multipoles. The functions $\Psi_2(r)$ and $\Psi_3(r)$ define the radial distribution of the current sources. Developing the function $g = f(t - R/c)/R$ over the spherical harmonics

$$g = 4\pi \sum \frac{1}{2l+1} g_l(r, r', t)\, Y_{lm}(\theta,\varphi)\, Y^*_{lm}(\theta',\varphi')$$

for the particular lm multipole we obtain

$$I_{lm}(\Psi) = \frac{4\pi}{2l+1} Y_{lm}(\theta,\varphi) \int g_l(r, r', t)\, \Psi_K(r')\, r'^2 dr' \quad (3.9)$$

(no sum over l,m here).

Transition to the point - like limit. Eqs. (3.9) define the integrals I_K for the finite spatial current distribution. It would not be right to put $\Psi_K(r') \sim \delta(r')$ in (3.9) to obtain the point current limit. There are two ways to reach it. The first one is essentially the same as used by E.G.P.Rowe / 14 / for the evaluation of the integral I_1 entering into the definition of Φ (see Eq.(3.5)). One simply puts

$$\Psi_K(\vec{r}) \sim Y_{lm}(-\vec{\nabla})\, \delta^3(\vec{r}) \quad (3.11)$$

It should be clarified what does $Y_{lm}(-\vec{\nabla})$ mean in the RHS of this equation. We write

$$Y_{lm}(x) = r^l\, Y_{lm}(\theta,\varphi) \quad (3.12)$$

where $Y_{em}(\theta,\varphi)$ is the usual spherical harmonic. Clearly, $Y_{em}(x)$ is the homogeneous function (of the order ℓ) of the cartesian variables x, y, z . For example,

$$Y_{20} \sim 2z^2 - x^2 - y^2$$

To obtain $Y_{em}(-\vec{\nabla})$ we change x_i by $(-\frac{\partial}{\partial x_i})$ in Eq. (3.12). For example,

$$Y_{20}(-\vec{\nabla}) \sim 2\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$$

Many of the properties of the functions $Y_{em}(x)$ and their physical applications may be found in ref. / 15 /. Now we substitute (3.11) into (3.4) and integrate by parts

$$I_K \sim Y_{em}(\vec{\nabla}) \cdot \frac{f(t-r/c)}{r} \tag{3.13}$$

An alternative way is to start from Eq. (3.9). But for this, one should know how the radial functions $\Psi(r)$ behave for small r . It turns out that in all cases which we were able to verify

$$\Psi(r) \sim r^{-\ell-2}\,\delta(r) \tag{3.14}$$

The function $f_e(r,r',t)$ for small r' falls as r'^{ℓ} (for example, it is proportional to the spherical Bessel function $j_e(kr')$ for $f(t)=\exp(i\omega t)$). Substituting (3.14) into (3.11) we get

$$I \sim Y_{em}(\theta,\varphi) \cdot \lim_{r'\to 0} [\, f_e(r,r',t)/r'^{\ell}] \tag{3.15}$$

However, we did not succeed in obtaining Eq.(3.15) in full generality.

New digression on the radiationless sources.

Having obtained the explicit expressions for the extended and point-like sources, we try now to construct the radiationless sources of higher multipolarities. Consider charge and current densities corresponding to the oscillating quadrupole moment:

$$\rho_q = f_q(t)\,[\,(\vec{n}\vec{\nabla})^2 - \tfrac{1}{3}\Delta]\,\delta^3(\vec{r}),$$
$$\vec{j}_q = -\dot{f}_q(t)\,[\,\vec{n}(\vec{n}\vec{\nabla}) - \tfrac{1}{3}\vec{\nabla}]\,\delta^3(\vec{r}) \tag{3.16}$$

They generate the following potentials and field strengths

$$\Phi_q = \frac{(\vec{n}\vec{r})^2 - \frac{1}{3}r^2}{c^2 r^3}\left[f_q^{(2)} + 3\frac{c}{r}f_q^{(1)} + 3\frac{c^2}{r^2}f_q\right],$$
$$\vec{A}_q = \frac{1}{c^2 r^2}\left[\vec{n}(\vec{n}\vec{r}) - \tfrac{1}{3}\vec{r}\right]\cdot\left[f_q^{(2)} + \frac{c}{r}f_q^{(1)}\right], \tag{3.17}$$
$$\vec{H}_q = \frac{1}{c^3 r^3}(\vec{n}\times\vec{r})(\vec{n}\vec{r})\cdot\left[f_q^{(3)} + 3\frac{c}{r}f_q^{(2)} + 3\frac{c^2}{r^2}f_q^{(1)}\right],$$
$$\vec{E}_q = -\frac{1}{c^3 r^2}\left[\vec{n}(\vec{n}\vec{r}) - \tfrac{1}{3}\vec{r}\right]\cdot\left[f_q^{(3)} + 3\frac{c}{r}f_q^{(2)} + 6\frac{c^2}{r^2}f_q^{(1)} + 6\frac{c^3}{r^3}f_q\right] -$$
$$- \frac{(\vec{n}\vec{r})^2 - \frac{1}{3}r^2}{c^3 r^4}\vec{r}\cdot\left[f_q^{(3)} + 6\frac{c}{r}f_q^{(2)} + 15\frac{c^2}{r^2}f_q^{(1)} + 15\frac{c^3}{r^3}f_q\right]$$

(The argument of the f functions, if not indicated, means t - r/c everywhere in this section). On the other hand, consider the current density (3.1) with

$$\Psi_1 = \Psi_2 = 0\ , \quad \Psi_3 = [\,(\vec{n}\vec{\nabla})^2 - \tfrac{1}{3}\Delta]\,\delta^3(\vec{r})$$

Or, explicitly,

$$\vec{j} = f_j(t)\cdot \mathrm{rot\,rot}\,(\vec{r}\,\psi_3) \tag{3.18}$$

The corresponding potentials and field strengths are given by

$$\varphi_j = 0,$$

$$\vec{A}_j = -\frac{2}{c^4 r^2}\left[(\vec{n}\vec{r})\vec{n} - \frac{1}{3}\vec{r}\right]\cdot\left[f_j^{(3)} + 3\frac{c}{r}f_j^{(2)} + 6\frac{c^2}{r^2}f_j^{(1)} + 6\frac{c^3}{r^3}f_j\right] +$$

$$+\frac{2}{c^4 r^4}\vec{r}\cdot\left[(\vec{n}\vec{r})^2 - \frac{1}{3}r^2\right]\cdot\left[f_j^{(3)} + 6\frac{c}{r}f_j^{(2)} + 15\frac{c^2}{r^2}f_j^{(1)} + 15\frac{c^3}{r^3}f_j\right],$$

$$\vec{E}_j = \frac{2}{c^5 r^2}\left[(\vec{n}\vec{r})\vec{n} - \frac{1}{3}\vec{r}\right]\cdot\left[f_j^{(4)} + 3\frac{c}{r}f_j^{(3)} + 6\frac{c^2}{r^2}f_j^{(2)} + 6\frac{c^3}{r^3}f_j^{(1)}\right] -$$

$$-\frac{2}{c^5 r^4}\vec{r}\cdot\left[(\vec{n}\vec{r})^2 - \frac{1}{3}r^2\right]\cdot\left[f_j^{(4)} + 6\frac{c}{r}f_j^{(3)} + 15\frac{c^2}{r^2}f_j^{(2)} + 15\frac{c^3}{r^3}f_j^{(1)}\right],$$

$$\vec{H}_j = -2\frac{(\vec{n}\vec{r})}{c^5 r^3}(\vec{n}\times\vec{r})\cdot\left[f_j^{(4)} + 3\frac{c}{r}f_j^{(3)} + 3\frac{c^2}{r^2}f_j^{(2)}\right] \tag{3.19}$$

From the comparison of Eqs. (3.17) and (3.19) we see that $\vec{E}_q = \vec{E}_j$ and $\vec{H}_q = \vec{H}_j$ when $f_q = -2\dot{f}_j/c^2$. Correspondingly, $\vec{E} = \vec{E}_q + \vec{E}_j = 0$ and $\vec{H} = \vec{H}_q + \vec{H}_j = 0$ if $f_q = 2\dot{f}_j/c^2$.

This means that the oscillating quadrupole charge-current configuration (3.16) and a pure current configuration (3.18) generate field strengths equal to zero everywhere (except for the origin). Nevertheless, the potentials are not zero:

$$\varphi = \varphi_q = 2\,\frac{(\vec{n}\vec{r})^2 - \frac{1}{3}r^2}{c^4 r^3}\left[f_j^{(3)} + 3\frac{c}{r}f_j^{(2)} + 3\frac{c^2}{r^2}f_j^{(1)}\right],$$

$$\vec{A} = \vec{A}_q + \vec{A}_j = -\frac{4}{c^4 r^3}\left[(\vec{n}\vec{r})\vec{n} - \frac{1}{3}\vec{r}\right]\cdot\left[f_j^{(2)} + 3\frac{c}{r}f_j^{(1)} + 3\frac{c^2}{r^2}f_j\right] +$$

$$+\frac{2}{c^4 r^4}\vec{r}\cdot\left[(\vec{r}\vec{n})^2 - \frac{1}{3}r^2\right]\cdot\left[f_j^{(3)} + 6\frac{c}{r}f_j^{(2)} + 15\frac{c^2}{r^2}f_j^{(1)} + 15\frac{c^3}{r^3}f_j\right] \tag{3.20}$$

Consider the particular time dependence. If $f_j = \mathrm{const}$ then

$$\dot{\vec{j}} = f_j\cdot \mathrm{rot\,rot}\,(\vec{r}\,\psi_3) \quad \text{and} \tag{3.21}$$

$$\vec{A} = -\frac{12}{c\,r^5}f_j\cdot\left[(\vec{n}\vec{r})\vec{n} - \frac{1}{3}\vec{r}\right] + \frac{30}{c\,r^7}\vec{r}f_j\left[(\vec{n}\vec{r})^2 - \frac{1}{3}r^2\right] \tag{3.22}$$

This VP falling at large distances as r^{-4} corresponds to $l = 3$ in Eq.(2.22). As we have mentioned, we did not succeed in identifying the finite static current configuration whose infinitesimal limit coincides with (3.21). The next in complexity case corresponds to octupole oscillations of the charge density

$$\rho = f(t)\,(\vec{n}\vec{\nabla})\left[(\vec{n}\vec{\nabla})^2 - \frac{3}{5}\Delta\right]\delta^3(\vec{r}),$$

$$\vec{j} = -\dot{f}(t)\cdot\vec{n}\left[(\vec{n}\vec{\nabla})^2 - \frac{3}{5}\Delta\right]\delta^3(\vec{r}) \tag{3.23}$$

The elementary toroidal current distribution giving the same ES (after readjustment of time dependences) corresponds to

$$\psi_1 = \psi_2 = 0, \quad \psi_3 \sim (\vec{n}\vec{\nabla})\cdot\left[(\vec{n}\vec{\nabla})^2 - \frac{3}{5}\Delta\right]\delta^3(\vec{r}) \tag{3.24}$$

The finite poloidal current distribution whose infinitesimal limit is given by Eq.(3.24) was obtained in ref. / 13 /. Its asymptotical behaviour is determined by Eq.(2.21). In general, there is one-to--one correspondence between the EMF generated by the oscillating charge distributions and the elementary toroidal current sources. But only half of them corresponds to the known finite distributions of poloidal currents inside the TS. As we have mentioned, they correspond to even values of l in Eq.(2.22). Since the electromagnetic strengths produced by the oscillating charge densities and the elementary toroidal sources are the same (if their time dependences and multipolarities are properly adjusted), particular

terms of the multipole expansions defining these strengths have the double names known in a physical literature as electric (see, e.g., Rose book / 16 /) or toroidal / 17 / multipoles. Despite the coincidence of the electromagnetic strengths, the corresponding potentials are essentially different. Particularly, the elementary toroidal sources (at least, part of them) are the limiting cases of the finite topologically nontrivial current configurations.
In conclusion, there is a hierarchy of the elementary charge--current configurations outside which the electromagnetic strentghs disappear but the potentials survive. The main problem is to find the corresponding configuration with finite dimensions. Up to now we have proved / 9, 13 / the existence of such finite configurations for the charge-current densities (2.13),(3.23) and (3.24). In this case, the corresponding electromagnetic potentials cannot be removed by the gauge transformation. Thus, they have the physical meaning and can be used for the carrying out time-dependent Aharonov-Bohm like experiments and the information transfer. It would be interesting to find finite charge-current time-dependent radiationless configurations corresponding to higher multipolarities.

4. Conclusion.

We briefly summarize the main results obtained:

1. There are found time-dependent charge-current sources outside which the electromagnetic field strengths disappear but the potentials survive. They can be used for performing time-dependent Aharonov-Bohm-like experiments and the information transfer (modulating the phase of the charge particle wave function).
2. The radiation field of toroidal-like current configurations is

investigated. For a given multipole there are two different representatives which generate essentially different electromagnetic fields.
3. Using the Neumann-Helmholtz parametrization of the current density we present the electromagnetic field of the arbitrary time-dependent charge-current density in a form convenient for applications. The contributions of different multipoles in it are explicitly separated.

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