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Низкоэнергетические теоремы в кварковой потенциальной модели

В рамках кварковой потенциальной модели рассматриваются низкоэнергетические теоремы для распадов пиона ($\pi \to \mu\nu$, $\pi \to \gamma\gamma$). Показано, что сепарабельное приближение воспроизводит эти теоремы с новым нелокальным определением кваркового конденсата, учитывающим форму потенциала.

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Low Energy Theorems in Quark Potential Models

Low energy theorems for pion decays ($\pi \to \mu\nu$, $\pi \to \gamma\gamma$) are considered in the framework of the quark potential models. It is shown that the separable approximation to the class of potentials leads to self-consistent reproducing of these theorems. It is found that the generalization of the approach to the arbitrary potential of the quark interaction requires redefinition of the quark condensate by taking into account the form of the potential.

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1 Introduction

The low energy theorems of hadron physics are independent of any model of hadrons. Consequently, they play a role of a pure tool to test the QCD-inspired models. These models (for example, the QCD sum rules [1], Nambu - Jona - Lasinio (NJL) model [2], QCD lattice calculations [3], nonrelativistic quark potential models [4, 5]) relate the PCAC in hadron physics, induced by the spontaneous chiral symmetry breakdown, to the quark condensate. It should be noted that in all these quark models the definition of the local quark condensate is used.

In the present paper we discuss the validity of the notion of the local condensate in the quark potential model within the low energy theorems. Since a wide class of nonlocal potentials used in the model lead to equations that are unsolvable by an analytical method, we investigate the problem by means of the separable approximation.

In section 2, we give formulae of the quark potential model used in this work. In section 3, we consider the model in the separable approximation and apply it to obtain the low energy theorems for pions. In section 4, we try to reproduce these theorems beyond the separable approximation, in the framework of the quark potential model, and discuss the arguments for redefinition of the quark condensate when the quark interaction is nonlocal.

2 Quark potential model

The quark potential model has originally been proposed in refs. [4] on the basis of the QCD effective Hamiltonian in the Coulomb gauge. In this model one considers the hadron as a bound state of the quarks formed due to the instantaneous dominant potential like an atom in quantum QED. One of the important issues of the model is that the same nonrelativistic potential used in the Schrödinger equation for describing heavy quarkonia leads to the spontaneous breakdown of the chiral symmetry which is a pure relativistic effect. In the framework of this model with the "confining" potential a qualitative description of the mass spectra of light mesons is obtained [5]. However, for the leptonic decay constants of pseudoscalar mesons and the quark condensate the estimations obtained in these works are considerably smaller than the available data. There are several attempts in the literature to solve this problem, in particular, by including the short-range Coulomb interaction as well as by choosing an "appropriate" confining potential [6, 7].

The relativistic covariant quark potential model has been proposed recently in refs. [8]. In the present work we use the formulae of these latest papers, namely, explicit forms of the Schwinger - Dyson (SD) and Bethe - Salpeter (BS) equations in the rest frame of the quark-antiquark bound state (in $^1S_0$- channel), and the amplitudes for the constants of pion decays $\pi \to \mu\nu$ and $\pi \to \gamma\gamma$.

In the quark potential model the "constituent" quarks and their bound states are described by the following SD and BS coupled equations [8]:

$$\begin{cases} E(p)\cos 2\vartheta(p) = m^0 + \frac{1}{2}\hat{I}_q\left[\cos 2\vartheta(p)\right] & (1a) \\ E(p)\sin 2\vartheta(p) = (\mid \mathbf{p} \mid) + \frac{1}{2}\hat{I}_q\left[\hat{p}\hat{q}\sin 2\vartheta(p)\right], & (2a) \end{cases}$$

$$\begin{cases} M_{\rm H}L_2(p) = E_t(p)L_1(p) - \hat{I}_q\left\{\left[c_p^{(-)}c_q^{(-)} + \hat{p}\cdot\hat{q}s_p^{(-)}s_q^{(-)}\right]L_1(q)\right\} \\ M_{\rm H}L_1(p) = E_t(p)L_2(p) - \hat{I}_q\left\{\left[c_p^{(+)}c_q^{(+)} + \hat{p}\cdot\hat{q}s_p^{(+)}s_q^{(+)}\right]L_2(q)\right\}; \end{cases} \quad (1)$$

Here m^0 is the "current" quark mass, $E_t = E_1 + E_2$, E_1 and E_2 are the energies, respectively, of the quark and antiquark, and $\vartheta(p)$ is the solution to the SD equation, L_1, L_2 *and* $M_{\rm H}$ are the eigenfunctions and eigenvalue of the BS equation, respectively, identified by the wavefunctions and mass of pseudoscalar meson (H), and the following abbreviations are used:

$$c_p^{(\pm)} \equiv \cos\left[\vartheta_1(p) \pm \vartheta_2(p)\right], \quad s_p^{(\pm)} \equiv \sin\left[\vartheta_1(p) \pm \vartheta_2(p)\right], \quad \hat{p}\cdot\hat{q} \equiv \frac{\mathbf{p}\cdot\mathbf{q}}{pq}, \quad p = |\mathbf{p}|,$$

$$\hat{I}_q F \equiv \int \frac{d\mathbf{q}}{(2\pi)^3}V(|\mathbf{p}-\mathbf{q}|)F$$

where $V(\mid \mathbf{p}-\mathbf{q} \mid)$ is the potential. We define the mass function as

$$m(p) = E(p)cos2(p). \quad (2)$$

The solutions of the BS equation satisfy the normalization condition

$$\frac{4N_c}{M_{\rm H}}\int \frac{d\mathbf{q}}{(2\pi)^3}L_1(q)L_2(q) = 1. \quad (3)$$

In the chiral limit, these equations admit the solutions satisfying the Goldstone theorem. In other words, if a solution to the SD equation for the massless quark ($m^0 = 0$) exists, then the same solution satisfies the BS equation for a massless ($M_\pi = 0$) pseudoscalar meson too:

$$L^{\rm G}(q) = L_1(q) = \frac{\cos 2\vartheta(p)}{c_{\rm G}}, \quad (4)$$

where $c_{\rm G}$ is the constant

$$c_{\rm G} = f_\pi \quad (5)$$

and f_π is the pion leptonic decay constant having the experimental value 132 MeV.

The latter equation is obtained from the amplitude of the decay $\pi \to \mu\ \nu$ in the chiral limit and normalization condition (4). In the potential model, the pion leptonic decay constant is defined as [8]

$$f_\pi = \frac{4N_c}{M_\pi}\int \frac{d\mathbf{q}}{(2\pi)^3}\cos 2\overline{\vartheta}(p)L_2(q) \quad (6)$$

where $\overline{\vartheta}(p) = \frac{\vartheta_1(p)+\vartheta_2(p)}{2}$, M_π is the pion mass.

Notice that in the limit of heavy quarks ($\frac{p}{m^0} << 1$) the BS equation reduces to the Schrödinger equation

$$(M_{\rm H} - m_1^0 - m_2^0 - \frac{p^2}{2\mu})\Psi(p) \ = \ -\hat{I}_q \cdot \Psi(p) \ ,$$

where

$$\Psi(p) \ = \ 2\sqrt{\frac{N_c}{M_{\rm H}}}L_1(p) \approx 2\sqrt{\frac{N_c}{M_{\rm H}}}L_2(p) \ ,$$

$$\mu \ = \ \frac{m_1^0 m_2^0}{m_1^0 + m_2^0} \ ,$$

whereas equation (7) coincides with the nonrelativistic definition of the meson decay constant

$$f_{\rm H}^{\rm NR} = \sqrt{\frac{N_c}{\mu}}\int \frac{d\mathbf{q}}{(2\pi)^3}\Psi(q) = \sqrt{\frac{N_c}{\mu}}\Psi(0) \ .$$

Thus, we can see that the quark potential model reflects the properties of the QCD at long and short distances.

3 Separable approximation to the potential

The separable approximation to the potential consists in the following factorization of the potential:

$$V(\mathbf{p}-\mathbf{q}) = \frac{c^2}{L^2}f(\frac{p}{L})\cdot f(\frac{q}{L}) \ , \ \ f(0) = 1 \ , \quad (7)$$

where c is the coupling constant, f is the form factor, and L is the cut-off parameter.

In this approximation one neglects the dependence of the spectra on the angular momentum, and the integral of equation (1b) equals zero due to the angular asymmetry. As a result, the SD equation takes the following simple form:

$$m(p) = m^0 + mf(\frac{p}{L}) \quad (8)$$

or

$$1-\frac{m_R^0}{m}=\frac{e^2}{L^2}<\frac{f^2}{2E}> \ , \quad m_R^0=m^0\frac{e^2}{L^2}<\frac{f}{2E}> \ , \tag{9}$$

where the following abbreviations are used:

$$m\equiv m(0), \ <F>=\int\frac{d\mathbf{q}}{(2\pi)^3}F \ .$$

It should be noted that the separable approximation prompts also insufficient role of the integral in equation (1b), which can be useful in investigating of the renormalization problem of the SD equation.

Let us consider the solution of this equation in the low energy limit, more exactly, when $m^0/L\to 0$. In this limit, equation (9) can be solved under the coupling constant

$$e^2=4\pi^2\left(\int_0^\infty dx\frac{x^2f^2(x)}{\sqrt{x^2+f^2(x)\gamma^2}}\right)^{-1}\sim 4\pi\left(\int_0^\infty dxxf^2(x)\right)^{-1} \ , \tag{10}$$

where the approximation for the integral is valid as the dependence on $\gamma\equiv m/L$ is weak.

We define the "averaged" quark Green function $G(q)$ with the density function f as follows:

$$<<q\bar{q}>>=4N_ci\mathrm{tr}\int\frac{dq}{(2\pi)^4}(f(q)G(q))=-4N_c<\frac{mf^2}{2E}>=-4N_c\frac{L^2}{e^2}m. \tag{11}$$

In the separable approximation, the BS equation also takes the form of the usual algebraic one

$$\frac{m_R^0}{m}=\frac{M_\pi^2}{4m^2}\left[\mathcal{D}^{(2)}+\frac{\mathcal{D}^{(3)2}}{1-\mathcal{D}^{(4)}}\right] \ ; \ \mathcal{D}^{(k)}=\frac{e^2}{L^2}<\frac{f^{(k)}}{2E^3}> \ . \tag{12}$$

In the low energy limit, $M_\pi << 4E^2(0)$, one obtains the following solutions to the equation:

$$L_1(p) = \frac{1}{f_\pi}\frac{m(p)}{E(p)}\left(1+\frac{M_\pi^2}{4E^2}\frac{\mathcal{D}^{(3)}}{1-\mathcal{D}^{(4)}}\right)\sim\frac{m(p)}{f_\pi E(p)} \ , \tag{13}$$

$$L_2(p) = \frac{M_\pi}{2E(p)}\left(1+f\frac{\mathcal{D}^{(3)}}{1-\mathcal{D}^{(4)}}\right)L_1(p) \ . \tag{14}$$

Now using eqs. (4),(10), and (12) we arrive at the following relation:

$$-4m_R^0<<q\bar{q}>>=f_\pi^2M_\pi^2 \ . \tag{15}$$

We see that this equation turns out into the well known low energy theorem of the local theory if one replaces the averaged Green function, $<<q\bar{q}>>$, with the quark condensate, $<q\bar{q}>$. The quantities included in this relation can be estimated if the form of the potential and the current quark mass are given.

To study the relation of the averaged Green function of the quark to the usual quark condensate, we consider the local NJL model type potential, $f(x)=\theta(1-x)$, and use the conventional values for the free parameters, namely,

$$L=950MeV, \quad m_{\mathrm{R}}^0=4MeV, \tag{16}$$

then we obtain the following numerical estimations

$$\begin{aligned} M_\pi &= 140\mathrm{MeV} \ ; \ f_\pi=132\mathrm{MeV} \ ; \\ <<q\bar{q}>> &= (-250\mathrm{MeV})^3 \ ; \ \mathcal{D}^{(k)}=\mathcal{D}=\frac{1}{6} \ ; \ \gamma^2=\frac{1}{12} \ . \end{aligned} \tag{17}$$

So, we see that these estimations are in agreement with the conventional values for the mass and decay constant of the pion as well as the quark condensate. We can conclude that equation (12) defines the quark condensate formed of the quark - antiquark pair due to the nonlocal interaction in the separable approximation.

Let us now consider the mode of the pion decay, $\pi\to\gamma\gamma$, induced by vector current. The amplitude of this process in the potential model is written as [9]

$$\begin{aligned} M(\pi^0\longrightarrow\gamma\gamma) &= <\gamma\gamma|W_{\mathrm{eff.}}|\pi^0>=\frac{(2\pi)^4\delta(\mathcal{P}-k_1-k_2)}{\sqrt{(2\pi)^98\mathcal{P}_0k_{10}k_{20}}} \cdot \\ & 2N_c(\epsilon_\mu^\alpha(k_1)\epsilon_\nu^\beta(k_2)\frac{1}{\sqrt{2}})(e_u^2-e_d^2)\epsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta I(k_1,k_2) \ , \end{aligned} \tag{18}$$

where $\epsilon_\mu^\alpha(k_i)$ is the photon polarization tensor ($i=1,2$), e_q are the quark charges, and N_c is the number of colors. The integral of this equation in the limit of zero photon momenta has the form

$$I(k_1=0,k_2=0)=\int\frac{d\mathbf{q}}{(2\pi)^3}\frac{m(q)}{E^2(q)}\frac{1}{E^2(q)-\frac{M_\pi^2}{16}}\frac{1}{2}[L_1(q)+\frac{4E(q)}{M_\pi}L_2(q)] \ , \tag{19}$$

where the approximation of equal masses of the quarks is used, $E(q)=\sqrt{q^2+m(q)^2}$ is the quark energy, P and k_1,k_2 are the 4 - momenta of the initial and final particles, respectively.

In the separable approximation the above integral takes the form

$$\begin{aligned} I(k_1=0,k_2=0)=\frac{1}{4\pi^2f_\pi} \ , J &= \left(\frac{3-\mathcal{D}}{1-\mathcal{D}}\right)\int_0^\infty dx\frac{f^2(\gamma x)}{\left(\sqrt{x^2+f^2(\gamma x)}\right)^5}= \\ &= \left(\frac{3-\mathcal{D}}{1-\mathcal{D}}\right)\left[\frac{1}{3(1+\gamma^2)^{3/2}}\right]=1.02 \ , \end{aligned} \tag{20}$$

where for the numerical estimation the values of the parameters (18) are used.

So, the separable approximation for the integral J leads to the the Bell - Jackiw theoretical estimation [10] ($J^{BJ} = 1$) and the experimental result ($J^{exp} \approx 1.04$).

4 The status of the local quark condensate in the potential model

Let us now consider the low energy theorems in the quark potential model by using equations (1), (2), (4) and (5).

We investigate whether one can self-consistently reproduce the low energy theorems for the pion decays using the conventional definition of the local quark condensate, namely,

$$< q\bar{q} >= iN_c \mathrm{tr} \int \frac{dq}{(2\pi)^4} G(q) = -2N_c \int \frac{d\mathbf{q}}{(2\pi)^3} cos(\vartheta(|\mathbf{p}|)) \ . \tag{21}$$

It is easy to see that the low energy theorem can be obtained from equations (4), (5), and (22) if the "small" component of the bound wave function ($L_2(p)$) in this energy scale has the behavior

$$L_2 = \frac{m_R^0}{2f_\pi M_\pi} = \mathrm{const} \ . \tag{22}$$

However, this asymptotic function contradicts the Bell-Jackiw estimation. Indeed, substitution of

$$L_1(p=0) \sim \frac{1}{F_\pi} \ , \ L_2 = 0 \ , \tag{23}$$

into (20) leads to

$$I(k=0) \ = \ \frac{1}{4\pi^2 F_\pi} J \ , \ J = \int_0^\infty dx \frac{x^2}{(1+x^2)^2} = \frac{\pi}{4} \ , \tag{24}$$

Notice that by using

$$L_1(q) = \frac{1}{f_\pi} \frac{m}{\sqrt{m^2 + \mathbf{q}^2}}$$

instead of the value (24) one would obtain $J = \frac{1}{3}$.

Thus, a straightforward application of the local quark condensate in the nonlocal potential model does not provide a self-consistent reproduction of the low energy theorems for the pion decays.

5 Conclusions

In the present paper, we have considered the low energy theorems for the pion decays ($\pi \to \mu \ \nu$, $\pi \to \gamma \ \gamma$) in the framework of the potential model.

The separable approximation to the potential indicates that the quark condensate in the nonlocal theory should be redefined by taking into account the form of the interaction potential. This redefinition provides the nonlocal generalization of the low energy theorems for the pion decays.

To summarize, we have verified that the potential model is able to reproduce the low energy theorems only if the nonlocal notion of the quark condensate is used.

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