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ON q-DEFORMED SPINNING RELATIVISTIC PARTICLE

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О q-деформированной релятивистской частице со спином

q-деформированная свободная релятивистская частица со спином обсуждается в рамках лагранжева формализма. Для этой системы получены три эквивалентных лагранжиана, обладающих q-деформированной локальной (супер)калибровочной симметрией и репараметрической инвариантностью. Показано, что эти симметрии эквивалентны только для $q = \pm 1$ при частном выборе трансформационных параметров. Требование q-коммутации двух суперсимметричных калибровочных трансформаций, генерирующих репараметризацию и суперсимметричную калибровочную трансформацию, приводит к аналогичному условию ($q = \pm 1$). Для специального выбора калибровки решения уравнений движения удовлетворяют $GL_{\sqrt{q}}(1|1)$ и $GL_q(2)$ инвариантности для произвольного значения эволюционного параметра, характеризующего квантовую супермировую линию.

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On q-Deformed Spinning Relativistic Particle

A q-deformed free spinning relativistic particle is discussed in the framework of the Lagrangian formalism. Three equivalent Lagrangians are obtained for this system which are endowed with q-deformed local (super) gauge symmetries and reparametrization invariance. It is demonstrated that these symmetries are on-shell equivalent only for $q = \pm 1$ under particular identification of the transformation parameters. The same condition ($q = \pm 1$) emerges due to the requirement that the q-commutator of two supersymmetric gauge transformations should generate a reparametrization plus a supersymmetric gauge transformation. For a specific gauge choice, the solutions for equations of motion respect $GL_{\sqrt{q}}(1|1)$ and $GL_q(2)$ invariances for any arbitrary value of the evolution parameter characterizing the quantum super world-line.

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There has been an upsurge of interest in the study of q-deformed (so-called quantum) groups [1,2] during the past few years. These q-deformed groups present examples of quasi-triangular Hopf algebras [3]. Together with the ideas of non-commutative geometry, it is expected that the understanding of quantum groups might provide a *fundamental length* [4] in the context of space-time quantization [5]. Thus, in addition to Planck's constant ($\hbar$) and the speed of light (c) that emerge from the study of quantum mechanics and the special theory of relativity, the derivation of a fundamental length in the context of quantum groups is conjectured to complete the *trio* of fundamental constants of nature. Despite remarkable progress, the ideas of quantum groups have not percolated to the level of multi-pronged physical applications. Few attempts have been made, however, to provide some physical meaning to these mathematical objects in the context of concrete physical examples [6]. These quantum groups have also been treated as gauge groups in an endeavour to develop q-deformed gauge theories [7]. To obtain Lorentz covariant q-deformed gauge theories, q-deformed path integral method, q-deformed field theories etc., it is of utmost importance to develop Lagrangian formulations for some known physical systems [8] in a cogent and consistent way.

The central theme of the present paper is to develop a Lagrangian formulation for a q-deformed free spinning relativistic particle moving on a quantum super world-line. We extend the prescription and methodology of Ref.[8] to this system in a hope of developing a general scheme for the discussion of other more complicated but realistic physical examples. We obtain three equivalent q-deformed Lagrangians which are found to be endowed with q-deformed local (super)gauge symmetries and reparametrization invariance. These symmetries are shown to be equivalent on-shell *only* for $q = \pm 1$ under specific identification of non-commuting gauge parameters in terms of the commuting diffeomorphism parameter. The same condition ($q = \pm 1$) also emerges due to the requirement that the q-commutator of two supersymmetric gauge transformations must produce the sum of a reparametrization and a supersymmetric gauge transformation as is essential for pure supergravity theories. One of the highlights of our work is the existence of $GL_{\sqrt{q}}(1|1)$ and $GL_q(2)$ invariant solutions for equations of motion under a specific gauge choice. This invariance persists at any arbitrary value of the evolution parameter. Since the emphasis in this work is laid on symmetry considerations in the framework of the Lagrangian formalism, we do not intend to discuss in detail the Hamiltonian formulation, $\hbar$-quantization, q-deformed Dirac Brackets, etc., for this system which is endowed with first-class as well as second-class constraints [9]. This issue together with the q-deformed BRST formalism for a scalar as well as a spinning particle will be reported elsewhere [10].

The simplest form of the local (super)gauge and reparametrization invariant undeformed (classical) Lagrangian that describes the free motion ($\dot{p}_\mu = 0$) of a massless relativistic spinning particle is [11]

$$L_F = p_\mu \dot{x}^\mu + \frac{i}{2}\psi_\mu \dot{\psi}^\mu - \frac{e}{2}p^2 + i\chi\psi_\mu p^\mu, \tag{1}$$

where x_μ, p_μ, e are *even* and ψ_μ, χ are *odd* elements of a Grassmann algebra. In the language of supergravity theories, x_μ and p_μ are canonically conjugate target space coordinates and momenta, e and χ ($\chi^2 = 0$) are Lagrange multipliers that are analogues of the vierbein and the Rarita-Schwinger fields and the Lorentz vector ψ_μ, which is the super partner of x_μ, presents spin degrees of freedom and anticommutes with itself ($\psi_\mu\psi_\nu + \psi_\nu\psi_\mu = 0$). The "velocities" $\dot{x}_\mu \equiv \frac{dx^\mu}{d\tau} = ep^\mu - i\chi\psi^\mu$ and $\dot{\psi}^\mu \equiv \frac{d\psi^\mu}{d\tau} = \chi p^\mu$ can be readily obtained from the above Lagrangian where the trajectory of a spinning particle is parametrized by an evolution variable τ. To obtain the q-analogue of the above Lagrangian we follow the discussion of a q-deformed free relativistic scalar particle [8] and generalize that prescription to a q-deformed spinning particle where the configuration space corresponding to the Minkowski supermanifold remains *flat* and *undeformed* ($x_\mu x_\nu = x_\nu x_\mu$, $\psi_\mu\psi_\nu + \psi_\nu\psi_\mu = 0$) but the cotangent supermanifold (momentum phase space) is *q-deformed* ($x_\mu p_\nu = q\, p_\nu x_\mu, x_\mu x_\nu = x_\nu x_\mu, p_\mu p_\nu = p_\nu p_\mu, \psi_\mu\psi_\nu + \psi_\nu\psi_\mu = 0$) in such a way that the Lorentz invariance is respected for any arbitrary ordering of μ and ν. Here all the dynamical variables are taken as hermitian elements of an algebra in involution ($|q| = 1$) and q is a non-zero c-number. As a consequence of the above deformation, the following *on-shell* and *(graded)associative* q-(anti)commutation relations emerge [1]

$$\begin{aligned}
&x_\mu\, x_\nu = x_\nu\, x_\mu, \quad \dot{x}_\mu\, \dot{x}_\nu = \dot{x}_\nu\, \dot{x}_\mu, \quad \dot{x}_\mu\, x_\nu = x_\nu \dot{x}_\mu, \quad x_\mu \dot{x}_\nu = \dot{x}_\nu x_\mu,\\
&p_\mu\, p_\nu = p_\nu\, p_\mu, \quad x_\mu\, p_\nu = q p_\nu\, x_\mu, \quad \dot{x}_\mu\, p_\nu = q\, p_\nu\, \dot{x}_\mu, \quad e\, x_\mu = q\, x_\mu\, e,\\
&e\, p_\mu = q\, p_\mu\, e, \quad e\, \dot{x}_\mu = q\, \dot{x}_\mu\, e, \quad e\, \psi_\mu = q\, \psi_\mu\, e, \quad e\, \chi = \chi\, e,\\
&\psi_\mu\, \psi_\nu + \psi_\mu \psi_\nu = 0, \quad \dot{\psi}_\mu\, \psi_\nu + \psi_\nu\, \dot{\psi}_\mu = 0, \quad \psi_\mu\, \dot{\psi}_\nu + \dot{\psi}_\nu\, \psi_\mu = 0,\\
&x_\mu\, \psi_\nu = q^{1/2} \psi_\nu\, x_\mu, \quad p_\mu\, \psi_\nu = q^{-1/2} \psi_\nu\, p_\mu, \quad p_\mu\, \dot{\psi}_\nu = q^{-1/2} \dot{\psi}_\nu\, p_\mu,\\
&\chi\, x_\mu = q^{1/2}\, x_\mu\, \chi, \quad \chi\, p_\mu = q^{1/2}\, p_\mu\, \chi, \quad \chi\, \psi_\mu = -q^{1/2}\, \psi_\mu\, \chi.
\end{aligned} \tag{2}$$

It is straightforward to see that in the limit when all the *odd* Grassmann variables are set equal to zero, we obtain q-commutation relations for a q-deformed scalar free relativistic particle [8]. Furthermore, in the limit $q \to 1$ the usual (anti)commutation relations among the dynamical variables of the Lagrangian (1) emerge automatically. Consistent with the q-(anti)commutation relations (2), the quantum super world-line, traced out by the free motion of a spinning relativistic particle can be defined

[1] These on-shell q-(anti)commutation relations emerge from basic (un)deformed relations on a deformed cotangent supermanifold and the equations of motion obtained from the (un)deformed Lagrangians (1) *or* (11). For instance, $\dot{x}_\mu\, \dot{x}_\nu = \dot{x}_\nu\, \dot{x}_\mu$ together with $e\, p_\mu = q\, p_\mu\, e$ and $e\, \dot{x}_\mu = q\, \dot{x}_\mu e$ leads to the relation $\chi(\psi_\mu\, p_\nu - \psi_\nu\, p_\mu) = q\, (p_\nu \chi\, \psi_\mu - p_\mu \chi\, \psi_\nu)$. The requirement of equality of this relation with the *similar one* emerging on-shell from $\dot{\psi}_\mu\, \psi_\nu + \psi_\nu\, \dot{\psi}_\mu + \dot{\psi}_\nu\, \psi_\mu + \psi_\mu\, \dot{\psi}_\nu = 0$ leads to: $p_\mu\, \psi_\nu = q^{-1/2} \psi_\nu\, p_\mu, \chi\, p_\mu = q^{1/2}\, p_\mu\, \chi$ and $\chi\, \psi_\mu = -q^{1/2}\, \psi_\mu\, \chi$ which are consistent with the undeformed case in the classical limit $q \to 1$.

in terms of the coordinate generator x^μ and the spin variable ψ^μ as

$$\begin{aligned}
x_\mu(\tau)\, \psi^\mu(\tau) &= q^{1/2}\, \psi_\mu(\tau)\, x^\mu(\tau),\\
(\psi^\mu(\tau))^2 &= 0.
\end{aligned} \tag{3}$$

Here repeated indices are summed over (i.e. $\mu = 0, 1, 2........D-1$) and the super world-line is parametrized by a real commuting variable τ. It is interesting to check that the quantum super world-line (3) remains form-invariant under the following transformations

$$\begin{aligned}
x_\mu &\to a\, x_\mu + \beta\psi_\mu,\\
\psi_\mu &\to \gamma x_\mu + d\, \psi_\mu,
\end{aligned} \tag{4}$$

if we assume the (anti)commutativity of the variables ψ^μ and x^μ with odd elements β, γ and even elements a, d of a 2×2 $GL_{\sqrt{q}}(1|1)$ matrix obeying the braiding relations in rows and columns as

$$\begin{aligned}
a\beta &= q^{1/2}\beta a, \quad d\beta = q^{1/2}\beta d, \quad \beta\,\gamma = -\gamma\,\beta, \quad \beta^2 = \gamma^2 = 0,\\
a\gamma &= q^{1/2}\gamma a, \quad d\gamma = q^{1/2}\gamma d, \quad ad - da = -(q^{1/2} - q^{-1/2})\beta\gamma.
\end{aligned} \tag{5}$$

It will be noticed that the $GL_{\sqrt{q}}(1|1)$ invariance is implied in component pairs: $(x_0, \psi_0)........(x_{D-1}, \psi_{D-1})$, namely;

$$\begin{pmatrix} x_i \\ \psi_i \end{pmatrix} \to \begin{pmatrix} a, & \beta \\ \gamma, & d \end{pmatrix} \begin{pmatrix} x_i \\ \psi_i \end{pmatrix}, \tag{6}$$

for $i = 0, 1, 2, 3.......D-1$ separately . The other candidate for the definition of the quantum super world-line: $\psi_\mu p^\mu = q^{1/2} p_\mu \psi^\mu$ has not been taken because $p_\mu \psi^\mu = 0$ is the constraint on the system under consideration. Moreover, it can be readily seen that the latter is contained in definition (3) due to the on-shell requirement in $\dot{x}_\mu \psi^\mu = q^{1/2} \psi_\mu \dot{x}^\mu$ and equation (2). It is worth noting that the on-shell condition in $x_\mu \dot{\psi}^\mu = q^{1/2} \dot{\psi}_\mu x^\mu$ leads to the definition of the $GL_q(2)$ invariant quantum world-line ($x_\mu(\tau)\, p^\mu(\tau) = q\, p_\mu(\tau)\, x^\mu(\tau)$) taken in the case of a free relativistic scalar particle [8] as it remains invariant under

$$\begin{aligned}
x_\mu &\to a\, x_\mu + b\, p_\mu,\\
p_\mu &\to c\, x_\mu + d\, p_\mu,
\end{aligned} \tag{7}$$

if we assume the commutativity of the phase variables x^μ and p^μ with the elements a, b, c, and d of a 2×2 $GL_q(2)$ matrix obeying following relationship

$$\begin{aligned}
ab &= qba, \quad ac = qca, \quad cd = qdc, \quad bd = qdb,\\
bc &= cb, \quad ad - da = (q - q^{-1})\, bc.
\end{aligned} \tag{8}$$

In the definition of the quantum world-line for a scalar particle, once again, repeated indices are summed over and the $GL_q(2)$ invariance is implied in the component pairs of the phase variables. It is gratifying to see that the $GL_q(2)$ invariant quantum world-line for a scalar relativistic particle emerges on-shell from the $GL_{\sqrt{q}}(1|1)$ invariant quantum super world-line (3) for a spinning relativistic particle.

The first-order Lagrangian, describing the free motion ($\dot{p}_\mu = 0$) of a massless q-deformed relativistic spinning particle, is

$$L_f = q^{1/2} p_\mu \dot{x}^\mu + \frac{i}{2} \psi_\mu \dot{\psi}^\mu - \frac{e}{1+q^2} p^2 + i\chi\psi_\mu p^\mu, \tag{9}$$

where the $q^{1/2}$ factor appears in the first term due to the Legendre transformation with q-symplectic metrices [8]

$$\Omega_{AB}(q) = \begin{pmatrix} 0, & -q^{-1/2} \\ q^{1/2}, & 0 \end{pmatrix} \quad \text{and} \quad \Omega^{AB}(q) = \begin{pmatrix} 0, & q^{-1/2} \\ -q^{1/2}, & 0 \end{pmatrix}. \tag{10}$$

In the last term of the Lagrangian (9), the variables p^μ and $\chi\psi^\mu$ are arranged in such a way that for the differentiation with respect to p^μ, one can exploit the $GL_q(2)$ invariant ($(\chi\psi_\mu)(\tau)\, p^\mu(\tau) = q\, p_\mu(\tau)\, (\chi\psi^\mu)(\tau)$) differential calculus [12]. Here the q-Hamiltonian for a free spinning particle has been taken to be: $H = \frac{e}{1+q^2} p^2 - i\chi\psi_\mu p^\mu$. One can include the mass term in the Lagrangian (9) by invoking another Lorentz scalar q-anticommuting spinor variable ψ_5 ($(\psi_5)^2 = -1$) as

$$L_f^m = q^{1/2} p_\mu \dot{x}^\mu + \frac{i}{2}(\psi_\mu \dot{\psi}^\mu - \psi_5 \dot{\psi}_5) - \frac{e}{1+q^2}(p^2 - m^2) + i\chi(\psi_\mu p^\mu - \psi_5 m), \tag{11}$$

where the τ independent mass term (m) obeys the following q-commutation relations with the rest of the dynamical variables

$$\begin{aligned} e\, m &= q m\, e, \quad \dot{x}_\mu\, m = q\, m\, \dot{x}_\mu, \quad x_\mu m = q\, m\, x_\mu, \quad p_\mu\, m = m\, p_\mu, \\ \chi m &= q^{1/2} m \chi, \quad \psi_\mu m = q^{1/2} m \psi_\mu, \quad \psi_5 m = q^{1/2} m \psi_5. \end{aligned} \tag{12}$$

The q-(anti)commutation relations of ψ_5 with the rest of the dynamical variables are the *same* as that of ψ_μ and both of these anticommute ($\psi_\mu \psi_5 + \psi_\mu \psi_5 = 0$). The equations of motion from the Lagrangian (11) are

$$\begin{aligned} \dot{x}_\mu &= q^{1/2}(e p_\mu - i\chi\psi_\mu) \\ \dot{\psi}_\mu &= q^{1/2} \chi p_\mu, \quad \dot{p}_\mu = 0, \\ \dot{\psi}_5 &= q^{1/2} \chi m, \end{aligned} \tag{13}$$

which satisfy the on-shell q-(anti)commutation relations (2) and (12). In the differentiation of the Lagrangian (11) with respect to $\dot{x}_\mu$ and p_μ, we have exploited

the $GL_q(2)$ invariant differential calculus developed in Ref.[12]. For instance, for $xy = qyx$, any monomial is arranged in the form $y^r\, x^s$ and then we use

$$\begin{aligned} \frac{\partial(y^r\, x^s)}{\partial x} &= y^r\, x^{s-1}\, q^r\, \frac{(1-q^{2s})}{(1-q^2)}, \\ \frac{\partial(y^r\, x^s)}{\partial y} &= y^{r-1}\, x^s\, \frac{(1-q^{2r})}{(1-q^2)}, \end{aligned} \tag{14}$$

where $r, s \in \mathbb{Z}$ are whole numbers (not fractions). For differentiations with respect to the odd Grassmann variables $\psi_\mu, \psi_5, \dot{\psi}_\mu, \dot{\psi}_5, \chi$, these variables are first brought to the left side by using q-(anti)commutation relations (2) and (12) in the corresponding expressions, and then, differentiation is carried out. Using the contravariant metric of equation (10) and the Hamiltonian (H), one can check that equations (13) can be rewritten as $\dot{x}_\mu = \Omega^{AB} \partial_A x_\mu \partial_B H \equiv q^{1/2}(e p_\mu - i\chi\psi_\mu)$, $\dot{p}_\mu = 0$ and $\dot{\psi}_\mu = -i\frac{\partial H}{\partial \psi^\mu}$.

The second-order Lagrangian (L_s^m), describing the motion of a spinning particle on the tangent manifold (velocity phase space), can be obtained from the first-order Lagrangian (11) by exploiting equations (2), (12) and (13) as given below:

$$L_s^m = \frac{q^2}{1+q^2}\, e^{-1}\, (\dot{x}_\mu + q^{1/2} i\chi\psi_\mu)^2 + \frac{e}{1+q^2}\, m^2 + \frac{i}{2}(\psi_\mu \dot{\psi}^\mu - \psi_5 \dot{\psi}_5) - i\chi\psi_5 m. \tag{15}$$

The consistent expression for the canonical momenta (p_μ) for the first- and second-order Lagrangians (11) and (15)

$$p_\mu = q^{-3/2}\Big(\frac{\partial L^m_{(f,s)}}{\partial \dot{x}^\mu}\Big) \equiv q^{-1/2}\, e^{-1}\, (\dot{x}_\mu + q^{1/2} i\chi\psi_\mu), \tag{16}$$

leads to its square as:

$$p_\mu\, p^\mu = e^{-2}(\dot{x}_\mu + q^{1/2} i\chi\psi_\mu)^2, \tag{17}$$

if we use the q-(anti)commutation relations (2). To see that the right hand side of equation (17) is the square of the mass, one has to exploit the $GL_q(2)$ invariant differential calculus to differentiate Lagrangian L_s^m with respect to e. For instance, the first term of (15) has to be first recast as $\frac{(\dot{x}_\mu + q^{1/2} i\chi\psi_\mu)^2 e^{-1}}{1+q^2}$ and then, differentiation with respect to e has to be performed. The final outcome

$$\frac{q^4}{1+q^2}\Big[m^2 - e^{-2}\, (\dot{x}_\mu + q^{1/2} i\chi\psi_\mu)^2\Big] = 0, \tag{18}$$

leads to the mass-shell condition for the q-deformed spinning relativistic particle as:

$$p_\mu\, p^\mu - m^2 = 0. \tag{19}$$

This equation is one of the Casimir invariants of the Poincaré group corresponding to the undeformed flat Minkowski space-time and it turns up here as the constraint condition. The other constraint condition $p_\mu \psi^\mu - m\psi_5 = 0$ appears because of the differentiation with respect to χ in the first-order Lagrangian (11) (which can be checked to be true for the Lagrangian (15) as well). These constraint conditions are in agreement with the discussion of the Klein-Gordon equation and the Dirac-equation derived from the representation theory of the Lorentz group and q-deformation of the Dirac-γ matrices [13]. To obtain the Lagrangian with a square root, it is essential to succinctly express (e, e^{-1}) in terms of mass (m) and the square root of $(\dot{x}_\mu + q^{1/2} i\chi\psi_\mu)^2$. It is not straightforward to obtain (e, e^{-1}) from $m^2 = e^{-2}(\dot{x}_\mu + q^{1/2} i\chi\psi_\mu)^2$ because of the non-commutativity of velocity, mass, χ and ψ_μ. A nice and simple way to compute these is to first start with

$$e^{-1} = f(q)\, m\, [(\dot{x}_\mu + q^{1/2} i\chi\psi_\mu)^2]^{-1/2}, \tag{20}$$

and insert it into (15) using q-(anti)commutation relations (2) and (12) such that e^{-1} and e occupy various positions in its first and second terms. The requirement of the equality of the resulting Lagrangians leads to [2]

$$f^4(q) = q^2. \tag{21}$$

This requirement is satisfied by four values of $f(q)$, namely; $f(q) = \pm q^{1/2}; \pm i q^{1/2}$. The key requirement, however, that the q-deformed Lagrangian should reduce to the usual undeformed (classical) Lagrangian in the limit $q \to 1$ restricts $f(q)$ to picking up *only* the value $q^{1/2}$. Ultimately the Lagrangian (L_0^m) with the square root turns out to be

$$L_0^m = q^{1/2} m [(\dot{x}_\mu + q^{1/2} i\chi\psi_\mu)^2]^{1/2} + \frac{i}{2}(\psi_\mu \dot{\psi}^\mu - \psi_5 \dot{\psi}_5) - i\chi\psi_5 m. \tag{22}$$

All the three Lagrangians (11), (15) and (22) are equivalent [3] as far as symmetry properties are concerned. They differ drastically, however, in the limit $m \to 0$.

The expression for the canonical momenta (16) is true for the Lagrangian (22) with square root as well. To see this, we require the $GL_q(2)$ invariant relation

$$[(\dot{x}_\mu + q^{1/2} i\chi\psi_\mu)^2]^{1/2}\, m = q\, m [(\dot{x}_\mu + q^{1/2} i\chi\psi_\mu)^2]^{1/2}, \tag{23}$$

that emerges from (20) and the equality $e^{-2} = m^2[(\dot{x}_\mu + q^{1/2} i\chi\psi_\mu)^2]^{-1}$. Finally we obtain,

$$p_\mu = q^{-3/2}\Big(\frac{\partial L_0^m}{\partial \dot{x}^\mu}\Big) \equiv (\dot{x}_\mu + q^{1/2} i\chi\psi_\mu)[(\dot{x} + q^{1/2} i\chi\psi)^2]^{-1/2}\, m, \tag{24}$$

[2] We have used equation (20) and $[(\dot{x}_\mu + q^{1/2} i\chi\psi_\mu)^2]^{1/2}\, m = f^2(q)\, m\, [(\dot{x}_\mu + q^{1/2} i\chi\psi_\mu)^2]^{1/2}$ in all three possible expressions in which both the terms can be recast. The term-by-term equality for all the possibilities yields a common condition $f^4(q) = q^2$.

[3] The last two-terms in (22) can be combined together to yield a more concise expression $\frac{i}{2}(\psi_\mu \dot{\psi}^\mu + \psi_5 \dot{\psi}_5)$ using the equation of motion $\psi_5 = q^{1/2}\chi\, m$. However, there are certain subtleties in the proof of equivalence of the resulting Lagrangian with the other two [9].

where the following chain rule has been used

$$\frac{\partial L_0^m}{\partial \dot{x}^\mu} = \frac{\partial(\dot{x} + q^{1/2} i\chi\psi)^2}{\partial \dot{x}^\mu}\, \frac{\partial[(\dot{x} + q^{1/2} i\chi\psi)^2]^{1/2}}{\partial(\dot{x} + q^{1/2} i\chi\psi)^2}\, \frac{\partial L_0^m}{\partial[(\dot{x} + q^{1/2} i\chi\psi)^2]^{1/2}}. \tag{25}$$

In the computation of the q-derivative of the q-variables with fractional power, one has to use

$$\frac{\partial}{\partial y}(y^{r/s}) = \frac{(1 - q^{2r})}{(1 - q^{2s})}\, y^{(r/s)-1}, \tag{26}$$

where r is *not* divisible by s $(r, s \in \mathbb{Z})$. The other constraint: $p \cdot \psi - m\, \psi_5 = 0$ emerges due to $\frac{\partial L_0^m}{\partial \chi} = 0$. The latter constraint and the mass-shell condition (19) are satisfied for both the left chain rule as well as the right chain rule of differentiation.

It is a well established fact that the existence of first-class constraints on a system implies underlying gauge symmetries [14]. For the system under consideration, there exist first-class as well as second-class constraints which can be seen for all the three equivalent Lagrangians (11), (15) and (22). The primary constaints momenta $\Pi_e \approx 0$ as well as $\Pi_\chi \approx 0$ and, corresponding secondary constraints $p^2 - m^2 \approx 0$, $p \cdot \psi - m\psi_5 \approx 0$, are first-class. However, the canonical momenta corresponding to the fields ψ_μ and ψ_5 are second-class. We shall not devote time here for the discussion of q-deformed Dirac brackets, subsequent $\hbar$-quantization schemes, etc., in the Hamiltonian formalism. Instead, we shall concentrate *only* on the first-order Lagrangian (11) and discuss local (super)gauge symmetries as well as reparametrization invariance that emerge due to the presence of the first-class constraints. For instance, the constraints $\Pi_e \approx 0$ and $p^2 - m^2 \approx 0$ produce the following infinitesimal local gauge symmetry transformations [14]

$$\begin{aligned} \delta_1 x^\mu &= q^{1/2}\, \xi\, p^\mu, \quad \delta_1 p^\mu = 0, \quad \delta_1 e = q^2\, \dot{\xi}, \\ \delta_1 \psi^\mu &= 0, \quad \delta_1 \psi_5 = 0, \quad \delta_1 \chi = 0, \end{aligned} \tag{27}$$

because the Lagrangian transforms as

$$\delta_1 L_f^m = \frac{d}{d\tau}\Big[\frac{\xi(p^2 + q^2 m^2)}{(1 + q^2)}\Big], \tag{28}$$

where ξ is an infinitesimal non-commuting gauge parameter ($\xi\, p_\mu = q\, p_\mu\, \xi$, *etc.*)

To remove the negative norm states from the physical spectrum (that might be generated due to the zero component of ψ^μ) one requires a local supergauge symmetry. The constraints $\Pi_\chi \approx 0$ and $p \cdot \psi - m\psi_5 \approx 0$ serve that purpose by generating the following supergauge transformations with the infinitesimal q-(anti)commuting parameter $\eta(\tau)(\eta^2 = 0)$

$$\begin{aligned} \delta_2 x^\mu &= q^{1/2}\, \eta\, \psi^\mu, \quad \delta_2 p^\mu = 0, \quad \delta_2 e = q^{1/2}(1 + q^2)\eta\chi, \\ \delta_2 \psi^\mu &= q^{1/2} i\eta\, p^\mu, \quad \delta_2 \psi_5 = q^{1/2} i\eta\, m, \quad \delta_2 \chi = i\, \dot{\eta}, \end{aligned} \tag{29}$$

where η obeys following q-(anti)commutation relations

$$\begin{aligned} \eta\, x_\mu &= q^{1/2}\, x_\mu\, \eta, \qquad \eta\, p_\mu = q^{1/2}\, p_\mu\, \eta, \qquad \eta\, \psi_\mu = -q^{1/2}\, \psi_\mu\, \eta, \\ \eta\, \psi_5 &= -q^{1/2}\, \psi_5\, \eta, \qquad \eta\, m = q^{1/2}\, m\, \eta, \qquad \eta\, \chi = -\,\chi\, \eta. \end{aligned} \tag{30}$$

As a consequence of the above symmetry transformations, the first-order Lagrangian transforms as

$$\delta_2 L_f^m = q^{1/2}\, \frac{d}{d\tau}\, \Big[\frac{\eta(p\cdot\psi + m\psi_5)}{2}\Big]. \tag{31}$$

The above Lagrangian is also invariant under the following reparametrization transformations

$$\begin{aligned} \delta_r\, x_\mu &= \epsilon\, \dot{x}_\mu, \qquad \delta_r\, p_\mu = \epsilon\, \dot{p}_\mu, \qquad \delta_r\, e = \frac{d}{d\tau}(\epsilon e), \\ \delta_r\, \psi_\mu &= \epsilon\, \dot{\psi}_\mu, \qquad \delta_r\, \psi_5 = \epsilon\, \dot{\psi}_5, \qquad \delta_r\, \chi = \frac{d}{d\tau}(\epsilon\chi), \end{aligned} \tag{32}$$

emerging due to the one-dimensional diffeomorphism $\tau \to \tau - \epsilon(\tau)$ (with the commuting infinitesimal parameter $\epsilon(\tau)$), because the Lagrangian undergoes the following change under (32):

$$\delta_r L_f^m = \frac{d}{d\tau}\Big[\epsilon\, L_f^m\Big]. \tag{33}$$

In the usual undeformed ($q = 1$) case of a free spinning relativistic particle, a linear combination of (super)gauge symmetries (29) and (27) is found to be equivalent on-shell to the reparametrization invariance (32) with the identification $\xi = \epsilon e$ and $\eta = \epsilon\chi$ [15]. However, in the deformed case, as it turns out, even for the above identification of the (super)gauge parameters and the on-shell condition (13), the following equality

$$(\delta_1 - i\delta_2)\Phi = \delta_r \Phi, \tag{34}$$

for the variables $\Phi \equiv x_\mu, p_\mu, \psi_\mu, \psi_5, e, \chi$ is true *only* for $q = \pm 1$. This is due to the fact that the transformations of the einbein field, in spite of the above identification, are not equal on-shell unless $q = \pm 1$. This condition ($q = \pm 1$) also turns up due to the requirement that the q-commutator of two supersymmetric gauge transformations must produce a reparametrization plus an additional supersymmetric gauge transformation as is essential in pure supergravity theories. It is not difficult to check that following equalities

$$\begin{aligned} [\delta_\kappa, \delta_\eta]_{q^2}\; x^\mu &= \epsilon \dot{x}^\mu + q^{1/2}\eta' \psi^\mu, \\ [\delta_\kappa, \delta_\eta]_{q^2}\; \psi^\mu &= \epsilon \dot{\psi}^\mu + q^{1/2} i \eta' p^\mu, \\ [\delta_\kappa, \delta_\eta]_{q^2}\; \psi_5 &= \epsilon \dot{\psi}_5 + q^{1/2} i \eta'\, m, \\ [\delta_\kappa, \delta_\eta]_{q^2}\; \chi &= \epsilon \dot{\chi} + \dot{\epsilon}\chi + i\, \dot{\eta}', \\ [\delta_\kappa, \delta_\eta]_{q^2}\; p^\mu &= \epsilon \dot{p}^\mu, \end{aligned} \tag{35}$$

are true *on-shell* for

$$\begin{aligned} \epsilon &= i q^{1/2}(1 + q^2)\eta\kappa e^{-1}, \\ \eta' &= i\epsilon\chi \equiv -q^{1/2}(1 + q^2)\eta\kappa e^{-1}\chi, \\ [\delta_\kappa, \delta_\eta]_{q^2} &\equiv \delta_\kappa\, \delta_\eta - q^2\, \delta_\eta\, \delta_\kappa. \end{aligned} \tag{36}$$

However, the validity of a similar equality in the case of the einbein field, namely, $(\delta_\kappa\, \delta_\eta - q^2\, \delta_\eta\, \delta_\kappa)e = \dot{\epsilon}\, e + \epsilon\dot{e} + q^{1/2}(1 + q^2)\eta'\chi$, requires

$$(\eta\dot{\kappa} - q^2\kappa\dot{\eta}) = \eta\kappa e^{-1}\dot{e}(1 - q^{-2}) + (\eta\dot{\kappa} - \kappa\dot{\eta}), \tag{37}$$

which is true only for $q = \pm 1$. [4] This only demonstrates that for arbitrary value of q, the supergravity requirement and the on-shell equivalence of (super)gauge and reparametrization symmetries are not true.

One can compute the conserved charges corresponding to the symmetry transformations (27) and (29) by applying the least action principle. This is illustrated below:

$$\delta S = 0 = \int d\tau \Big(\delta[q^{1/2} p_\mu \dot{x}^\mu + \frac{i}{2}\psi\cdot\dot{\psi} - \frac{i}{2}\psi_5\dot{\psi}_5 - \mathcal{H}_c(x, p, \psi, \psi_5)] - \frac{d}{d\tau} g(\tau)\Big), \tag{38}$$

where S is the action corresponding to the Lagrangian (15), $\mathcal{H}_c$ is the canonical Hamiltonian and $g(\tau) = \frac{\xi(p^2 + q^2 m^2)}{1 + q^2}$ and $\frac{q^{1/2}\eta(p\cdot\psi + m\psi_5)}{2}$, respectively, for the above symmetry transformations. Using anticommutation relations for ψ_μ, ψ_5 and q-commutation relations $\delta\dot{x}^\mu p_\mu = q p_\mu \delta\dot{x}^\mu$, we obtain Hamilton equations of motion and conservation laws. For validity of the following Hamilton equations of motion

$$\dot{x}^\mu = q^{-1/2}\frac{\partial\mathcal{H}_c}{\partial p^\mu},\ \dot{p}^\mu = -q^{1/2}\frac{\partial\mathcal{H}_c}{\partial x^\mu},\ \dot{\psi}^\mu = -i\frac{\partial\mathcal{H}_c}{\partial\psi^\mu},\ \dot{\psi}_5 = i\frac{\partial\mathcal{H}_c}{\partial\psi_5}, \tag{39}$$

we obtain a general expression for the conserved charge (Q) as:

$$Q = q^{-1/2}\delta x^\mu p_\mu + \frac{i}{2}\delta\psi_5\psi_5 - \frac{i}{2}\delta\psi_\mu\psi^\mu - g(\tau). \tag{40}$$

In the case of the global version of symmetry transformations (27) and (29), this yields the following charges:

$$Q_\xi = \frac{q^2(p^2 - m^2)}{1 + q^2} \quad \text{and} \quad Q_\eta = q^{1/2}(p\cdot\psi - m\psi_5). \tag{41}$$

One can readily check that both of these charges are conserved due to the equations of motion (13). The latter one is conserved on the constrained submanifold where the first-class constraint $p^2 - m^2 = 0$ is valid.

[4] In the computation of $\dot{\epsilon}$, we have used $\dot{e}^{-1} = (\partial e^{-1}/\partial e)\dot{e} \equiv -q^{-2}\, e^{-2}\dot{e}$. Here η and κ are supersymmetric transformation parameters and the transformation $\delta_\kappa[\eta\psi_\mu = -q^{1/2}\psi_\mu\eta]$ leads to $\eta\kappa = -\kappa\eta$ due to $\eta p_\mu = q^{1/2} p_\mu \eta$.

It is rather difficult to extract out a general solution for equation (13) when all the variables are arbitrary functions of the evolution parameter τ. However, due to gauge symmetry transformations (27) and (29), one can choose an analogue of the Lorentz gauge, namely; $\dot{e} = \dot{\chi} = 0$. Under such a gauge choice, one obtains

$$\begin{aligned} x_\mu(\tau) &= x_\mu(0) + q^{1/2}[e(0)p_\mu(0) - i\chi(0)\psi_\mu(0)]\,\tau, \\ \psi_\mu(\tau) &= \psi_\mu(0) + q^{1/2}\,\chi(0)p_\mu(0)\,\tau, \\ \psi_5(\tau) &= \psi_5(0) + q^{1/2}\,\chi(0)\,m\,\tau, \\ p_\mu(\tau) &= p_\mu(0), \end{aligned} \tag{42}$$

which satisfy all the q-(anti)commutation relations (2), the $GL_q(2)$ invariant quantum world-line $x_\mu(\tau)p^\mu(\tau) = q\;p_\mu(\tau)x^\mu(\tau)$ and the q-super world-line (3) at any arbitrary value of the evolution parameter τ, if we assume the validity of relations (2) and (12) at the initial "time" $\tau = 0$.

Unlike the q-dependent (anti)commutation relations among the variables in equations (2), (12) and (30), there are some *q-independent* (anti)commutation relations that emerge automatically due to (graded)associativity conditions *or* infinitesimal gauge transformations on q-dependent relations. For instance, one can easily see the commutativity of e and χ that is present in equation (2). This emerges due to the on-shell condition $\dot{\psi}_\mu = q^{1/2}\chi p_\mu$ in $e\dot{\psi}_\mu = q^{1/2}\dot{\psi}_\mu e$ with $ep_\mu = qp_\mu e$. The commutativity of mass parameter m and momenta p_μ in equation (12) is mainly due to the mass-shell condition which can be also checked by extracting out the expression for p_μ from equations of motion (13) and using equations (2) as well as (12). More comments about this commutativity can be found in Ref.[8]. The q-independent anticommutativity of η and χ in equation (30) emerges due to $\chi\delta_2\psi_\mu = -q^{1/2}\delta_2\psi_\mu\chi$ when we use $\chi p_\mu = q^{1/2}p_\mu\chi$.

It is now a very interesting venture to develop a q-deformed BRST quantization scheme on a quantum (super)world-line for a spinning and a scalar relativistic particle, as they present a prototype example of an Abelian gauge theory. These examples would provide the simplest laboratory for the development of q-deformed gauge theory, q-deformed Hamiltonian formulation, q-deformed constraint analysis and q-deformed Dirac brackets, etc., in the *undeformed* Minkowski space-time manifold. It would be worthwhile to extend these models to the case when Minkowski space-time manifold and cotangent manifold *both* are q-deformed. In addition, one can generalize the second-order Lagrangian (15) to the corresponding q-deformed Neveu-Schwarz-Ramond model for q-deformed string theory. These are some of the issues for future investigations.

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