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DEVIATION EQUATIONS IN SPACES WITH A TRANSPORT ALONG PATHS

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1. INTRODUCTION

This paper starts a series of applications of the transports along paths introduced in [1,2] in fibre bundles to certain physical problems.

All considerations in the present work are made in a (real) differentiable manifold M [3,4] whose tangent bundle $(T(M),\pi,M)$ is endowed with a transport along paths [1]. Here $T(M):=\bigcup_{x\in M} T_x(M)$, $T_x(M)$ being the tangent to M space at $x\in M$ and $\pi:T(M)\longrightarrow M$ is such that $\pi(V):=x$ for $V\in T_x(M)$.

The set of all sections of a fibre bundle ξ [3,4] is denoted by $\mathrm{Sec}(\xi)$; e.g. $\mathrm{Sec}(T(M),\pi,M)$ is the module of vector fields on M.

By J and $\gamma:J\longrightarrow M$ denoted are, respectively, an arbitrary real interval and a path in M. If γ is of class C^1, its tangent vector is written as $\dot{\gamma}$.

The transport along paths in $(T(M),\pi,M)$ (cf. [1]) is a map $I:\gamma\longmapsto I^\gamma$, $I^\gamma:(s,t)\longmapsto I^\gamma_{s\to t}$, $s,t\in J$ being the transport along γ, where $I^\gamma_{s\to t}:T_{\gamma(s)}(M)\longrightarrow T_{\gamma(t)}(M)$, satisfy the equalities

$$I^\gamma_{t\to r}\circ I^\gamma_{s\to t}=I^\gamma_{s\to r},\quad r,s,t\in J, \tag{1.1}$$

$$I^\gamma_{s\to s}=id_{T_{\gamma(s)}(M)},\quad s\in J. \tag{1.2}$$

Here id_X is the identity map of the set X.

A linear transport (L-transport) along paths L in $(T(M),\pi,M)$ satisfies, besides (1.1) and (1.2), the equality (cf. [2])

$$L^\gamma_{s\to t}(u^i e_i(s))=H^i_{\cdot j}(t,s;\gamma)u^j e_i(t),\quad s,t\in J,\ u^i\in\mathbb{R}. \tag{1.3}$$

Here and henceforth in our text the Latin indices run from 1 to $n:=\dim(M)$ and summation from 1 to n is assumed over repeated indices on different levels; $\{e_i(s)\}$ is a basis in $T_{\gamma(s)}(M)$; and $H(t,s;\gamma):=\|H^i_{.j}(t,s;\gamma)\|$ is the matrix of the L-transport, in terms of which (1.2) reads

$$H(s,s;\gamma)=\mathbb{1}:=\mathrm{diag}(1,\ldots 1):=\|\delta^i_j\|, \tag{1.2'}$$

δ^i_j being the Kronecker's delta symbols.

This work is organized as follows. In Sect. 2, based on the ideas of [5,6], a strict definition is given of the displacement vector in a manifold with a transport along paths in its tangent bundle. The deviation vector between two paths with respect to a third one is introduced on the ground of this concept. Sect. 3, which follows the works [5,7-10], is devoted to the deviation equation, satisfied by the deviation vector, which is a generalization of the geodesic deviation equation (known also in the mathematical literature as a Jacobi equation). Special cases of this equation are considered. In particular, it is proved that it generalizes the equation of motion of two point particles, i.e. the second Newton's low of mechanics.

2. DISPLACEMENT AND DEVIATION VECTORS

Let in the tangent fibre bundle $(T(M),\pi,M)$ to the differentiable manifold M there be given a transport along paths I and $\gamma:J\longrightarrow M$ be a smooth, of class C^1, path in M. We define maps

$$d^\gamma_s:J\longrightarrow T_{\gamma(s)}(M)=\pi^{-1}(\gamma(s)),\ s\in J, \tag{2.1a}$$

such that

$$d^\gamma_s(t):=\int_s^t(I^\gamma_{u\to s}\dot\gamma(u))du,\ s,t\in J. \tag{2.1b}$$

Proposition 2.1. If I coincides with some linear transport along paths L, then

$$d^\gamma_r(s)=d^\gamma_r(t)+L^\gamma_{t\to r}(d^\gamma_t(s)),\ r,s,t\in J. \tag{2.2}$$

Proof. (2.2) follows from (2.1b) and (1.1):

$$d^\gamma_r(s):=\int_r^s(L^\gamma_{u\to r}\dot\gamma(u))du=\int_r^t(L^\gamma_{u\to r}\dot\gamma(u))du+\int_t^s(L_{t\to r}\circ L^\gamma_{u\to t}\dot\gamma(u))du=$$

$$=d^\gamma_r(t)+L^\gamma_{t\to r}(d^\gamma_t(s)),\ r,s,t\in J.\ \blacksquare$$

Definition 2.1. The vector $d^\gamma_s(t)$ will be called a displacement vector of $\gamma(t)$ with respect to $\gamma(s)$ if d^γ_s, $s\in J$ maps J homeomorphicly onto its image $d^\gamma_s(J)$.

Generally, for an arbitrary transport along paths and a path γ the condition in this definition is not fulfilled. But it happens that under sufficiently general conditions there exist suitable combinations of I and γ for which it is true. Without going into details of this problem, we shall present only two examples for such cases. They are expressed by the proved below proposition 2.3 and corollary 2.1 and practically include all combinations essential for physical application. For the first example we need

Definition 2.2. Let I be a transport along paths in $(T(M),\pi,M)$. An I-path is a smooth, of class C^1, path $\gamma:J\longrightarrow M$ the tangent vector field $\dot\gamma\in\mathrm{Sec}(T(\gamma(J)),\pi,\gamma(J))$ of which is transported by means of I along γ, i.e.

$$\dot\gamma(t)=I^\gamma_{s\to t}\dot\gamma(s),\ s,t\in J. \tag{2.3}$$

The existence problem for the I-paths in the case of L-transports along paths, i.e. for L-paths, is shortly formulated in Ref. [11].

As the theory of I-paths is not in the main direction of this investigation, we shall only remark that an evident special case of the I-paths (in affine parameterization) in manifolds with connection is the geodesic paths, whose tangent vector undergoes a parallel transport (defined by the manifold's connection) along themselves [3,4].

Proposition 2.2. If $\gamma:J\longrightarrow M$ is an I-path, then

$$d_s^\gamma(t)=(t-s)\dot\gamma(s),\quad s,t\in J. \tag{2.4a}$$

Proof. (2.4a) follows from the substitution of (2.3) for s=u and t=s into (2.1b).■

Corollary 2.1. If $\gamma:J\longrightarrow M$ is a regular I-path, then $d_s^\gamma(t)$ is a displacement vector of $\gamma(t)$ with respect to $\gamma(s)$, i.e. the condition in definition 2.1 is fulfilled.

Proof. From (2.4a) it follows that for any $s\in J$ the mapping (2.1a) is linear, so, due to the regularity of γ (i.e. $\dot\gamma(s)\neq 0$), it is a diffeomorphism, and consequently homeomorphism, from J onto $d_s^\gamma(J)$.■

Proposition 2.3. If γ is a C^1 path without self-intersections, then in the case of L-transports along paths the mappings d_s^γ, $s\in J$ map J locally homeomorphicly on its image $d_s^\gamma(J)$, i.e. locally $d_s^\gamma(t)$ is a displacement vector of $\gamma(t)$ with respect to $\gamma(s)$.

Remark. In this case the word "locally" means in some part of (or over the whole) set $\gamma(J)$ in a neighborhood of which there exist local coordinates with the properties described in [12], lemma 7. (See also below the proof of this proposition.)

Proof. Firstly we shall prove that d_s^γ, $s\in J$ are locally injective, i.e. if $t_1\neq t_2$, then $d_s^\gamma(t_1)\neq d_s^\gamma(t_2)$. In fact, for linear transports along paths, by proposition 3.1 of [2] in $(T(M),\pi,M)$ along γ there exists a basis $\{E_{i'}\}$, which by [12], lemma 7 is (locally) holonomic and in which the matrix of the transport is

$\|H^{i'}_{\cdot\cdot j'}(t,s;\gamma)\|=\mathbb{1}=\|\delta^i_j\|$. In this basis, if $\gamma(s)$ and $\gamma(t)$ belong to one and the same coordinate neighborhood, we have

$$(d_s^\gamma(t))^{i'}=\int_s^t H^{i'}_{\cdot\cdot j'}(s,u;\gamma)\dot\gamma^{j'}(u)du=\int_s^t\dot\gamma^{i'}(u)du=\gamma^{i'}(t)-\gamma^{i'}(s), \tag{2.4b}$$

where the validity of the last equality follows from the fact that γ is without self-intersections. Consequently, if $\gamma(s)$, $\gamma(t_1)$ and $\gamma(t_2)$ belong to one and the same coordinate neighborhood, then $d_s^\gamma(t_1)\neq d_s^\gamma(t_2)$ is equivalent to $\gamma^{i'}(t_1)\neq\gamma^{i'}(t_2)$, $t_1,t_2\in J$, which is equivalent to $t_1\neq t_2$ only if the path γ is without selfintersections in the mentioned coordinate neighborhood, as is supposed here.

The maps d_s^γ, evidently, are locally (in the above neighborhood) unique and differentiable, besides, due to (2.1b), we have $\frac{d}{dt}(d_s^\gamma(t))=I^\gamma_{t\longrightarrow s}\dot\gamma(t)$. The existence and the continuity of $(d_s^\gamma)^{-1}:d_s^\gamma(J)\longrightarrow J$ follows from the representation (2.4b) of $d_s^\gamma(t)$ in the basis $\{E_{i'}\}$.

Analogously the proposition can be proved when only the points $\gamma(t_1)$ and $\gamma(t_2)$ lie in the same coordinate neighborhood with the needed properties. The only difference now is that if $\gamma(s)$ is out of this neighborhood, then in the right-hand-side of the last equality in (2.4b) there appear terms independent of t, which does not change the validity of the above considerations.■

Remark. If the path γ has self-intersections, in the right-hand-side of (2.4b) the term $\sum_k\oint_{\gamma_k}\dot\gamma^{i'}(u)du$ appears, where the summation is taken over all closed loops γ_k formed by the restriction of γ on the interval $[\min(s,t),\max(s,t)]$. Therefore in the general case, from (2.4b) it does not follow that d_s^γ are injective maps.

Further, the transport I and the path γ are supposed to be chosen so that the condition in definition 2.1 would be true, i.e. $d_s^\gamma(t)$ would be a displacement vector of $\gamma(t)$ with respect to $\gamma(s)$.

The displacement vector, a direct generalization of the difference of two Euclidean radius-vectors (see below Sect. 4), finds application due to the property that it has the meaning of "vector relative coordinate" on the one-dimensional submanifold $\gamma(J)$, i.e. if a point $\gamma(s)\in\gamma(J)$ for fixed $s\in J$ is given, then from the knowledge of the displacement vector $d_s^\gamma(t)$ for any $t\in J$ one can define (recover) the point $\gamma(t)$ and vice versa. In fact, if $t\in J$, then by (2.1b) to it there corresponds a unique vector $d_s^\gamma(t)\in T_{\gamma(s)}(M)$ and on the opposite, if $\Delta\in d_s^\gamma(J)$, then as $d_s^\gamma: J \longrightarrow d_s^\gamma(J)$ are homeomorphisms there exists a unique $t\in J$, and so a point $\gamma(t)\in d_s^\gamma(J)$, with the property $d_s^\gamma(t):=\Delta$. For the same reason, with the help of a displacement vector there can be defined also a (global) chart on $\gamma(J)$: because $d_s^\gamma: J \longrightarrow d_s^\gamma(J)$ is a homeomorphism, the set $d_s^\gamma(J)\subset T_{\gamma(s)}(M)$ is one dimensional submanifold and, hence, there exists a homeomorphism $\varphi_s: d_s^\gamma(J) \longrightarrow \mathbb{R}^1$, as a consequence of which $(\gamma(J), \varphi_s\circ d_s^\gamma\circ\gamma^{-1})$, where $\gamma^{-1}: \gamma(J)\longrightarrow J$ and $\gamma^{-1}(\gamma(t)):=t$, $t\in J$, is a global chart on $\gamma(J)$.

Using the displacement vector one can construct the so-called deviation vector between two paths with respect to a third one. This is done as follows.

Let there be given paths $x_a: J_a \longrightarrow M$, $a=1,2$ and $x: J\longrightarrow M$. Let there be fixed one-to-one maps $\tau_a: J\longrightarrow J_a$, $a=1,2$. (These maps always exit as all real intervals are equipollent.) Let also be given the one parameter families of paths $\{\gamma_s:\ \gamma_s: J'_s \longrightarrow M,\ s\in J\}$ and $\{\eta_s:\ \eta_s: J''_s \longrightarrow M,\ s\in J\}$ having the properties $\gamma_s(r'_s):=x_1(\tau_1(s)):=\eta_s(t'_s)$, $\gamma_s(r''_s):=x_2(\tau_2(s))$ and $\eta_s(t''_s):=x(s)$ for some $r'_s, r''_s\in J'_s$ and $t'_s, t''_s\in J''_s$, $s\in J$. The paths γ_s, $s\in J$ are supposed smooth and such that the maps $d_r^{\gamma_s}$, $r\in J'_s$, $s\in J$ determined by them from (2.1) define corresponding displacement vectors.

Definition 2.3. The deviation vector of x_2 with respect to x_1

relatively to x at the point $x(s)$, $s\in J$ is the vector

$$h_{21}:=h_{21}(s;x):=\left(I^{\eta_s}_{t'_s\longrightarrow t''_s}\circ d^{\gamma_s}_{r'_s}\right)(r''_s)=$$

$$=I^{\eta_s}_{t'_s\longrightarrow t''_s}\int_{r'_s}^{r''_s}(I^{\gamma_s}_{u\longrightarrow r'_s}\dot\gamma_s(u))du\in T_{x(s)}(M). \qquad (2.5)$$

The deviation vector and the objects involved in its definition can be interpreted from the view point of the physical applications as follows. (Anything written below needs many additional definitions and precise statements as to have a strict meaning. For this reason one may think that M in it is the 4-dimensional space-time V_4 of general relativity - see e.g. [13].) We can interpret the paths x_1 and x_2 as trajectories (world lines) of two observed point particles, the path x - as a trajectory of an observer "studying" their behavior. The parameters $s_1\in J_1$, $s_2\in J_2$ and $s\in J$ may be considered as "proper times" of the corresponding particles. The maps τ_1 and τ_2 give the connection between these proper times, define the "observation process" in this concrete situation, and, in a certain sense, they give some "simultaneity" between all particles: τ_1 and τ_2 define a simultaneity between the observer and the observed particles and $\tau_2\circ\tau_1^{-1}$ - between the observed particles. For a fixed $s\in J$ the paths γ_s and η_s can be regarded as trajectories (world lines) of "signals" which "physically realize" the maps $\tau_2\circ\tau_1^{-1}$ and τ_1. (For instance, in V_4 if γ_s and η_s are isotropic geodesic paths, then the above described construction corresponds to the definition of simultaneity with the help of light signals - see [13].) In this context the deviation vector has the meaning of a vector describing the relative position of the second observed particle with respect to the first one as this is "seen" from an observer.

At the end of this section we want to present the lowest and, respectively, the most used approximations when one works with the displacement and deviation vectors.

If the transport I has a continuous dependence on (one of) its parameters, then using the formula $\int_a^b f(u)du=f(a)(b-a)+O((b-a)^2)$ for any continuous function $f:[a,b]\longrightarrow\mathbb{R}$, from (2.1b) and (1.2), we find

$$d_s^{\gamma}(t)=(t-s)\dot{\gamma}(s)+O((t-s)^2). \qquad (2.6)$$

If the points $\gamma(s)$ and $\gamma(t)$ are "sufficiently" (infinitesimally) close, then the vector

$$\zeta_s^{\gamma}(t):=(t-s)\dot{\gamma}(s) \qquad (2.7)$$

is a "good" (of first order with respect to t-s) approximation to the displacement vector (2.1b). By definition it is called the *infinitesimal displacement vector*. Evidently, in the case of I-paths, due to proposition 2.2, the vector (2.7) coincides (globally) with the displacement vector.

From (2.5) and (2.6), we find the following representation of the deviation vector

$$h_{21}=I^{\eta_s}_{t'_s\longrightarrow t''_s}[\dot{\gamma}_s(r'_s)(r''_s-r'_s)+O((r''_s-r'_s)^2)], \qquad (2.8)$$

which for L-transports in local coordinates, as a consequence of (1.2), is equivalent to

$$h^i_{21}=\dot{\gamma}^i_s(r'_s)(r''_s-r'_s)+O(t''_s-t'_s)+O((r''_s-r'_s)^2). \qquad (2.9)$$

Here we see that within the quantities of first order with respect to $(t''_s-t'_s)$ and second order with respect to $(r''_s-r'_s)$ the vector

$$\zeta_{21}:=(r''_s-r'_s)\dot{\gamma}_s(r'_s), \qquad (2.10)$$

which, though beiing defined at another point, by its components is an approximation to the deviation vector (2.5). In this case the vector (2.10) is called the *infinitesimal deviation vector* [13].

3. DEVIATION EQUATIONS

Let the manifold M be endowed with an affine connection with local coefficients $\{\Gamma^i_{.jk}(x)\}$ and let ∇ denote it covariant differentiation defined by (cf. [3,4]). If $X,Y,Z\in\mathrm{Sec}(T(M),\pi,M)$, then the tensors (operators) of torsion T and curvature R are

$$T(X,Y):=\nabla_X Y-\nabla_Y X-[X,Y], \qquad (3.1)$$

$$R(X,Y)Z:=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z, \qquad (3.2)$$

where [X,Y] is the commutator of X and Y, and in any local basis $\{E_i\}$ we have $[E_i,E_j]=:C^k_{.ij}E_k$, so their components, respectively, are:

$$T^i_{.jk}=-2\Gamma^i_{.[jk]}-C^i_{.jk}, \qquad (3.3)$$

$$R^i_{.jkl}=-2\Gamma^i_{.j[k,l]}-2\Gamma^m_{.j[k}\Gamma^i_{.|m|l]}-\Gamma^i_{jm}C^m_{.kl}. \qquad (3.4)$$

Let us express $\nabla_x Z$ from (3.1) for Y=Z, substitute the obtained result into (3.2), and put in the thus found equality $X=\xi$, $Y=Z=U$ for $\xi,U\in\mathrm{Sec}(T(M),\pi,M)$. Thus using the skewsymmetry of T, R and the commutator on their first two arguments, we get the equality

$$\nabla^2_U\xi=R(U,\xi)U+\nabla_\xi(\nabla_U U)+\nabla_U(T(U,\xi))+\nabla_U[U,\xi]+\nabla_{[U,\xi]}U. \qquad (3.5)$$

In [14] this equality is called the "basic equation" as from it by imposing additional condition on the quantities involved in it the deviation equations used in the literature can be obtained (for geodesic as well as for nongeodesic paths) [14,7,8].

The physical meaning and interpretation of the equality (3.5) can be obtained as follows.

Let besides the connection in $(T(M),\pi,M)$ there be defined a transport along paths I and there be given the construction of paths, $\mathbb{R}$-intervals and maps between them appearing in definition 2.3 of the deviation vector (2.5), for which we suppose to have a C^2 dependence on $s\in J$.

Let us put U in (3.5) to be the tangent vector field to the path x and ξ to be the field of the deviation vector of x_2 with respect to x_1, i.e.

$$U_{x(s)}=\dot{x}(s),\ \ \xi_{x(s)}=h_{21}(s;x). \tag{3.6}$$

Then on $x(J)$

$$\nabla_U=\frac{D}{ds}\Big|_x \tag{3.7}$$

is the covariant differentiation along x and (3.5) takes the form

$$\left(\frac{D}{ds}\Big|_x\right)^2 h_{21}=R(U,h_{21})U+\nabla_{h_{21}}\left(\frac{D}{ds}\Big|_x U\right)+\frac{D}{ds}\Big|_x(T(U,h_{21}))+$$

$$+\frac{D}{ds}\Big|_x[U,h_{21}]+\nabla_{[U,h_{21}]}U. \tag{3.8}$$

The equality (3.8) is called the *generalized deviation equation*. In the local case, i.e. when h_{21} is an infinitesimal vector, which usually is identified with the infinitesimal deviation vector (2.10), this name was introduced in [15,8,14], and in the global case, i.e. for an arbitrary deviation vector h_{21}, in [5].

The physical interpretation of the generalized deviation equation (3.8) may be found, for example, in [5,8,9] and it is based on the physical interpretation of the deviation vector given in Sect. 2. Due to it h_{21}, $\nabla_U h_{21}$ and $\nabla^2_U h_{21}$ are interpreted, respectively, as relative coordinate, velocity and acceleration (or, more precisely, these are the deviation vector, the deviation velocity and the deviation acceleration, but for the moment this is not essential) of

the second observed particle with respect to the first one relatively to the observer. The quantities U and $\nabla_U U$ are interpreted, respectively, as the velocity and force per unit mass acting on the observer. As a consequence of this we can say that the generalized deviation equation (3.8) gives the relative acceleration $\nabla^2_U h_{21}$ between the observed particles as a function of the characteristics of the manifold M (R, T and $\Gamma^i_{.jk}$), the trajectory (the world line) of an observer (s, x, U and $\nabla_U U$) and the relative movement of the observed particles (h_{21} and $\nabla_U h_{21}$).

Example 3.1. Now, analogously to the investigations in [9], on the basis of the above general considerations, we shall derive the nonlocal (noninfinitesimal) deviation equation of the geodesics.

Let $y:\Lambda_2\longrightarrow M$, where Λ_2 is a neighborhood in $\mathbb{R}^2$, be a C^2 congruence of geodesics (with respect to the connection of M) paths. This means that the tangent vectors U and V, respectively, to the u-paths $y(\cdot,v)$, v=const and v-paths $y(u,\cdot)$, u=const, $(u,v)\in\Lambda_2$ which are geodesics, satisfy the equalities

$$\nabla_U U\Big|_{(u,v)}=f_v(u)U\Big|_{(u,v)},\ \ \nabla_V V\Big|_{(u,v)}=g_u(v)V\Big|_{(u,v)}. \tag{3.9}$$

Here the restriction $\Big|_{(u,v)}$ means that the corresponding quantities are taken at the point $y(u,v)$, $(u,v)\in\Lambda_2$ and the functions f_v and g_u depend only on the choice of the parameters u and v (for instance, if u is an affine parameter, then by definition $f_v(u)\equiv 0$).

We have to find the deviation equation of two arbitrary u-paths from the family $y(\cdot,v)$, v=const, $(u,v)\in\Lambda_2$. For this purpose, in the above general construction, we substitute: $y(\cdot,v_1)$ and $y(\cdot,v_2)$ for some fixed values v_1 and v_2 of the parameter v, respectively, for x_1 and x_2; $y(\cdot,v_1)$ for x (and consequently $\tau_1=\tau_2=id$); and $y(u,\cdot)$ for γ_s. As a concrete and "most natural" realization of the transport I we shall use the parallel transport defined by the connection of M.

As $y(u,\cdot)$ is a geodesic, we have

$$I^{y(u,\cdot)}_{v_0\longrightarrow v}(V|_{(u,v_0)})=\mu_u(v_0,v)V|_{(u,v)}$$

for some scalar function μ of u, v_0 and v, which due to (1.2) has the property $\mu_u(v_0,v_0)=1$. On the other hand, (cf. [2], proposition 4.1), the fact that $I^{y(u,\cdot)}$ is a parallel transport along $y(u,\cdot)$ leads to $\nabla_V|_{(u,v)}\circ I^{y(u,\cdot)}_{v_0\longrightarrow v}\equiv 0$. Combining these equalities with the second equation from (3.9), we get $\mu_u(v_0,v)=\exp(-\int_{v_0}^{v}g_u(w)dw)$. Due to this from (2.5) we find the deviation vector of $y(\cdot,v_2)$ with respect to $y(\cdot,v_1)$ at the point $y(u,v_1)$ as

$$h:=h(u,v_1,v_2):=\lambda\cdot V|_{(u,v_1)}, \tag{3.10}$$

$$\lambda:=\lambda_u(v_1,v_2):=\int_{v_1}^{v_2}\left(\exp(-\int_{v_1}^{v}g_u(w)dw)\right)dv=\frac{a_u(v_2)-a_u(v_1)}{\partial a_u(v_1)/\partial v_1}, \tag{3.11}$$

where $a_u(v):=C_1(u)\int_{v_0}^{v}\left(\exp(-\int_{v_0}^{t}g_u(w)dw)\right)dt+C_2(u)$, with $C_1\neq 0$, and C_2 being arbitrary functions, is any affine parameter of $y(u,\cdot)$.

The form of the deviation equation (3.8) in the considered case is defined by two additional conditions. First, on $y(\cdot,v_1)$ the first equation of (3.9) gives

$$\nabla_U U|_{(u,v_1)}=f_{v_1}(u)U|_{(u,v_1)}. \tag{3.12}$$

Second, as u and v are independent parameters of the C^2 congruence y, in local coordinates, we get $\partial^2y^i(u,v)/\partial u\partial v= =\partial^2y^i(u,v)/\partial v\partial u$, which on $y(\cdot,v_1)$ reduces to

$$[h,U]|_{(u,v_1)}=L_hU|_{(u,v_1)}=-\lambda' V|_{(u,v_1)}=-h\lambda'/\lambda, \tag{3.13}$$

where $\lambda':=\partial\lambda/\partial u$ and $L_hU:=[h,U]$ is the commutator of h and U (or the

Lie derivative of U with respect to h). Substituting (3.12) and (3.13) into (3.8) and using the notation $\lambda'':=\partial\lambda'/\partial u$ and the relationship $L_hU=\nabla_hU-\nabla_Uh-T(h,U)$, which is true for any vector fields h and U (see (3.1)), we find the geodesic deviation equation as

$$\left(\frac{D}{du}\Big|_{y(\cdot,v_1)}\right)^2 h|_{(u,v_1)}=R(U,h)U|_{(u,v_1)}+\Big[f_{v_1}(u)T(U,h)+(\nabla_UT)(U,h)+$$
$$+T(U,\frac{D}{du}\Big|_{y(\cdot,v_1)}h)\Big]\Big|_{(u,v_1)}+\lambda\cdot\left(\frac{D}{dv}\Big|_{y(u,\cdot)}(f_v(u)U)\right)\Big|_{(u,v_1)}+$$
$$+\frac{\lambda'}{\lambda}\cdot[2\frac{D}{du}\Big|_{y(\cdot,v_1)}h-2\frac{\lambda'}{\lambda}\cdot h+T(h,U)]\Big|_{(u,v_1)}+\frac{\lambda''}{\lambda}\cdot h|_{(u,v_1)}. \tag{3.14}$$

If the parameter v is affine, then (by definition) $g_u(v)=0$, so now (3.10), (3.11) and (3.14) take, respectively, the form:

$$h=(v_2-v_1)V|_{(u,v_1)},\quad \lambda=v_2-v_1, \tag{3.15}$$

$$\left(\frac{D}{du}\Big|_{y(\cdot,v_1)}\right)^2 h|_{(u,v_1)}=R(U,h)U|_{(u,v_1)}+\Big[f_{v_1}(u)T(U,h)+(\nabla_UT)(U,h)+$$
$$+T(U,\frac{D}{du}\Big|_{y(\cdot,v_1)}h)\Big]\Big|_{(u,v_1)}+(v_2-v_1)\cdot\left(\frac{D}{dv}\Big|_{y(u,\cdot)}(f_v(u)U)\right)\Big|_{(u,v_1)}. \tag{3.16}$$

If, besides, u is affine too, then (by definition) $f_v(u)=0$ and (3.16) reduces to the equation

$$\left(\frac{D}{du}\Big|_{y(\cdot,v_1)}\right)^2 h|_{(u,v_1)}=R(U,h)U|_{(u,v_1)}+(\nabla_UT)(U,h)|_{(u,v_1)}+$$
$$+T(U,\frac{D}{du}\Big|_{y(\cdot,v_1)}h)|_{(u,v)}. \tag{3.17}$$

Analogously one can get the deviation equation for the congruence $y:\Lambda_2\longrightarrow M$ in the case when only the v-paths $y(u,\cdot)$ are geodesics. Then, as there remains only the additional condition (3.13) the deviation vector is also given by (3.10)-(3.11). So

$$\left(\frac{D}{du}\Big|_{y(\cdot,v_1)}\right)^2 h|_{(u,v_1)}=R(U,h)U|_{(u,v_1)}+\frac{D}{du}\Big|_{y(\cdot,v_1)}[T(U,h)|_{(u,v_1)}+$$

$$+ \left(\frac{D}{du}\Big|_{y(\cdot,v)}\frac{D}{dv}\Big|_{y(u,\cdot)}U\right)\Big|_{(u,v_1)} + \frac{\lambda'}{\lambda}\cdot[2\frac{D}{du}\Big|_{y(\cdot,v_1)}h - 2\frac{\lambda'}{\lambda}\cdot h +$$

$$+ T(h,U)]\Big|_{(u,v_1)} + \frac{\lambda''}{\lambda}\cdot h\Big|_{(u,v_1)}, \tag{3.18}$$

If we impose also the condition (3.12), we see that (3.18) reduces to (3.14).

Example 3.2. In this example, based on the work [10], we shall show that the deviation equation (3.8) contains as its special case the equation of relative motion of two point particles. In this sense the deviation equation is a generalization of the second Newton's law of the dynamics.

Let the C^2 trajectory $x:J\longrightarrow M$ of the observer be given as a solution of the following initial-value problem:

$$\nabla_U U\big|_{x(s)}=F(s,x,U)\in T_{x(s)}(M),\ \ U:=\dot{x},\ \ s\in J, \tag{3.19a}$$

$$x(s_0)=x_0\in M,\ \ U_{x_0}:=\dot{x}(s_0)=U_0\in T_{x(s_0)}(M),\ \ s_0\in J, \tag{3.19b}$$

where x_0 and U_0 are fixed, and F is a continuous function of its arguments. Physically this means to consider a (point) observer that passes through the point x_0 with velocity U_0 and undergoes a force per unit mass F.

Let the family of C^2 paths $\{\gamma_s:\ s\in J\}$ be given as the unique solution of the following initial-value problem:

$$\nabla_{\gamma'}\gamma'\big|_{\gamma_s(r)}=F_s(r):=F_s(r,\gamma_s(r),\gamma_s'(r))\in T_{\gamma_s(r)}(M),\ \ s\in J, \tag{3.20a}$$

$$\gamma_{s_0}(r)=\chi(r)\in M,\ \ \gamma'_{s_0}(r)=\varphi(r)\in T_{\chi(r)}(M),\ \ s_0\in J, \tag{3.20b}$$

where γ' is the tangent vector field to the s-paths $\gamma_{\cdot}(r)$, $r=\text{const}\in J_s'$, $s\in J$ (i.e. $(\gamma'\big|_{\gamma_s(r)})^i:=\partial\gamma_s^i(r)/\partial s$), and F_s, χ and φ are continuous functions of their arguments. Physically F_s is interpreted as a force field (force per unit mass) acting in the two-dimensional region $\{\gamma_s(r),\ r\in J_s,\ s\in J\}$.

Let us remind (see Sect. 2) that by definition $\gamma_s(r_s'):=x_1(\tau_1(s)):=\eta_s(t_s')$, $\gamma_s(r_s''):=x_2(\tau_2(s))$ and $\eta_s(t_s''):=x(s)$ for some $r_s',r_s''\in J_s'$ к $t_s',t_s''\in J_s''$, $s\in J$.

Further in this example we suppose that the transport I is linear, i.e. we shall work with L-transports (see [2]).

The following purpose is to write, in the considered case, the deviation equation for the deviation vector h_{21} of x_2 with respect to x_1 relatively to x in the form of equation of motion that is "most close" to the second law of the Newton's mechanics. It "more clearly" shows the dependence of the relative (deviation) acceleration between the observed particles on the force fields F and F_s. (This intention comes from the above given physical interpretation of the deviation equation.)

To write certain formulae compactly, we shall generalize the operation of differentiation of vector fields along paths (see e.g. (3.7)). Let $p,q\geq 0$ be integers, $z_a:J\longrightarrow M$, $a=1,\dots,p+q$ be C^1 paths, $z:J\longrightarrow M\times\cdots\times M$ (p+q times) with $z(s):=(z_1(s),\dots,z_{p+q}(s))$, $s\in J$ and $T^p_{\ q}(z(s);M):=T_{z_1(s)}(M)\otimes\cdots\otimes T_{z_p(s)}(M)\otimes T^*_{z_{p+1}(s)}(M)\otimes\cdots\otimes T_{z_{p+q}(s)}(M)$.

For every $s\in J$, we define the map

$$\frac{D}{ds}:\ \mathrm{Sec}(\bigcup_{t\in J}T^p_{\ q}(z(t);M),\pi,z(J))\longrightarrow T^p_{\ q}(z(s);M),$$

where $\pi(A_{z(s)}):=z(s)$ for $A_{z(s)}\in T^p_{\ q}(z(s);M)$, in such a way as for $A\in\mathrm{Sec}(\bigcup_{t\in J}T^p_{\ q}(z(t);M),\pi,z(J))$, in local coordinates

$$\left(\frac{D}{ds}A\right)^{i_1\dots i_p}_{j_1\dots j_p}:=\frac{d}{ds}\left(A^{i_1\dots i_p}_{j_1\dots j_p}(z(s)\right) +$$

$$+\sum_{a=1}^{p}\Gamma^{i_a}_{\ \ ..kl}(z_a(s))A^{i_1\dots i_{a-1}k i_{a+1}\dots i_p}_{j_1\dots j_p}(z(s))\dot{z}_a^l(s) -$$

$$-\sum_{b=1}^{q}\Gamma^{k}_{.j_b l}(z_{p+b}(s))A^{i_1\dots i_p}_{j_1\dots j_{b-1}kj_{b+1}\dots j_q}(z(s))\dot{z}^{l}_{p+b}(s), \tag{3.21}$$

where $\dot{z}_a$ is the tangent vector field to z_a, $a=1,\dots,p+q$.

With the help of (3.21) it is easy to check that D/ds is a derivation of the (many-point) tensor algebra over z(J), i.e. this operator is linear, commutes with the contraction operator (defined now only on indices referring to dual spaces) and satisfies the relation $D/ds(A\otimes B)=(DA/ds)\otimes B+A\otimes(DB/ds)$.

If $p+q=1$, then from (3.21) follows $D/ds=d/ds|_{z_1}$, i.e. when acting on vector fields or 1-forms defined over $z_1(J)$ the above defined operator reduces to a covariant differentiation along z_1.

Let the L-transport along $\gamma:J\longrightarrow M$ from s to t, $s,t\in J$ in $(T(M),\pi,M)$ be defined by the matrix $\|H^{i}_{.j}(t,s;\gamma)\|$ through (1.3) and $\{E_i|_y\}$ be a basis in $T_y(M)$, $y\in M$. We put

$$H:=H^{i}_{.j}(t''_s,t'_s;\eta_s)E_i|_{\eta_s(t''_s)}\otimes E^j|_{\eta_s(t'_s)}\in T_{\eta_s(t''_s)}(M)\otimes T^{*}_{\eta_s(t'_s)}(M),$$

$$H^{-1}:=H^{i}_{.j}(t'_s,t''_s;\eta_s)E_i|_{\eta_s(t'_s)}\otimes E^j|_{\eta_s(t''_s)}\in T_{\eta_s(t'_s)}(M)\otimes T^{*}_{\eta_s(t''_s)}(M),$$

$$\Lambda(r):=H^{i}_{.j}(r'_s,r;\gamma_s)E_i|_{\gamma_s(r'_s)}\otimes E^j|_{\gamma_s(r)}\in T_{\gamma_s(r'_s)}(M)\otimes T^{*}_{\gamma_s(r)}(M).$$

For brevity with a point ($\cdot$) the contracted tensor product will be denoted, i.e. if $X\in T^{p}_{.q}|_y(M)$, $Y\in T^{p'}_{..q'}|_y(M)$, $p',q\geq 0$ and $p,q'\geq 1$, then $X\cdot Y:=C^{p}_{q+1}(X\otimes Y)$, C^{p}_{q} being the contraction operator on the p-th super- and q-th subscript.

Using the above notions, we can write the deviation vector (2.5) as

$$h=H\cdot\int_{r'_s}^{r''_s}\Lambda(u)\cdot\dot{\gamma}_s(u)du. \tag{3.22}$$

Hance, we find:

$$\left(\frac{D}{ds}\Big|_x\right)^2 h=\left(\frac{D}{ds}\right)^2 h=\frac{D^2H}{ds^2}\cdot H^{-1}\cdot h\ +\ 2\frac{DH}{ds}\cdot\int_{r'_s}^{r''_s}\left(\frac{D\Lambda(u)}{ds}\cdot\dot{\gamma}_s(u)+\Lambda(u)\cdot\frac{D\dot{\gamma}_s(u)}{ds}\right)du+$$

$$+H\cdot\int_{r'_s}^{r''_s}\left(\frac{D^2\Lambda(u)}{ds^2}\cdot\dot{\gamma}_s(u)+2\frac{D\Lambda(u)}{ds}\cdot\frac{D\dot{\gamma}_s(u)}{ds}+\Lambda(u)\cdot\frac{D^2\dot{\gamma}_s(u)}{ds}\right)du\ +\rho, \tag{3.23}$$

where

$$\rho:=H\cdot\left\{\frac{D}{ds}\left[\frac{dr''_s}{ds}\Lambda(r''_s)\cdot\dot{\gamma}_s(r''_s)-\frac{dr'_s}{ds}\dot{\gamma}_s(r'_s)\right]+\frac{dr''_s}{ds}\left[\frac{D}{ds}(\Lambda(u)\cdot\dot{\gamma}_s(u))\right]\Big|_{u=r''_s}-\right.$$

$$\left.-\frac{dr'_s}{ds}\left[\frac{D}{ds}(\Lambda(u)\cdot\gamma_s(u))\right]\Big|_{u=r'_s}\right\}+2\frac{DH}{ds}\cdot\left[\frac{dr''_s}{ds}\Lambda(r''_s)\cdot\dot{\gamma}_s(r''_s)-\frac{dr'_s}{ds}\dot{\gamma}_s(r'_s)\right]$$

arises from the differentiation with respect to s of the boundaries of integration r'_s and r''_s. Let us note that usually [13-16] the statement of the problem is such that r'_s and r''_s do not depend on s, therefore $\rho=0$.

By its essence the equation (3.23) gives an answer to the problem stated above. In particular, if we write the term D^2H/ds^2 in detail, we shall see the "major" dependence of the relative acceleration D^2h/ds^2 on the force F acting on the observer. But more essential is the dependence on the force field F_s and to write it we shall transform the derivative $D^2\dot{\gamma}/ds^2$ in (3.23) as follows.

Taking into account the evident equality $\nabla_{\gamma'}\dot{\gamma}=D\dot{\gamma}/ds$, from the basic equation (3.5) for $U=\gamma'$ and $\xi=\dot{\gamma}$, we get

$$D^2\dot{\gamma}/ds^2=\nabla^2_{\gamma'}\dot{\gamma}=R(\gamma',\dot{\gamma})\gamma'+\nabla_{\dot{\gamma}}(\nabla_{\gamma'}\gamma')+\nabla_{\gamma'}(T(\gamma',\dot{\gamma}))+\nabla_{\gamma'}[\gamma',\dot{\gamma}]+\nabla_{[\gamma',\dot{\gamma}]}\gamma'.$$

In this equality the last two terms are zeros because of $[\gamma',\dot{\gamma}]=0$. (In fact, the i-th component of this commutator at $\gamma_s(r)$ is

$$([\gamma',\dot{\gamma}]|_{\gamma_s(r)})^i=(\gamma'(\dot{\gamma}^i_s)-\dot{\gamma}(\gamma'^i))|_{\gamma_s(r)}=\partial^2\gamma^i_s(r)/\partial r\partial s-\partial^2\gamma^i_s(r)/\partial s\partial r\equiv 0,$$

where we suppose a C^2 dependence of $\gamma_s(r)$ on s and r.) So, using this, $(\nabla_{\gamma'}\gamma')=D\gamma'/ds|_{\gamma_s(r)}=F_s(r)$ (see (3.20a)) and $\nabla_{\dot{\gamma}}F_s(r)=$

$=DF_s(r)/dr$, we find

$$\frac{D^2\dot\gamma}{ds^2}\Big|_{\gamma_s(r)}=R(\gamma',\dot\gamma)\gamma'\big|_{\gamma_s(r)}+\frac{D}{dr}F_s(r)+T(F_s,\dot\gamma_s)\big|_{\gamma_s(r)}+$$
$$+\left(\frac{DT}{ds}(\gamma',\dot\gamma)\right)\Big|_{\gamma_s(r)}+T(\gamma',\frac{D\dot\gamma}{ds})\big|_{\gamma_s(r)}$$

and consequently (3.23) takes the form

$$\left(\frac{D}{ds}\Big|_x\right)^2 h=\frac{D^2H}{ds^2}\cdot H^{-1}\cdot h + 2\frac{DH}{ds}\cdot\int_{r'_s}^{r''_s}\left(\frac{D\Lambda(u)}{ds}\cdot\dot\gamma_s(u) + \Lambda(u)\cdot\frac{D\dot\gamma_s(u)}{ds}\right)du +$$
$$+ H\cdot\int_{r'_s}^{r''_s}\Big\{\frac{D^2\Lambda(u)}{ds^2}\cdot\dot\gamma_s(u)+2\frac{D\Lambda(u)}{ds}\cdot\frac{D\dot\gamma_s(u)}{ds}+\Lambda(u)\cdot[(R(\gamma',\dot\gamma)\gamma'+\frac{DT}{ds}(\gamma',\dot\gamma) +$$
$$+T(\gamma',\frac{D\dot\gamma}{ds}))\Big|_{\gamma_s(u)}+\frac{D}{du}F_s(u)+T(F_s,\dot\gamma_s)\Big|_{\gamma_s(u)}]\Big\}du + \rho. \tag{3.24}$$

This equation is the answer of the problem stated in this example problem. It represents the deviation equation in the form of an equation of motion in the considered case.

From a dynamical point of view the most important terms in (3.24) are those containing explicitly the force $F_s(r)$, i.e.

$$H\cdot\int_{r'_s}^{r''_s}\Lambda(u)\cdot[\frac{D}{du}F_s(u)+T(F_s,\dot\gamma_s)\Big|_{\gamma_s(u)}]du=L^{\eta_s}_{t'_s\to t''_s}(L^{\gamma_s}_{r''_s\to r'_s}F_s(r''_s)-$$
$$-F_s(r'_s))+H\cdot\int_{r'_s}^{r''_s}[\Lambda(u)\cdot T(F_s,\dot\gamma_s)\Big|_{\gamma_s(u)}-\frac{D\Lambda(u)}{ds}\cdot F_s(u)]du, \tag{3.25}$$

where we have done an evident integration by parts of the integrand $\Lambda(u)\cdot\frac{D}{du}F_s(u)$. Let us note that the first term in (3.25), which is written explicitly by a transport L is simply the difference defined by means of L at the point x(s) of the forces $F_s(r'')$ and $F_s(r')$ acting on the observed particles.

At the end, we are going to consider two important special cases of (3.24).

First, in the Euclidean case (3.24) reduces to the second law of the Newtonian mechanics. In fact, in this case we can put $M=\mathbb{R}^n$, $dr'_s/ds=dr''_s/ds=0$ and $H=\Lambda(u)=\delta$, where δ is the unit tensor with components the Kronecker deltas δ^i_j (see (1.2′)), and if we use a basis in which $\Gamma^i_{.jk}=0$, then (3.24) becomes

$$\frac{d^2h}{ds^2}=\left(\frac{D}{ds}\Big|_x\right)^2 h=\int_{r'_s}^{r''_s}\frac{d}{du}F_s(u)du=F_s(r'')-F_s(r'). \tag{3.26}$$

Second, in the infinitesimal case (3.24) reduces to the equation known, e.g. from [16], for the relative motion of two "sufficiently near" point particles.

For brevity and simplicity we shall suppose $dr'_s/ds=dr''_s/ds=0$. As a consequence of (1.2), we have

$$H=\delta+O(t''_s-t'_s),\ H^{-1}=\delta+O(t''_s-t'_s),\ \Lambda(r)=\delta+O(r-r'_s).$$

Using these equalities, the formula $\int_{r'}^{r''}f(u)du=f(r')(r''-r')+$ $+O((r''-r')^2)$ for any C^1 function $f:[r',r'']\longrightarrow\mathbb{R}$, and the infinitesimal deviation vector $\zeta:=\zeta(s):=\dot\gamma_s(r'_s)(r''_s-r'_s)$ (see (2.10)) from (3.24), we obtain

$$\frac{D^2\zeta}{ds^2}=R(\gamma',\zeta)\gamma'\Big|_{\gamma_s(r')}+\frac{DF_s(r)}{dr}\Big|_{r=r'}(r''_s-r'_s)+T(F_s,\zeta)\Big|_{\gamma_s(r'_s)}+$$
$$+\frac{DT}{ds}(\gamma',\zeta)\Big|_{\gamma_s(r'_s)}+T(\gamma',\frac{D\zeta}{ds})\Big|_{\gamma_s(r'_s)}+O(t''_s-t'_s)+O((r''_s-r'_s)^2). \tag{3.27}$$

If here we neglect the terms $O(t''_s-t'_s)$ and $O((r''_s-r'_s)^2)$ and put T=0, we get the equation derived in [16], ch. 8, sect. 1 for relative motion of two "nearly" moving point particles.

4. CONCLUDING REMARKS

The displacement vector introduced in Sect. 2 is a direct generalization of the difference of two Euclidean (radius-)vectors. To show this, we consider the (pseudo-)Euclidean transport generated by Cartesian coordinates in $M=\mathbb{R}^n$ or $M=E^n$, which is insignificant now (see [2], definition 3.1), i.e. as a concrete realization of I we shall use, the parallel transport in $\mathbb{R}^n$ will be used. Then in any basis, we have $(I^{\gamma}_{s\to t}u)^i=u^i$ for any path $\gamma: J\to\mathbb{R}^n$, every $u\in T_{\gamma(s)}(\mathbb{R}^n)$ and arbitrary $s,t\in J$. Hence in this case (1.1b) gives

$$(d^{\gamma}_{s}(t))^i := \int_s^t \dot\gamma^i(u)du = \gamma^i(t)-\gamma^i(s),\quad s,t\in J, \tag{4.1}$$

which proves the above statement.

As it is known to the author, the equality (2.5) is published for the first time in [7] (see therein equation (1) in which a slightly different notation is used). Its full derivation in local coordinates, with the usage of Lie derivatives, is presented in [8] (see therein section 1 and the appendix). More precisely, in [7] the equation (2.8) is given for an arbitrary path x and vector h_{21} (with the usage of $[U,h_{21}]=L_U h_{21}$ and $F:=\nabla_U U$), the proof of which has been published later in [8]. As a consequence of the arbitrariness of x and h_{21}, for which in [8] (2.8) is proved, in this case the qualities (2.8) and (2.5) are equivalent.

Independently, the equality (2.5) is found in [14] from where the presented here its derivation is taken.

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