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THE BACKGROUND-FIELD METHOD
AND NONINVARIANT RENORMALIZATION¹

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1 Introduction

To obtain meaningful results in quantum field theory, one has to remove ultraviolet and infrared divergencies. This goal can be achieved by a renormalization procedure, that is, a proper subtraction of singularities. In a general case, ultraviolet renormalization involves three steps:

1. Regularization of Feynman amplitudes by introducing some parameter which converts divergencies into singularities as this parameter tends to a particular limit value (say, zero or infinity). There should exist a smooth limit of taking the regularization off: any amplitudes that were finite without it should not be distorted.
2. Renormalization of the parameters of the theory (coupling constants, masses, etc) in order to absorb the singularities into a redefinition of these parameters, which is achieved by introducing some local counterterms.
3. The choice of a renormalization scheme which fixes the finite arbitrariness left after the regularization is taken off.

In gauge theories, one has to be very careful because the renormalization procedure may violate the gauge invariance on the quantum level, thus destroying the renormalizability of the theory. Therefore, when dealing with gauge theories, one is bound to apply an invariant renormalization. By this we mean a renormalization that preserves all the relevant symmetries of the model on the quantum level, that is, preserves all the Ward identities [1] for the renormalized Green functions.

On the one hand, this can be achieved by applying an invariant regularization first (respecting the symmetries in the regularized theory) and then using, for instance, the minimal subtraction scheme [2, 3] to fix the finite arbitrariness. On the other hand, when the regularization is noninvariant or no explicit regularization is introduced at all, there is no automatic preservation of the symmetries. Then one has to take care of that directly. For example, one can use some noninvariant regularization and consecutively choose certain finite counterterms to restore the invariance (the symmetry of the renormalized Green functions) order by order in perturbation theory [4].

When treating theories with nonlinear realization of a symmetry, like two-dimensional σ models or quantum gravity, one faces extraordinary complexity of perturbative calculations. To simplify them, one usually applies the so-called background-field method [5] which allows one to handle all the calculations in a strictly covariant way. This method was successfully applied to multiloop calculations in various gauge and scalar models, being combined with the minimal subtraction scheme based on some invariant regularization.

The most popular and handy regularization used in these calculations was the dimensional regularization [6]. It has been proved to be an invariant regularization, preserving all the symmetries of the classical action that do not depend explicitly on the space-time dimension [3, 7]. Moreover, any formal manipulations with the dimensionally regularized integrals are allowed. However, an obvious drawback of this regularization is the violation of the axial invariance and of supersymmetry. That is why there are numerous attempts to find some other regularization equally convenient and efficient. Among such schemes the recently proposed differential renormalization [8] is discussed.

In the present paper we investigate the compatibility of the background-field formalism with various regularizations and renormalization prescriptions. Our conclusion is that the background-field method necessarily requires one to use an invariant renormalization procedure. As the invariance does not hold, the method gives incorrect results. We demonstrate this by an example of the two-dimensional $O(n)$ σ -model, comparing the dimensional regularization, the cut-off regularization within the minimal subtraction scheme, and the differential renormalization method.

2 Invariant Renormalization in the Background-Field Formalism

To preserve all the symmetries on the quantum level, one has to apply an invariant renormalization procedure. The simplest way to construct such a procedure is to use an invariant regularization and the minimal subtraction scheme. An invariant regularization should permit any formal manipulations with the functional integral that are needed to ensure the Ward identities. Let us list the properties of an invariant regularization [9]:

1. translational invariance $\int d^D x f(x+y) = \int d^D x f(x)$;
2. unambiguity of the order of integrations
 $\int d^D x \int d^D y f(x,y) = \int d^D y \int d^D x f(x,y)$;
3. linearity $\int d^D x \sum_j a_j f_j(x) = \sum_j a_j \int d^D x f_j(x)$;
4. Lorentz covariance;
5. integration by parts, neglecting the surface terms;
6. possibility of canceling the numerator with the denominator;
7. commutativity of the space-time or momentum integration and differentiation with respect to an external parameter.

The only known regularization obeying all these requirements is the dimensional regularization. Combined with the minimal subtraction scheme, it makes an invariant renormalization for which the action principle is valid [7]. Using the dimensional renormalization in conjunction with the background-field method leads to covariant results of multiloop calculations in any theory unless its symmetry properties depend on the particular number of dimensions.

On the other hand, if one applies some noninvariant regularization, the initial symmetry may be violated, and one has to explore the possibility of using such a regularization in the framework of the background-field method.

In multiloop calculations by this method an invariant regularization automatically provides some implicit correlations between different diagrams, which may be essential as the formal background-field expansion of the action is performed. Any violation of these fine correlations by a noninvariant regularization or by an improper choice of finite counterterms may result in wrong answers, although covariant in form.

3 Two-dimensional nonlinear $O(n)$ σ model

Let us consider the two-dimensional σ model of the $O(n)$ principal chiral field (n field) and calculate the two-loop β function, using various approaches. The model is described by the lagrangian

$$\mathcal{L} = \frac{1}{2h} (\partial_\mu \mathbf{n})^2, \quad \mathbf{n}^2 = 1. \quad (1)$$

It can be treated as a special case of the generic bosonic σ model

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi^j) g_{jk}(\phi) \partial_\mu \phi^k, \quad (j, k = 1, 2, \dots, n-1), \quad (2)$$

where the metric is of the form

$$g_{jk}(\phi) = \delta_{jk} + \frac{h \phi_j \phi_k}{1 - h \phi^2}. \quad (3)$$

The background-field expansion of the action can be done in a strictly covariant fashion [10].

To separate the ultraviolet and infrared divergencies, we add an auxiliary mass term to the initial lagrangian (2)

$$\mathcal{L}_m = \frac{1}{2} m^2 \phi^j g_{jk}(\phi) \phi^k. \quad (4)$$

This additional term serves only for eliminating infrared divergencies, naively present in any two-dimensional theory with massless scalars. After the calculation of the ultraviolet logarithms, one should set $m^2=0$.

The σ model (2) with the particular choice of the metric (3) becomes renormalizable. All the covariant structures that may appear as counter-terms are reducible to the metric, so that the only thing that happens is a renormalization of the kinetic term. By rescaling the fields the renormalization can be absorbed into the charge. The invariant charge $\bar{h}=Z^{-1}h$ is defined through the field renormalization constant Z . To calculate Z within the background-field method, one has to consider the one-particle-irreducible diagrams with two external lines of the background field, and quantum fields inside the loops. Up to two loops the relevant diagrams [11] are shown in fig. 1. Their contributions to Z are obtained by normalizing to the tree term $\left[-\frac{1}{2} (\partial_\mu \phi^j) g_{jk} \partial_\mu \phi^k\right]$.

The Riemann and Ricci tensors are the functionals of the background field. In our model with the metric given by eq. (3) they are evaluated to

$$R_{abjk} = h (g_{aj} g_{bk} - g_{ak} g_{bj}), \quad R_{jk} = (n-2) h g_{jk}.$$

this way are shown in fig. 3.

$$\begin{aligned}
& \frac{1}{2} R_{aj} R^a{}_k (\partial_\mu \phi^j) (\partial_\mu \phi^k) (-\mathcal{K}\mathcal{R}' \text{ fig. 1a}) \quad \text{(a)} \\
& -\frac{1}{2} R_{jabk} R^{ab} (\partial_\mu \phi^j) (\partial_\mu \phi^k) (-\mathcal{K}\mathcal{R}' \text{ fig. 1a}) \quad \text{(b)} \\
& \frac{1}{2} m^2 R_{jabk} R^{ab} (\partial_\mu \phi^j) (\partial_\mu \phi^k) (-\mathcal{K}\mathcal{R}' \text{ fig. 2}) \quad \text{(c)}
\end{aligned}$$

Figure 3: The diagrams emerging from the background-field expansion of the one-loop counterterms.

One can find the β function by requiring independence of the invariant charge on the normalization point. This leads to the following expression for the β function through the finite wave-function renormalization constant:

$$\beta(h) = h \left(\mu^2 \frac{\partial Z}{\partial \mu^2} \right) \left[\left(1 - h \frac{\partial}{\partial h} \right) Z \right]^{-1}. \quad (5)$$

Using the dimensional regularization and the minimal subtraction scheme to calculate the diagrams presented above, one obtains the following well-known expression for the two-loop β function of the \mathbf{n} -field model (1) [12, 11]:

$$\beta_{\text{dim}} = -(n-2) \frac{h^2}{4\pi} \left(1 + 2 \frac{h}{4\pi} \right). \quad (6)$$

As it has already been mentioned, the dimensional regularization and the minimal subtraction scheme provide us with an invariant renormalization procedure within the background-field method. Hence, the obtained expression for the β function is correct, and we can use it as a reference expression to compare with other approaches. Owing to the presence of just one coupling constant in the model, the β function should be renormalization-scheme independent up to two loops and should coincide with eq. (6).

To check the validity of the background-field method in conjunction with other regularizations and renormalization prescriptions, let us consider the calculation of the β function within two schemes: the cut-off regularization and the differential renormalization.

3.1 The Cut-Off Regularization

We start with the regularization that uses a cut-off in the momentum space. All the integrals over the radial variable in the Euclidean space are cut at an upper limit Λ . Strictly speaking, this is not a very promising regularization, since it explicitly breaks the Lorentz, as well as gauge, invariance. However, we use it here to realize what may happen when a noninvariant regularization is applied.

We are going to use the minimal subtraction scheme which respects the invariance properties of the applied intermediate regularization, keeps them intact, as they are.

When the regularization parameter has the dimension of a mass, the minimal subtraction procedure can be defined [13] so as just to convert the logarithms of the (infinite) cut-off Λ into the logarithms of a finite renormalization point μ which appears in the theory after renormalizations:

$$\mathcal{K} \ln^n(\Lambda^2) = \ln^n(\Lambda^2) - \ln^n(\mu^2), \quad \mathcal{K} \Lambda^n = \Lambda^n, \quad (7)$$

so that

$$\mathcal{R} \ln^n(\Lambda^2) = (1 - \mathcal{K}) \ln^n(\Lambda^2) = \ln^n(\mu^2), \quad \mathcal{R} \Lambda^n = 0. \quad (8)$$

In case of overlapping divergencies, which generate powers of the logarithms, one ought to perform the standard renormalization procedure prior to the subtractions. However, if only the final renormalized answers are of interest, one can simply drop all the contributions of the minimally subtracted counterterms (7), since they will be annihilated by $(1 - \mathcal{K})$, eq. (8), irrespective of any powers of the logarithms from the residual graphs with contracted subgraphs. The same will happen to all the diagrams of fig. 3, generated by re-expanding the counterterms.

Thus, it is sufficient to calculate the regularized diagrams of figs. 1 and 2, up to Λ -power corrections and ultraviolet-finite two-loop contributions, and then to replace Λ^2 by μ^2 and m^2 in the one-loop diagram of

fig. 1a by $m^2 Z_{m^2}$, including the correction from fig. 2. The contributions of individual diagrams are

$$\begin{aligned}
Z(\text{fig. 1a}) &= -(n-2) \hbar / (4\pi) \ln(\mu^2/m^2), \\
Z(\text{fig. 1b}) &= \frac{1}{3} (n-2) (n+1) \hbar^2 / (4\pi)^2 \ln^2(\mu^2/m^2), \\
Z(\text{fig. 1c}) &= -\frac{1}{3} (n-2)^2 \hbar^2 / (4\pi)^2 [\ln^2(\mu^2/m^2) - \ln(\mu^2/m^2)], \\
Z(\text{fig. 1d}) &= -\frac{2}{3} (n-2) \hbar^2 / (4\pi)^2 \ln^2(\mu^2/m^2), \\
Z(\text{fig. 1e}) &= -\frac{1}{3} (n-2) \hbar^2 / (4\pi)^2 \ln^2(\mu^2/m^2), \\
Z_{m^2}(\text{fig. 2}) &= -\frac{1}{3} (n-2) \hbar / (4\pi) \ln(\mu^2/m^2).
\end{aligned}$$

The charge-renormalization constant proves then to be

$$Z_{em} = 1 - (n-2) \frac{\hbar}{4\pi} \ln \frac{\mu^2}{m^2} + 0 \cdot \hbar^2, \quad (9)$$

so that eq. (5) gives the β function

$$\beta_{em}(\hbar) = -(n-2) \frac{\hbar^2}{4\pi} (1 + 0 \cdot \hbar). \quad (10)$$

The difference between this result and that obtained in dimensional renormalization (6) is a direct manifestation of the noninvariance of the cut-off regularization, which violates the translational invariance. However, one needs to explain the reason for the failure to reproduce the correct β function in the present case. Although the cut-off regularization is noninvariant, still it has been successfully used to perform multiloop calculations in scalar field theories and in the quantum electrodynamics both within the background-field method and by the conventional diagram technique.

The point is that those theories were renormalizable in the ordinary sense, that is, they had a finite number of types of divergent diagrams. In contrast, the \mathbf{n} -field model is renormalizable only in the generalized sense. The total number of divergent structures here (with various external lines) is infinite, but they are related to each other by general covariance of the renormalized theory (in case of an invariant renormalization). So the number of independent structures remains finite. Expanding the lagrangian, we get an infinite number of terms; however, the renormalization constants are not arbitrary but mutually related. Although the background-field method

formally preserves the covariance of the model, the use of a noninvariant renormalization would break the intrinsic connection between various diagrams (and between their renormalization constants), thus leading to wrong results.

Therefore, we conclude that in generalized renormalizable (as well as nonrenormalizable) theories it is not allowed to use the cut-off regularization with the minimal subtractions in the framework of the background-field method.

We now want in the same way to check the invariance properties of the differential renormalization method.

3.2 Differential Renormalization

The idea of the differential renormalization traces back to the foundations of the renormalization procedure [14] as a redefinition of the product of distributions at a singular point. The method suggests to work in the coordinate space, where the free Green functions are well defined, although their product at coinciding points suffers from ultraviolet divergencies. The divergencies manifest themselves as singular functions which have no well-defined Fourier transform. The recipe of the differential renormalization [8] consists in rewriting a singular product in the form of a differential operator applied to a nonsingular expression:

$$f(x_j, \dots, x_k) = D(\square_{x_j}^{\sigma_j}, \dots, \square_{x_k}^{\sigma_k}) g(x_j, \dots, x_k), \quad (11)$$

Eq. (11) should be understood in the sense of distributions, that is, in the sense of integration with a test function. Then one ignores any surface terms on rearranging the derivatives via integration by parts. The nonsingular function $g(x_j, \dots, x_k)$ is obtained by solving a differential equation, and hence, involves an obvious arbitrariness. The latter can be identified with the choice of a renormalization point and a renormalization scheme. In this respect the differential renormalization does not differ from any other renormalization prescription.

In the absence of a primary regularization this prescription might preserve all the needed invariances and, what is important for applications, seems to renormalize ultraviolet singularities in the integer dimension. On the other hand, the absence of any intermediate regularization prevents one from using the standard scheme: invariant regularization + minimal sub-

tractions. Therefore, to verify the invariance properties of the differential renormalization, one has to deal with renormalized amplitudes directly.

Two-loop calculations of the renormalization constant in the two-dimensional σ model in the framework of the differential renormalization have been performed in ref. [15]. The authors have used the concept of the infrared $\tilde{\mathcal{R}}$ operation to handle the infrared divergencies. In the present case the infrared-renormalized free propagator in the co-ordinate representation has the form

$$\tilde{\mathcal{R}} \Delta_0(x) = -\frac{1}{4\pi} \ln(x^2 N^2), \quad (12)$$

where N^2 is an infrared renormalization scale.

An important role in the calculations plays the tadpole diagram (fig. 1a). In four dimensions, diagrams of this type diverge quadratically and can be consistently renormalized to zero, as it was originally done in the method of the differential renormalization [8]. However, in two dimensions the leading one-loop contribution to the β function comes from this very diagram. Hence, the tadpole should be different from zero in any renormalization. This means that we have to define the two-dimensional tadpole diagram in a self-consistent way in addition to the recipe of the differential renormalization. Such an extension has been discussed in detail in ref. [15], where the following expression for the massless tadpole has been suggested:

$$\mathcal{R}^* \Delta_0(0) = \frac{1}{4\pi} \ln \frac{M^2}{N^2}. \quad (13)$$

The parameter M^2 is an ultraviolet scale, and \mathcal{R}^* denotes the complete infrared and ultraviolet renormalization.

According to this modification of the differential renormalization rules, the expression for the β function has been found to be

$$\beta_{\text{diff}} = -(n-2) \frac{h^2}{4\pi} + 0 \cdot h^3. \quad (14)$$

Thus, the definition of the tadpole via eq. (13) in the massless case gives the correct expression for the *one-loop* β function, but fails in two loops.

Therefore, we would like to circumvent possible ambiguities of combining the infrared $\tilde{\mathcal{R}}$ operation with the differential renormalization. We

are going to apply the method to the massive model in which no infrared difficulties ever appear.

Introducing the mass term according to eq. (4), we obtain the free propagator of the form

$$\Delta_m(x) = \frac{1}{2\pi} K_0(m|x|). \quad (15)$$

where K_0 is the MacDonald function, obeying the equation

$$(\partial^2 - m^2) K_0(m|x|) = -2\pi \delta_{(2)}(x). \quad (16)$$

Bearing in mind the known expansion of the MacDonald function

$$K_0(x) = -\ln\left(\frac{x}{2}\right) \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{x}{2}\right)^{2k} + \sum_{k=0}^{\infty} \frac{\psi(k+1)}{(k!)^2} \left(\frac{x}{2}\right)^{2k},$$

we come to the following natural generalization of eq. (13) to the massive case:

$$\mathcal{R}\left[\delta(x) \Delta_m(x)\right] = \delta(x) \frac{1}{4\pi} \ln \frac{M^2}{m^2}. \quad (17)$$

In due course of the calculation we shall also need to define the product of two tadpoles. In the spirit of the consistent \mathcal{R} operation, the squared tadpole (fig. 1b) should be defined as the square of the renormalized value (17) for fig. 1a, that is,

$$\mathcal{R}\left[\delta(x) \Delta_m^2(x)\right] = \delta(x) \left(\frac{1}{4\pi} \ln \frac{M^2}{m^2}\right)^2. \quad (18)$$

Now we are in a position to complete the calculation of the β function. We present it in more detail for the diagrams of fig. 1c and fig. 1d. The simple tadpole subgraph of fig. 1c has already been defined via eq. (17). Consider another subgraph, with the numerator,

$$-\int d^2y \left[\frac{\partial}{\partial y_\nu} \Delta_m(x-y) \right]^2. \quad (19)$$

Integrating by parts, ignoring the surface term, and then using eq. (16), we get

$$\int d^2 y \left[m^2 \Delta_m^2(x-y) - \delta(x-y) \Delta_m(x-y) \right].$$

The integral of the first term is finite and known (the normalization integral for K_0). On the other hand, the second term is just reduced to the basic tadpole (17). Thus, the result for eq. (19) is $1/(4\pi) [1 - \ln(M^2/m^2)]$.

Generally speaking, the systematic differential renormalization to all orders [16] allows for introducing different ultraviolet renormalization scales in different diagrams (all the scales varying proportionally to each other under renormalization-group transformations). The ratio of these parameters can then be specially chosen [8] to satisfy the Ward identities. Let us denote the scale that appears in the tadpole with the numerator by M_1 .

Proceed now to fig. 1d. Its contribution to the effective action is

$$\frac{1}{3} \int d^2 x \int d^2 y R^{ab}_{c}{}_{d}(x) R_{abck}(y) [\partial_\mu \phi^c(x)] [\partial_\nu \phi^k(y)] \Delta_m^2(x-y) \partial_\mu \partial_\nu \Delta_m(x-y).$$

Picking out the trace and traceless parts according to ref. [8], we get

$$\begin{aligned} & \frac{1}{3} \int d^2 x \int d^2 y R^{ab}_{c}{}_{d}(x) R_{abck}(y) [\partial_\mu \phi^c(x)] [\partial_\nu \phi^k(y)] \Delta_m^2(x-y) \times \\ & \left[(\partial_\mu \partial_\nu - \frac{1}{2} \delta_{\mu\nu} \partial^2) \Delta_m(x-y) + \frac{1}{2} m^2 \delta_{\mu\nu} \Delta_m(x-y) + \frac{1}{2} \delta_{\mu\nu} (\partial^2 - m^2) \Delta_m(x-y) \right]. \end{aligned}$$

The first term, which is traceless, is finite and does not generate any ultraviolet scale; one can easily establish this fact in the momentum representation. The second term vanishes as $m^2 \rightarrow 0$. Thus, we are left with the last term. Via eq. (16) it is reduced to eq. (18), that is, gives only the square of the logarithm. However, again the renormalization scale M_2 in the new diagram may differ from M .

Below we present the contributions of all the diagrams to the renormalization constants:

$$\begin{aligned} Z(\text{fig. 1a}) &= -(n-2) h/(4\pi) \ln(M^2/m^2), \\ Z(\text{fig. 1b}) &= \frac{2}{3} (n-2) (n+1) h^2/(4\pi)^2 \ln^2(M^2/m^2), \\ Z(\text{fig. 1c}) &= -\frac{1}{3} (n-2)^2 h^2/(4\pi)^2 \ln(M_1^2/m^2) [\ln(M_1^2/m^2) - 1], \\ Z(\text{fig. 1d}) &= -\frac{2}{3} (n-2) h^2/(4\pi)^2 \ln^2(M_2^2/m^2), \\ Z(\text{fig. 1e}) &= -\frac{1}{3} (n-2) h^2/(4\pi)^2 \ln^2(M_3^2/m^2), \\ Z_{m^2}(\text{fig. 2}) &= -\frac{1}{3} (n-2) h/(4\pi) \ln(M^2/m^2). \end{aligned}$$

This gives the β function

$$\beta_{\text{diff}} = -(n-2) \frac{h^2}{4\pi} \left\{ 1 + \frac{h}{4\pi} \left[\frac{n-2}{3} \ln \frac{M_1^2}{M^2} + \frac{4}{3} \ln \frac{M_2^2}{M^2} + \frac{2}{3} \ln \frac{M_3^2}{M^2} \right] \right\}. \quad (20)$$

We see that the result explicitly depends on the ratio of the renormalization scale parameters in different diagrams. Such a dependence on the details of the renormalization prescription is beyond the usual scheme arbitrariness. It would never occur to two loops in the conventional perturbation theory for ordinary renormalizable one-charge models. There the arbitrariness would be completely absorbed into a finite number of counterterms which are of the operator types present in the tree lagrangian. Hence, we should try to fix the parameters of the differential renormalization by imposing some additional requirements. In the quantum electrodynamics the gauge Ward identities could be used to this end [8]. For the σ model in the background-field formalism the situation is not so clear.

The parameter M_1 that appears in the one-loop tadpole subgraph of fig. 1c with the numerator can be fixed as follows. In the momentum representation we can easily see that the sum of this diagram and the simple tadpole (fig. 1a) is just an ultraviolet-finite integral which equals $1/(4\pi)$. The value will be correctly reproduced by the differential renormalization if we choose the same scale for both tadpole graphs: $M_1=M$. Thus, for these diagrams the renormalization seems to be automatically invariant.

Let us point out that this simple check is by no means trivial. For example, the straightforward Feynman regularization of the quantum-field propagator in the momentum space $1/(p^2+m^2) \rightarrow 1/(p^2+m^2) - 1/(p^2+M^2)$ would not stand the test. As a result, the coefficient of the lower-order logarithm generated by fig. 1c would be incorrect, and a contribution proportional to $(n-2)^2$ would be left in the two-loop β function [as for $M_1 \neq M$ in eq. (20)]. The Feynman cut-off is therefore a noninvariant regularization and cannot be freely combined with the background-field method.

Expecting that the differential renormalization is automatically invariant, we would set $M_2=M_3=M$ as well. However, then eq. (20) would again give us the wrong result (14) obtained under the assumption of the automatic invariance in the massless theory via the infrared $\tilde{\mathcal{R}}$ operation. Hence, the ratio of the renormalization parameters ought to be somehow tuned in order to restore the invariance.

The identical situation was encountered in a nonrenormalizable chiral

theory already at the one-loop level for physical observables [17]. Inside the differential renormalization, one finds no *a priori* internal criterion for choosing the ratios of auxiliary masses, to get reliable results. Of course, comparing eq. (20) to eq. (6) in the dimensional renormalization [or the results for fig. 1(d,e) individually], we can infer the values that would ensure the invariance: $\ln(M_2^2/M^2) = \ln(M_3^2/M^2) = 1$. But by itself the differential renormalization remains ambiguous if we apply it to a theory that is not renormalizable in the ordinary sense, and is not directly compatible with the background field method.

4 Conclusion

Our examples show that the background-field formalism requires one to use an invariant renormalization procedure in order to obtain valid results in a generalized-renormalizable theory. A noninvariant regularization or renormalization may break an implicit correlation between different diagrams, which is essential as one formally expands the action in the background and quantum fields.

We have demonstrated by direct two-loop calculations that the regularization via a cut-off in the momentum space is noninvariant and gives a wrong result for the β function of the n -field model within the background-field formalism.

We have also found that the differential renormalization is not automatically invariant. The result depends on the ratio of the auxiliary scale parameters beyond the allowed scheme arbitrariness in the second order of perturbation theory. We can partially fix the ambiguity by imposing a condition on divergent one-loop tadpole-type diagrams a combination of which should be finite. But this is not enough, and there seems to be no algorithm of generalizing such conditions to more complicated graphs.

We would like to stress once more that the calculations in nonlinear models like the σ model or supergravity are hardly possible without the background-field formalism. Thus, the need in the regularization that preserves the underline symmetries and is practically usefull at the same time is of vital importance. The example considered above clearly demonstrate the problems arising when using a non-invariant procedure.

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