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QUANTUM GROUP COVARIANT NONCOMMUTATIVE GEOMETRY

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1 Introduction

Noncommutative geometry [1] has started to play a significant role in mathematical physics for the last few years. One of the nontrivial examples of noncommutative geometry is given by quantum groups [2, 3]. After the paper [4], the differential geometric aspects of the theory of quantum groups have been intensively investigated recently (see e.g. [5]-[8]). Using these investigations, various approaches to formulate quantum group gauge theories have been developed [9],[10]-[13].

In this paper, we continue researches of quantum group covariant noncommutative geometry proposed in [9, 14]. In Sect.2, we describe how to revise the usual commutative geometry (e.g., the geometry of the principal fibre bundle) and introduce differentials covariant under the special quantum group co-transformation interpreted as a local (structure) transformation. Here, a quantum group is an exterior extension of $GL_q(N)$. Then, we define the corresponding geometrical objects such as noncommutative 1-form connections and curvature 2-forms. We show that these noncommutative geometrical objects generate $GL_q(N)$-covariant quantum algebras. In Sect.3, we discuss noncommutative geometry related to the coset space $GL_q(N+1)/(GL_q(N)\otimes GL(1))$. This geometry yields a nontrivial explicit example of algebraic constructions considered in Sect.2. Then, in Sect.4, we compose from the generators of the $GL_q(N)$-covariant quantum algebras a set of $GL_q(N)$-local invariants which could be considered as noncommutative images of the well-known gauge invariant Lagrangians (e.g., discrete gauge theories and Einstein gravity). Some of these invariants are nothing but noncommutative analogs of the Chern characters. We would like to stress, however, that this analogy with the conventional Lagrangians is rather formal and, strictly speaking, it may not lead to q-deformations of the corresponding field theories.

We use the notation and methods of the paper [2] in which the R-matrix formulation of quantum groups has been elaborated. Some further development [15] of the R-matrix notation, considerably simplifying the calculations, is also employed. According to the results obtained in [13] one can reformulate our algebraic construction of the noncommutative geometry for the case of unitary structure groups $U_q(N)$. Moreover, we believe that using Brzezinski's theorem [16] (and its generalization to the braided case [17]) about exterior Hopf algebras, one can apply our construction in the case of any quasitriangular Hopf algebra with bicovariant first order differential calculus. In the Conclusion we briefly discuss this possibility and make some other remarks.

2 $GL_q(N)$-covariant derivatives, noncommutative connections and curvature

Let us consider a Z_2-graded finite dimensional Zamolodchikov algebra (denoted by Ω_Z) generated by the operators $\{e^i,\ (de)^j\}$, $(i,j=1,2,\dots,N)$ with the following

commutation relations:

$$\mathbf{R}ee' = cee'\ ,\ \ (\pm)c\mathbf{R}(de)e' = e(de)'\ ,\ \ \mathbf{R}(de)(de)' = -\frac{1}{c}(de)(de)', \tag{2.1}$$

where $e=e_1$ is a q-vector in the first space, $e'=e_2$ is a q-vector in the second space, $\mathbf{R}=P_{12}R_{12}$ is a matrix which acts in the first and second spaces simultaneously, $P_{12}=\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}$ is the permutation matrix and

$$\begin{gathered} R_{12} = R^{i_1,i_2}_{j_1,j_2} = \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}(1+(q-1)\delta^{i_1 i_2}) + (q-q^{-1})\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\Theta_{i_1 i_2}, \\ \Theta_{ij} = \{1 \text{ if } i>j,\ 0 \text{ if } i\le j\} \end{gathered} \tag{2.2}$$

is the $GL_q(N)$ R–matrix satisfying the Hecke relation ($\lambda = q-q^{-1}$).

$$\mathbf{R}^2 = \lambda\mathbf{R} + \mathbf{1}. \tag{2.3}$$

Here $\mathbf{1}$ is a $(N^2\times N^2)$ unit matrix. We imply the wedge product in the multiplication of the differential forms in the formulas (2.1) (we also omit $\wedge$ in all formulas below). One can recognize in the relations (2.1) (for $(\pm)=+1$) the Wess-Zumino formulas of the covariant differential calculus on the bosonic ($c=q$) and fermionic ($c=-1/q$) quantum hyperplanes [18] where e^i are the coordinates of the quantum hyperplane while $(de)^i$ are the associated differentials. The choice $(\pm)=-1$ corresponds to the case when e^i are bosonic ($c=-1/q$) and fermionic ($c=q$) veilbein 1-forms. Note that there is the second version of the algebra (2.1) obtained by means of the replacement $\mathbf{R}\to\mathbf{R}^{-1}$, $c\to c^{-1}$. Below, we concentrate only on the consideration of the algebra (2.1) (the other type can be treated analogously).

It has been proposed in [20, 19, 9] to consider the algebra Ω_Z (2.1) as a comodule with respect to the coaction of the Z_2-graded quantum group $\Omega_{GL_q(N)}$ with the $GL_q(N)$-generators $\{T^i_j\}$ and additional generators $\{(dT)^k_l\}$ $(i,j,k,l=1,2,\dots,N)$ which are the basis of the differential 1-forms on the quantum group $GL_q(N)$. This coaction $\Omega_Z \xrightarrow{g_l} \Omega_{GL_q(N)}\otimes\Omega_Z$ conserves the grading and can be written down as a homomorphism:

$$e^i \xrightarrow{g_l} \widetilde{e}^i = T^i_j\otimes e^j, \tag{2.4}$$

$$(de)^i \xrightarrow{g_l} (\widetilde{de})^i = (dT)^i_j\otimes e^j + T^i_j\otimes(de)^j. \tag{2.5}$$

Here $\otimes$ denotes the graded tensor product: $a\otimes b = (-1)^{\hat{a}\hat{b}}(1\otimes b)(a\otimes 1)$, where $\hat{f}=deg(f)$ and $a\in\Omega^{(\hat{a})}_{GL_q(N)}$, $b\in\Omega^{(\hat{b})}_Z$. We recall that the algebra Ω_Z with the generators (2.1) has the following expansion $\Omega_Z=\bigoplus_{n=0}\Omega^{(n)}_Z$, where $\Omega^{(n)}_Z$ denotes the subspace of the differential n-forms and there exists a similar expansion for the Z_2-graded quantum group $\Omega_{GL_q(N)}=\bigoplus_{n=0}\Omega^{(n)}_{GL_q(N)}$. Substituting the transformed algebra $\{\widetilde{e}^i,\ (\widetilde{de})^i\}$ into the commutation relations (2.1), we obtain the following equations for the generators $\{T^i_j,\ (dT)^i_j\}$:

$$(\mathbf{R}-c)TT'(\mathbf{R}+c^{-1})=0\ ,\ \ (\mathbf{R}(dT)T' - T(dT)'\mathbf{R}^{-1})(\mathbf{R}+c^{-1})=0\ , \tag{2.6}$$

$$(\mathbf{R}+c^{-1})(dT)(dT)'(\mathbf{R}+c^{-1})=0\,,\quad (\mathbf{R}+c^{-1})((dT)T'\mathbf{R}-\mathbf{R}^{-1}T(dT)')=0\,,\tag{2.7}$$

where $T = T_1 = T \otimes I$ while $T' = T_2 = I \otimes T$ and I is a $(N \times N)$ unit matrix. The relations (2.6), (2.7) have to be fulfilled both for $c = q$ and $c = -q^{-1}$, therefore we deduce from them the following q-commutation relations for the bicovariant differential complex on $GL_q(N)$ (see [19, 7, 8]):

$$\begin{aligned}
\mathbf{R}TT' &= TT'\mathbf{R}\,, &(2.8)\\
\mathbf{R}(dT)T' &= T(dT)'\mathbf{R}^{-1}\,, &(2.9)\\
\mathbf{R}(dT)(dT)' &= -(dT)(dT)'\mathbf{R}^{-1}\,. &(2.10)
\end{aligned}$$

We stress that (2.10) follows from (2.9) if the differential d is nilpotent $d^2 = 0$ and obeys the graded Leibnitz rule $d(fg) = d(f)g + (-1)^{\hat{f}} f d(g)$. It is interesting to note (see [9]) that the algebra $\Omega_{GL_q(N)}$ (2.8)-(2.10) is the Hopf algebra. The comultiplication Δ, the counit ϵ and the antipode S are defined by

$$\begin{array}{c}
\Delta(T) = T \otimes T\,,\ \ \epsilon(T) = I\,,\ \ S(T) = T^{-1}\,,\\
\Delta(dT) = dT \otimes T + T \otimes dT\,,\ \ \epsilon(dT) = 0\,,\ \ S(dT) = -T^{-1}dTT^{-1},
\end{array}\tag{2.11}$$

and satisfy all the axioms of the Hopf algebra. Thus, the algebra $\Omega_{GL_q(N)}$ yields a special example of the general exterior Hopf algebras discussed in [16]. We stress that the example of the $GL_q(N)$-exterior Hopf algebra proposed in [16] has slightly different comultiplication comparing with the Hopf algebra $\Omega_{GL_q(N)}$ (2.8)-(2.11) independently introduced in [8, 9]. One can show that it is possible to extend the action of the differential d over the tensoring and apply d to the algebra $\Omega_{GL_q(N)} \otimes \Omega_Z$ in such a way that: $d(g \otimes \Omega_Z) = d(g) \otimes \Omega_Z + (-1)^k g \otimes d(\Omega_Z)$, where $g \in \Omega^{(k)}_{GL_q(N)}$ and $d^2 = 0$.

Now we would like to interpret formulas (2.4) and (2.5) as a local (structure) quantum group transformation of the comodule e^i. Here the matrix T^i_j is understood as a noncommutative analog of a local (structure) group element. In view of this, it is natural to consider the appearing of the additional term $(dT)^i_j \otimes e^j$ in (2.5) as a noncovariance of the comodule $(de)^i$ under the "gauge" rotation (2.4) (or as an indication that the differentials $(de)^i$ describe "nonparallel transporting" of the vector e^i). To restore the covariance let us introduce a covariant differential ∇ in such a way that the transformations (2.4), (2.5) are rewritten in the form

$$e^i \xrightarrow{g_l} \tilde{e}^i = T^i_j \otimes e^j,\tag{2.12}$$

$$(\nabla e)^i \xrightarrow{g_l} \widetilde{(\nabla e)}^i = T^i_j \otimes (\nabla e)^j\,.\tag{2.13}$$

In general $(\nabla e)^i \notin \Omega_Z$ and, hence, the action of the operator ∇ enlarges the algebra Ω_Z up to some new algebra $\Omega_{\bar{Z}}$. The operator d can be induced (as a differential) onto the whole algebra $\Omega_{\bar{Z}}$ and this algebra is naturally decomposed as $\Omega_{\bar{Z}} = \bigoplus_{n=0} \Omega^{(n)}_{\bar{Z}}$

where $\Omega^{(n)}_{\bar{Z}}$ is the subspace of n-forms. Then, we postulate that the elements $(\nabla e)^i \in \Omega^{(1)}_{\bar{Z}}$ are expanded over the generators $\{e^i,\ (de)^j\}$ of Ω_Z in the following way:

$$(\nabla e)^i = (de)^i - A^i_j e^j,\tag{2.14}$$

It is clear that the coefficients A^i_j belong to the subspace $\Omega^{(1)}_{\bar{Z}}$ and it is natural to consider them as noncommutative analogs of the connection 1-forms. Under the transformations (2.12) and (2.13) the 1-forms A^i_j are transformed as:

$$A^i_k \xrightarrow{g_l} \widetilde{A^i_k} = T^i_j(T^{-1})^l_k \otimes A^j_l + dT^i_j(T^{-1})^j_k \otimes 1 \equiv (TAT^{-1})^i_k + (dTT^{-1})^i_k\,,\tag{2.15}$$

Here $\widetilde{A}^i_j \in \Omega_{GL_q(N)} \otimes \bar{Z}$. In the last part of (2.15) a short notation is introduced to be used below. The second action of the covariant derivative ∇ on the expression (2.14) leads to the definition of the curvature 2-forms $F^i_j \in \Omega^{(2)}_{\bar{Z}}$:

$$\nabla(\nabla e) = -\left(d(A) - A^2\right) e = -Fe.\tag{2.16}$$

The quantum co-transformation (2.15) for the curvature 2-forms F^i_j is represented as an adjoint coaction

$$F^i_j \xrightarrow{g_{ad}} \widetilde{F}^i_j = T^i_k(T^{-1})^l_j \otimes F^k_l \equiv T^i_k F^k_l (T^{-1})^l_j\,.\tag{2.17}$$

The curvature tensor F^i_j is a reducible adjoint representation of $GL_q(N)$ and it is possible to decompose it into the scalar $F^0 = Tr_q(F)$ and the q-traceless tensor:

$$\tilde{F}^i_j = F^i_j - \delta^i_j Tr_q(F)/Tr_q(I).$$

Here, we have introduced the q-deformed trace [2, 7, 9, 21] for the $GL_q(N)$-group

$$F^0 = Tr_q(F) \equiv Tr(DF) \equiv \sum_{i=0}^{N} q^{-N-1+2i} F^i_i\,.\tag{2.18}$$

Below we need the feature of invariance of the q-trace:

$$Tr_q(TET^{-1}) = Tr_q(E)\tag{2.19}$$

where $[T_{ij},\ E_{kl}] = 0$ and $T^i_j \in GL_q(N)$. In particular, we have

$$Tr_{q2}(\mathbf{R}E\mathbf{R}^{-1}) = Tr_{q2}(\mathbf{R}^{-1}E\mathbf{R}) = Tr_q(E)\tag{2.20}$$

Here $Tr_{q2}(.)$ denotes the quantum trace over the second space. We also use the relations

$$Tr_q(\mathbf{R}^{\pm 1}) = q^{\pm N},\ Tr_q(I) = \frac{q^N - q^{-N}}{q - q^{-1}} \equiv [N]_q\tag{2.21}$$

The next action of the covariant derivative on formula (2.16) yields the Bianchi identities that are represented in the classical form

$$d(F) = [A,\ F].$$

To complete the definition of the algebra Ω_Z we have to deduce the commutation relations of the new generators $\{A^i_j,\ F^i_j, \ldots\}$ and their cross-commutation relations with the generators $\{e^i,\ (de)^j\}$. First of all, let us note that the choice of the connection in the pure gauge form (see (2.15))

$$A^i_j = dT^i_k (T^{-1})^k_j \otimes 1\ , \tag{2.22}$$

leads to the conclusion that the generators A^i_j could satisfy the following q-deformed anticommutation relations:

$$\mathbf{RARA} + \mathbf{ARAR}^{-1} = 0\ , \tag{2.23}$$

where $\mathbf{A} = A_1 = A \otimes I$. These relations for the noncommutative 1-form connections (gauge fields) have been postulated in [9, 12]. Note, however, that in the right hand side of Eq.(2.23) one may add an arbitrary linear combination of the curvature 2-forms $F = dA - A^2$ which vanishes on the solution (2.22). Thus, the general covariant commutation relations for A^i_j are

$$\mathbf{RARA} + \mathbf{ARAR}^{-1} = a(\mathbf{R})(\mathbf{FR} + \mathbf{R}^{-1}\mathbf{F}) + \kappa(\mathbf{R})F^0 \equiv \Delta(F)\ , \tag{2.24}$$

where $\mathbf{F} = F_1 = F \otimes I$, $a(\mathbf{R}) = a_1 + a_2\mathbf{R}$ and for convenience we choose the parameter $\kappa(\mathbf{R})$ in the form: $\kappa(\mathbf{R}) = (\kappa_1 + \kappa_2\mathbf{R})(\mathbf{R} + \mathbf{R}^{-1})$.

Special form of the right hand side of Eq.(2.24) is dictated by the symmetry properties of the q-anticommutator appeared in the left-hand side of this equation ($c = \pm q^{\pm 1}$):

$$(\mathbf{R} - c)(\mathbf{RARA} + \mathbf{ARAR}^{-1})(\mathbf{R} + c^{-1}) = 0\ .$$

We stress that the anticommutation relations (2.24) are covariant under the transformations (2.15) and (2.17). Moreover, one can extract from the relations (2.24) subsets of covariant relations using the methods proposed in [15]. Namely, applying $Tr_{q(2)}(\ldots)$ and $Tr_{q(2)}(\ldots\mathbf{R})$ to (2.24) and using (2.21) we obtain two sets of relations transformed as adjoint comodules

$$\begin{aligned}\lambda q^N A^2 + \{A^0,\ A\} &= [a_1(q^N + q^{-N}) + a_2([N]_q + \lambda q^N)]F + a_2 F^0 + \\ &+ [\kappa_1(q^N + q^{-N}) + \kappa_2(2[N]_q + \lambda q^N)]F^0\ ,\end{aligned} \tag{2.25}$$

$$\begin{aligned}q^N A^2 + (A * A) &= [a_1([N]_q + \lambda q^N) + a_2 q^N (q^2 + q^{-2})]F + (a_1 + \lambda a_2)F^0 + \\ &+ \left[\kappa_1(2[N]_q + \lambda q^N) + \kappa_2(q^N(q^2 + q^{-2}) + \lambda [N]_q)\right] F^0\ ,\end{aligned} \tag{2.26}$$

where $(A * A) = Tr_{q(2)}(\mathbf{RARAR})$, $F^0 = Tr_q(F)$, $A^0 = Tr_q(A)$. Then, applying $Tr_{q(1)}(\ldots)$ to (2.25) and (2.26) we obtain two scalar relations ($q^2 \neq -1$)

$$Tr_q(A^2) = \left[(a_1 + \kappa_1)q^{-N}[N]_q + (a_2 + \kappa_2)\right] F^0\ , \tag{2.27}$$

$$(A^0)^2 = [(a_1 + \kappa_1)q^{-N} + (a_2 + \kappa_2)[N]_q]F^0\ . \tag{2.28}$$

We see that in the noncommutative case Eqs.(2.27)-(2.28) give additional relations of the 1-form connections A and 2-form curvatures $F \equiv dA - A^2$.

Arbitrary parameters a_i, κ_i introduced in Eq.(2.24) depend on the choice of the noncommutative geometry and have to be fixed partially by the consistency conditions (with respect to the two ways of ordering of any cubic monomial) for the algebra Ω_Z. It is amusing to note that the additional nonzero term included into the right-hand side of (2.24) looks similar to the quantum anomaly terms arising in the (anti)commutators of fields (or currents) in certain conventional quantum field theories.

In order to find commutation relations A^i_j with the generators $\{e^i,\ (de)^j\}$ we postulate that the coordinates of the comodule (2.14) commute in the same way as the components of the 1-forms $(de)^i$ (see (2.1))

$$\mathbf{R}(\nabla e)(\nabla e)' = -\frac{1}{c}(\nabla e)(\nabla e)' \tag{2.29}$$

$$(\pm)(c - b)\mathbf{R}(\nabla e)e' = e(\nabla e)'\ . \tag{2.30}$$

where b is a constant to be fixed below. Let us stress that Eqs.(2.29),(2.30) are not general covariant relations of that kind. For example, one can add to (2.29) terms of the type $(Fe)e'$. We, however, prefer to consider here the simplest case of the relations (2.29),(2.30). From (2.1) and (2.30) we deduce covariant commutation relations of A and e:

$$(\pm)e\mathbf{A}' = \mathbf{RAR}e + b\mathbf{R}(\nabla e) \tag{2.31}$$

Considering the consistency condition for the reordering (in two different ways) of the monomial $ee'\mathbf{A}'' \equiv e_1 e_2 A_3$, we obtain only two solutions for the parameter b:

$$A.)\ \ b = 0 \quad B.)\ \ b = \lambda\ . \tag{2.32}$$

Thus, we have two variants for Eq.(2.31)

$$A.)\ (\pm)e\mathbf{A}' = \mathbf{RAR}e\ ,\quad B.)\ (\pm)e\mathbf{A}' = \mathbf{RAR}^{-1}e + \lambda\mathbf{R}(de)\ . \tag{2.33}$$

Note that in the paper [9] we have considered only the first case A.): $b = 0$. Taking into account (2.29), one can obtain the corresponding commutation relations for (de) and A

$$(\pm)(de)\mathbf{A}' = -\mathbf{R}^{-1}\mathbf{AR}(de) + (b - \lambda)\mathbf{AR}(\nabla e) + \tilde{a}(\mathbf{R})\Delta(F)e\ , \tag{2.34}$$

where

$$\tilde{a}(\mathbf{R}) = \frac{1+\gamma(\mathbf{R}-c)}{1+c^2} \tag{2.35}$$

and γ is a new arbitrary constant to be fixed below. Type A.) and type B.) commutation relations (2.31), (2.34) are covariant under the gauge coactions (2.4), (2.5) and (2.15) and both cases lead to the same covariant commutation relation for (∇e) and A:

$$(\pm)(\nabla e)\mathbf{A}' = -\mathbf{RAR}(\nabla e) + (\tilde{a}(\mathbf{R})-1)\Delta(F)e\ , \tag{2.36}$$

Differentiating (2.31) and, then, using (2.34) one can derive

$$\begin{aligned} e\mathbf{F}' &= \mathbf{RF}(\mathbf{R}-b)e + \tilde{\tilde{a}}(\mathbf{R})\Delta(F)e = \\ &= (\mathbf{R}+\tilde{\tilde{a}}(\mathbf{R})a(\mathbf{R}))\mathbf{FR}e + (\tilde{\tilde{a}}(\mathbf{R})a(\mathbf{R})\mathbf{R}^{-1} - b\mathbf{R})\mathbf{F}e + \tilde{\tilde{a}}(\mathbf{R})\kappa(\mathbf{R})F^0 e \end{aligned} \tag{2.37}$$

where we define

$$\tilde{\tilde{a}}(\mathbf{R}) = -(1+b\mathbf{R})\tilde{a}(\mathbf{R}) + (b-\lambda)\mathbf{R}. \tag{2.38}$$

Considering the reordering of the monomials $ee'\mathbf{F}''$ in two possible ways and comparing the results, we obtain for both types A.) $b=0$ and B.) $b=\lambda$ the restrictions

$$1.)\ \tilde{\tilde{a}}(\mathbf{R})a(\mathbf{R}) = 0\ ,\ \ \tilde{\tilde{a}}(\mathbf{R})\kappa(\mathbf{R}) = 0, \tag{2.39}$$

which lead to the commutation relation:

$$e\mathbf{F}' = \mathbf{RF}(\mathbf{R}-b)e. \tag{2.40}$$

Note, that for the type A.) $(b=0)$ we have an additional solution

$$2.)\tilde{\tilde{a}}(\mathbf{R})a(\mathbf{R}) = -\lambda\ ,\ \ \tilde{\tilde{a}}(\mathbf{R})\kappa(\mathbf{R}) = 0$$

equivalent to the relation: $e\mathbf{F}' = \mathbf{R}^{-1}\mathbf{FR}^{-1}e$. This relation, however, is consistent with the algebra (2.24), (2.31) and (2.36) only if some additional relations on the generators of $\Omega_{\mathcal{Z}}$ are fixed. One can prove this by considering two different ways of reordering of the monomials $e\mathbf{R}'\mathbf{A}'\mathbf{R}'\mathbf{A}'$ where $\mathbf{R}' = P_{23}R_{23}$.

Taking into account the conditions (2.39) we obtain from the definitions (2.38) and (2.35) the following solutions for the parameters $a(\mathbf{R})$ and γ

$$\begin{aligned} &1.)\ a(\mathbf{R}) = 0\ ,\ \ \kappa(\mathbf{R}) = 0 \Rightarrow \Delta(F) = 0, \\ &2.)\ a(\mathbf{R}) = a_0(\mathbf{R}-c),\ \kappa(\mathbf{R}) = \kappa_0(\mathbf{R}-c),\ \gamma = \tfrac{1}{c+c^{-1}} + (b-\lambda) \Rightarrow \\ &(\mathbf{R}-c)\tilde{a}(\mathbf{R}) = \tfrac{(\lambda-b)}{c}(\mathbf{R}-c),\ \ (\mathbf{R}-c)\tilde{\tilde{a}}(\mathbf{R}) = \tfrac{b(\lambda-b)}{c^2}(\mathbf{R}-c) \equiv 0. \end{aligned} \tag{2.41}$$

Here $a_0 \neq 0,\ \kappa_0 \neq 0$ are constants.

Now, we deduce the covariant commutation relations for the generators F^i_j postulating the following natural quantum hyperplane condition

$$(\mathbf{R}-c)(\mathbf{F}e)(\mathbf{F}'e') = 0\ . \tag{2.42}$$

Using (2.40) one can obtain from (2.42) the following relations

$$(\mathbf{R}-c)\mathbf{FRF}(\mathbf{R}+c^{-1}) = 0\ . \tag{2.43}$$

The commutation relations for the curvature 2-form F^i_j have to be independent of the class of the comodule $\{e^i\}$ and, therefore, of the choice of the parameter $c = \pm q^{\pm 1}$. So, we deduce from Eqs.(2.43) the commutation relations

$$\mathbf{RFRF} = \mathbf{FRFR}\ . \tag{2.44}$$

These relations are known, first, as reflection equations [22], second, as the commutation relations for invariant vector fields on $GL_q(N)$ [7, 8] and, third, as the defining relations for the braided algebras [23].

To complete the definition of the algebra $\Omega_{\mathcal{Z}}$ one can deduce the following cross-commutation relation for F and A:

$$\mathbf{FRAR} = \mathbf{RARF}\ . \tag{2.45}$$

This is the simplest relation covariant under the coactions presented in (2.15) and (2.17) and allowing one to push the operators F through the operators A.

Thus, leaving aside the commutation relations with the generators $\{e,\ de\}$, we come to the following algebra with the generators A (1-form connection) and $F = dA - A^2$ (2-form curvature):

$$\begin{aligned} \mathbf{FRAR} = \mathbf{RARF}\ ,\ \ &\mathbf{RFRF} = \mathbf{FRFR}\ , \\ \mathbf{RARA} + \mathbf{ARAR}^{-1}\ &= a(\mathbf{R})(\mathbf{FR} + \mathbf{R}^{-1}\mathbf{F}) + \kappa(\mathbf{R})F^0\ , \end{aligned} \tag{2.46}$$

where $a(\mathbf{R}) = (\mathbf{R}-c)a_0$ and $\kappa(\mathbf{R}) = (\mathbf{R}-c)\kappa_0$ (see Eqs.(2.41)). Note, that for the case $a_0 \neq 0$ and $\kappa_0 \neq 0$ the consistence conditions for the whole covariant algebra $\Omega_{\mathcal{Z}}$ give some additional constraints on the generators of this algebra. In particular, one can deduce

$$(\mathbf{R}-c)\tilde{\mathbf{F}}\mathbf{R}e = 0, \tag{2.47}$$

where $\tilde{F} = F - \frac{\kappa_0}{a_0(c+c^{-1})}F^0$.

3 $GL_q(N+1)/(GL_q(N)\otimes GL(1))$ noncommutative geometry

In this section we present an explicit realization of such a covariant algebra $\Omega_{\mathcal{Z}}$ where parameters a_0, κ_0 and additional relations (of the type (2.47)) on the generators will be fixed. We consider differential geometry on the group $GL_q(N+1)$ [19, 20, 7, 8] and interpret it as noncommutative geometry on the total space of the principal fibre bundle with the base space $GL_q(N+1)/(GL_q(N)\otimes GL(1))$ and the structure group being $GL_q(N)\otimes GL(1)$.

Let us introduce Z_2-graded extension of the $GL_q(N+1)$ quantum group (exterior Hopf algebra) with the generators $\{T^I_J,\ dT^I_J\}$ $(I,J=0,1,\dots N)$ satisfying the commutation relations (2.8)-(2.10) where the $GL_q(N+1)$ R-matrix acts in the space $Mat(N+1)\times Mat(N+1)$. Then, we consider the following left coaction of the group $GL_q(N)\otimes GL(1)$ on the group $GL_q(N+1)$:

$$T^I_J \to \left(\begin{array}{c|c} t & 0 \\ \hline 0 & T^i_k \end{array}\right)\otimes\left(\begin{array}{c|c} T^0_0 & T^0_j \\ \hline T^k_0 & T^k_j \end{array}\right) \tag{3.1}$$

where as usual $i,j,k=1,2,\dots N$ and t $([t,\ T^i_j]=0)$ is a dilaton generator of $GL(1)$. It is evident (from the commutation relations for the $GL_q(N+1)$-generators) that the elements T^i_j generate the quantum group $GL_q(N)$. The noncommutative coordinates for the "base space" $GL_q(N+1)/(GL_q(N)\otimes GL(1))$ could be related with the generators T^0_i and T^j_0. For the Cartan 1-forms on the $GL_q(N+1)$-group:

$$\Omega^I_J = dT^I_K(T^{-1})^K_J = \left(\begin{array}{c|c} \omega & \Omega^0_j = <\bar{e}|_j \\ \hline \Omega^i_0 = |e>^i & A^i_j \end{array}\right) \tag{3.2}$$

the coaction (3.1) is represented in the form:

$$\left(\begin{array}{c|c} \omega & <\bar{e}| \\ \hline |e> & A \end{array}\right) \to \left(\begin{array}{c|c} \omega + dtt^{-1} & <\bar{e}|T^{-1}t \\ \hline t^{-1}T|e> & TAT^{-1}+dTT^{-1} \end{array}\right) \tag{3.3}$$

where the short notation has been used (see e.g. (2.15)). By comparing these transformations with the transformations (2.12) and (2.15), it becomes clear that the Cartan 1-forms $|e>$ and A, ω can be interpreted as veilbein 1-forms and connection 1-forms, respectively. Then, the generators $<\bar{e}|$ are nothing but contragradient veilbein 1-forms. The Maurer-Cartan equation $d\Omega^I_J = \Omega^I_K\Omega^K_J$ leads to the following constraints on the noncommutative differential 1-forms Ω^I_J:

$$\left(\begin{array}{c|c} d\omega-\omega^2-<\bar{e}|e> & d<\bar{e}|-<\bar{e}|A-\omega<\bar{e}| \\ \hline d|e>-A|e>-|e>\omega & dA-A^2-|e><\bar{e}| \end{array}\right) = 0\,. \tag{3.4}$$

The q-deformed commutators for the noncommutative Cartan 1-forms (3.2) are deduced from the $N+1$-dimensional analog of the relations presented in (2.23). Taking into account the Maurer-Cartan equations (3.4) and using the notation (3.2) we rewrite these relations in the form:

$$\mathbf{RARA}+\mathbf{ARAR}^{-1} = -\lambda(\mathbf{RF}+\mathbf{FR}^{-1}) \tag{3.5}$$

$$-e\mathbf{A}' = \mathbf{RAR}e+\lambda\mathbf{R}(de-\mathbf{A}e)\,,\ -\mathbf{A}'\bar{e} = \bar{e}\mathbf{RAR}+\lambda(d\bar{e}-\bar{e}\mathbf{A})\mathbf{R} \tag{3.6}$$

$$\bar{e}\mathbf{R}e = -qe'\bar{e}',\quad \mathbf{R}ee' = -q^{-1}ee',\quad \bar{e}'\bar{e}\mathbf{R} = -q^{-1}\bar{e}'\bar{e}, \tag{3.7}$$

$$\omega^2=0,\quad [\omega,e]_+ = [\omega,\bar{e}]_+ = 0,\quad [A,\omega]_+ = q\lambda|e><\bar{e}| = q\lambda F. \tag{3.8}$$

Here we have also introduced the notation for the curvature 2-form

$$F = dA - A^2 = |e><\bar{e}| = -q^{-1} <\bar{e}|_1\mathbf{R}|e>_1\ . \tag{3.9}$$

The last two equalities follow from Eqs.(3.4) and (3.7) and reveal the dependence of the curvature 2-forms and the veilbein 1-forms. Note that for this form (3.9) of the curvature one can directly prove the identity (2.47) (for $\kappa_0=0$) using the relations (3.7). Now, we find (applying the commutation relations (3.5)-(3.8) and Eq.(3.9)) that the following relations for F and A hold:

$$\mathbf{RFRF} = \mathbf{FRFR},\quad \mathbf{RARF} = \mathbf{FRAR}+\lambda(\mathbf{RF}\omega-\mathbf{F}\omega\mathbf{R}) \tag{3.10}$$

We would like to compare these relations with the relations (2.46) but at this stage we cannot do this in view of appearing in (3.10) of an additional scalar generator ω which is nothing but the $GL(1)$-connection 1-form (see (3.3)). To exclude from the consideration these scalar connection 1-form, we introduce a new total $GL_q(N)\otimes GL(1)$ connection:

$$A_t = A-\omega I\,, \tag{3.11}$$

for which we have

$$\nabla_t e = \nabla_t\bar{e} = 0\,, \tag{3.12}$$

(see (3.4)) and the corresponding curvature 2-forms

$$F_t = q^2F - <\bar{e}|e>\cdot I = q^2F+q^{1-N}F^0\cdot I \tag{3.13}$$

satisfy the conditions

$$F_t|e> = <\bar{e}|F_t = 0\,. \tag{3.14}$$

The scalar 2-form $F^0 = Tr_q(F)$ in (3.13) is defined by Eq.(2.18) and is invariant under the adjoint coaction (2.17). Finally, we find from Eqs.(3.6)-(3.8) and (3.10) that the elements $\{e,\ A_t,\ F\}$ generate the following closed algebra:

$$\begin{gathered} \mathbf{RFRF} = \mathbf{FRFR},\quad \mathbf{RA}_t\mathbf{RF} = \mathbf{FRA}_t\mathbf{R}, \\ \mathbf{RA}_t\mathbf{RA}_t+\mathbf{A}_t\mathbf{RA}_t\mathbf{R}^{-1} = a_0(\mathbf{FR}^{-1}+\mathbf{RF})(\mathbf{R}-c), \\ -e\mathbf{A}'_t = \mathbf{RA}_t\mathbf{R}e,\quad e\mathbf{F}' = \mathbf{RFR}e \end{gathered} \tag{3.15}$$

where $a_0 = 1-q^2$ and $c=-q^{-1}$.

Comparing the commutation relations (3.7) and (3.15) with the relations (2.1), (2.31) and (2.46), one can infer that we have explicitly realized the defining relations for the covariant quantum algebra Ω_Z of the type A.) (2.32), (2.33) in terms of the algebraic objects related to the $GL_q(N+1)/GL_q(N)\otimes GL(1)$-geometry. To be precise, we have to consider an algebra of the type (3.15) with the substitution $F \leftrightarrow F_t$. The corresponding defining relations are

$$\begin{gathered} \mathbf{R}\mathbf{F}_t\mathbf{R}\mathbf{F}_t = \mathbf{F}_t\mathbf{R}\mathbf{F}_t\mathbf{R}, \quad \mathbf{R}\mathbf{A}_t\mathbf{R}\mathbf{F}_t = \mathbf{F}_t\mathbf{R}\mathbf{A}_t\mathbf{R}, \\ \mathbf{R}\mathbf{A}_t\mathbf{R}\mathbf{A}_t + \mathbf{A}_t\mathbf{R}\mathbf{A}_t\mathbf{R}^{-1} = a_0(\mathbf{F}_t\mathbf{R}^{-1} + \mathbf{R}\mathbf{F}_t)(\mathbf{R} - c) + \tfrac{a_0(c+c^{-1})(\mathbf{R}-c)}{q^3[N]_q} F_t^0, \\ -e\mathbf{A}_t' = \mathbf{R}\mathbf{A}_t\mathbf{R}e, \quad e\mathbf{F}_t' = q^{-2}\mathbf{R}^{-1}\mathbf{F}_t\mathbf{R}e \end{gathered} \tag{3.16}$$

One can note that this type of algebras, in view of the last relation in (3.16), have not been presented in the general consideration of Sect.2. The explanation of this fact is that in Sect.2 we essentially use the conditions $\nabla_t e \neq 0$, $F_t|e> \neq 0$ which are not fulfilled here (see (3.12) and (3.14)). That is why we have not received in Sect.2 the cross-commutation relations for F and e presented in (3.16).

4 $GL_q(N)$-local co-invariants and Chern characters

Our final aim is to define composite elements $\mathcal{L}$ for the extended algebra Ω_Z which are co-invariant $\mathcal{L} \to 1 \otimes \mathcal{L}$ under the $GL_q(N)$ local transformations (2.4), (2.5), (2.15) and (2.17). We would like to interpret these elements $\mathcal{L}$ as noncommutative Lagrangians. However, we stress that this interpretation is rather formal because the elements $\mathcal{L}$ are not the usual Lagrangians for certain field theories. To write down such noncommutative Lagrangians we further extend the algebra Ω_Z described in Sect.2 by introducing the Z_2-graded contragradient comodule $(\bar{e}_i, d\bar{e}_j)$ with the following commutation relations:

$$\begin{gathered} \bar{e}' \cdot \bar{e}\mathbf{R} = c\bar{e}'\bar{e}\ , \quad (d\bar{e})'\bar{e} = (\pm)c\bar{e}'(d\bar{e})\mathbf{R}, \\ (d\bar{e})'(d\bar{e})\mathbf{R} = -\tfrac{1}{c}(d\bar{e})'(d\bar{e}). \end{gathered} \tag{4.1}$$

Note that contragradient q-vectors have naturally appeared in the context of the explicit example of the $GL_q(N)$-covariant noncommutative geometry considered in Sect.3. The quantum group local (structure) transformation of the vector $(\bar{e}_i, d\bar{e}_j)$ is expressed as the following homomorphism of the algebra (4.1):

$$\begin{aligned} (\bar{e},\ d\bar{e}) &\xrightarrow{g_l} ((T^{-1})_i^k \otimes \bar{e}_k\ ,\ \ d(T^{-1})_j^k \otimes \bar{e}_k + (T^{-1})_j^k \otimes d\bar{e}_k) \equiv \\ &\equiv (\bar{e},\ d\bar{e}) \cdot \begin{pmatrix} T^{-1}, & -T^{-1}dTT^{-1} \\ 0, & T^{-1} \end{pmatrix}, \end{aligned} \tag{4.2}$$

where in the last equality of (4.2) we have used the short notation (see (2.15), (2.17)) and the operators T_j^i and dT_l^k are the same as in Eqs.(2.8)-(2.10). The commutation relations for the coordinates of the contragradient q-vectors $\{\bar{e}_i,\ d\bar{e}_j\}$ with the former generators of Ω_Z can be found by using covariance of these relations

under the gauge coactions (2.4), (2.5), (2.15), (2.17) and (4.2). For example, one can assume relations of the type appeared in the explicit construction of Sect.3:

$$e'\bar{e}' = c\bar{e}\mathbf{R}e\ , \quad (\pm)(de)'\bar{e}' = c(\bar{e}\mathbf{R}(de) + \lambda\bar{e}\mathbf{R}\mathbf{A}\mathbf{R}e)\ , \tag{4.3}$$

$$A'\bar{e} = (\pm)\bar{e}\mathbf{R}\mathbf{A}\mathbf{R}\ , \quad F'\bar{e} = \bar{e}\mathbf{R}\mathbf{F}\mathbf{R}\ . \tag{4.4}$$

These relations are not unique covariant relations for the generators $\{e, \bar{e}, A, F, \ldots\}$. There are other choices corresponding to another noncommutative geometry. For example, in our paper [9] we have proposed noncommutative geometry with different relations (4.3).

Now one can define the co-invariant elements of Ω_Z transformed under the local co-transformations as $\mathcal{L} \to 1 \otimes \mathcal{L}$. For example, using the noncommutative generators e^i, $\bar{e}_i$ and A_i^j we construct the co-invariant

$$\mathcal{L} = \bar{e}_i\left(de^i - A_j^i e^j\right). \tag{4.5}$$

We call these composite elements of the algebra Ω_Z the noncommutative (algebraical) Lagrangians bearing in mind the formal similarity of (4.5) to the Lagrangians for the one dimensional discrete gauge models (see e.g. [24]).

In order to write down other local quantum group co-invariants, it is convenient to use the curvature 2-form F transformed as the adjoint comodule (2.17). As an example we present noncommutative analogs of the Chern characters. For this, let us consider a special case of the closed algebra (2.46) with the generators A and F where the parameters $a(\mathbf{R}) = 0$ and $\kappa(\mathbf{R}) = 0$. Here, as we have explained above, A_j^i are noncommutative analogs of the connection 1-forms while F_j^i are interpreted as curvature 2-forms. By analogy with the classical case (see e.g. [25]), we consider as invariant characters the following expressions:

$$C_k = Tr_q(F^k) = D_j^i F_{j_1}^j \cdots F_i^{j_{k-1}}, \tag{4.6}$$

where we have used the q-deformed trace defined in (2.18). Using (2.19) we immediately obtain that $2k$-forms C_k (4.6) are invariant under the adjoint coaction (2.17). Moreover, C_k are the closed $2k$-forms. Indeed, from the Bianchi identities $dF = [A, F]$ we deduce

$$dC_k = Tr_q(AF^k - F^k A) = 0, \tag{4.7}$$

where we have taken into account (see Eqs.(2.46), (2.20) and (2.21))

$$\begin{gathered} Tr_q(AF^k) = q^{-N} Tr_{q1}(Tr_{q2}(\mathbf{R}^{-1}\mathbf{R}\mathbf{A}\mathbf{R}\mathbf{F}^k)) = \\ q^{-N} Tr_{q1}(Tr_{q2}(\mathbf{F}^k\mathbf{R}\mathbf{A})) = Tr_q(F^k A). \end{gathered}$$

We believe that C_k have to be presented as the exact form $C_k = dL_{CS}^{(k)}$, where the Chern-Simons $(2k-1)$-forms $L_{CS}^{(k)}$ are represented as

$$L_{CS}^{(k)} = Tr_q\{A(dA)^{k-1} + \frac{1}{h_1^{(k)}} A^3(dA)^{k-2} + \ldots + \frac{1}{h_{\ldots}^{(k)}} A^{2k-1}\} \tag{4.8}$$

and the constants $h_i^{(k)}$ depend on the deformation parameter q. We do not have explicit formulas for all parameters $h_{...}^{(k)}$ (in the classical case $q=1$ these formulas are known [26]), but for the case $k=2$ one can obtain a noncommutative analog of the three-dimensional Chern-Simons term in the form:

$$L_{CS}^{(2)} = Tr_q\{AdA - \frac{1}{h_1^{(2)}}A^3\}\ , \quad h_1^{(2)} = 1 + \frac{1}{q^2+q^{-2}}\ . \tag{4.9}$$

We would like to note that it is extremely interesting to write the Chern characters for the general case of the algebra (2.46) when the parameters $a(\mathbf{R}) \neq 0$ and $\kappa(\mathbf{R}) \neq 0$.

At the end of this section we propose a way how to find an algebraical Lagrangian corresponding to the field theoretical Lagrangian for the Einstein gravity. First, we take the four generators of the underlying Zamolodchikov algebra (2.1), (4.1) in the form of the 2×2 matrix e^{ij} $(i,j=1,2;\ \ e^\dagger = e)$ interpreted as the spinorial representation for the 4-dimensional veilbein 1-forms. The differential complex Ω_Z for this algebra is the anticommuting version $((\pm)=+1)$ of the differential complex for the q-Minkowski space [28, 29]

$$\mathbf{R}e\mathbf{R}e + e\mathbf{R}e\mathbf{R}^{-1} = 0\,, \tag{4.10}$$

$$\mathbf{R}\,de\,\mathbf{R}\,e - (\pm)e\,\mathbf{R}\,de\,\mathbf{R} = 0\,, \tag{4.11}$$

$$\mathbf{R}\,de\,\mathbf{R}\,de - de\,\mathbf{R}\,de\,\mathbf{R} = 0\,. \tag{4.12}$$

Note that there is another consistent differential complex with the choice of eq.(4.11) in the form $\mathbf{R}\,e\,\mathbf{R}\,de = (\pm)de\,\mathbf{R}\,e\,\mathbf{R}$. Here we do not consider this possibility which is absolutely parallel. The factor $(\pm)=-1$ corresponds to the fermionic version of the q-Minkowski space. The algebra (4.10)-(4.12) is covariant under the q-Lorentz global transformations

$$e \to Te\tilde{T}^{-1}\,, \tag{4.13}$$

$$de \to Tde\tilde{T}^{-1}\,. \tag{4.14}$$

where $\{e,\ de\}$ commute with $\{T,\ \tilde{T}\}$, and elements of the matrices T and $\tilde{T} = (T^\dagger)^{-1}$ are the generators of the two $SL_q(2)$-groups with the following crossing-commutation relations:

$$\mathbf{R}T\tilde{T}' = \tilde{T}T'\mathbf{R}\,, \tag{4.15}$$

This formulation of the q-Lorentz group has been proposed and investigated in [27]-[29]. Using the q-trace (2.18) one can construct from the generators e^{ij} the contragradient veilbein 1-forms $\bar{e}_{ij}$:

$$\bar{e}_{ij} = e^{ij} - q^{-1}Tr_q(e)\delta_{ij}\,. \tag{4.16}$$

The co-transformation (4.13),(4.14) for $\bar{e}$ reads

$$\bar{e} \to \tilde{T}\bar{e}T^{-1}\,, \quad d\bar{e} \to \tilde{T}d\bar{e}T^{-1}\,. \tag{4.17}$$

Further, we need a differential calculus on $SL_q(2)$. Up to now we do not have an appropriate calculus on $SL_q(N)$ (see however [30]). Therefore, we will consider the case of extended Lorentz symmetry generated by $\Omega_{GL_q(2)}$. In this case, one can consider the local version of the transformation (4.14)

$$de \to dTe\tilde{T}^{-1} + Tde\tilde{T}^{-1} - (\pm)Ted\tilde{T}^{-1}\,, \tag{4.18}$$

where $\{T,\ dT\}$ and $\{\tilde{T},\ d\tilde{T}\}$ are two isomorphic $GL_q(2)$-exterior algebras (2.8)-(2.10) with the cross-commutation relations defined by eq.(4.15) and

$$\begin{aligned} \mathbf{R}T\,d\tilde{T}' &= d\tilde{T}\,T'\mathbf{R}\,, \\ \mathbf{R}dT\,\tilde{T}' &= \tilde{T}\,dT'\mathbf{R}\,, \\ \mathbf{R}dT\,d\tilde{T}' &= -d\tilde{T}\,dT'\mathbf{R}\,, \end{aligned} \tag{4.19}$$

Note that formulas (2.8)-(2.10),(4.15) and (4.19) for the $GL_q(N)$ R-matrix define the differential complex on $GL_q(N,C)$. Then, one can introduce the covariant derivative

$$(\nabla e) = de - Ae - e\tilde{A} \tag{4.20}$$

where the connection 1-forms A and $\tilde{A}$ are transformed as

$$A \to TAT^{-1} + dTT^{-1}\,, \quad \tilde{A} \to \tilde{T}\tilde{A}\tilde{T}^{-1} + d\tilde{T}\tilde{T}^{-1}\,. \tag{4.21}$$

For the consistence we demand that $\tilde{A} = A^\dagger$. The corresponding curvature 2-forms F and $\tilde{F}$ are defined as usual

$$F = dA - A^2\,, \quad \tilde{F} = d\tilde{A} - \tilde{A}^2\,. \tag{4.22}$$

We assume that 2-forms F and $\tilde{F}$ admit the expansion over the basis of the veilbein 1-forms (cf. with (3.9))

$$F_1 = Tr_{q2}(\bar{e}_2F_{12}e_2) \to \tilde{F}_1 = Tr_{q2}(e_2\tilde{F}_{12}\bar{e}_2)\,. \tag{4.23}$$

The noncommutative scalar curvature could be obtained as a real combination of the coefficients F_{12}, $\tilde{F}_{12}$:

$$\mathcal{F} = Tr_{q1}Tr_{q2}(F_{12} + \tilde{F}_{12})\,, \tag{4.24}$$

and the corresponding algebraical version of the Einstein Lagrangian reads

$$\mathcal{L} = \mu(e^{ij})\cdot\mathcal{F}$$

where the invariant 4-dimensional real measure μ can be chosen in the form

$$\mu = i\,(Tr_q(e\bar{e}e\bar{e}) - Tr_q(\bar{e}e\bar{e}e))\,.$$

Here $\bar{e}_i$ are contragradient veilbein 1-forms transformed as in (4.17).

5 Discussion and Conclusion

To conclude the paper we would like to make some remarks and comments.

1. We note that there is a realization of the differential complex (2.8)-(2.10) with the usual differential $d = dz\partial_z + d\bar{z}\partial_{\bar{z}}$ over the classical 2-dimensional space $\{z,\ \bar{z}\}$. Indeed, let us consider the algebra

$$\begin{array}{rcl} \mathbf{R}TT' &=& TT'\mathbf{R}\,, \\ TM' = \mathbf{R}M\mathbf{R}T &,& \bar{M}T' = T'\mathbf{R}^{-1}\bar{M}\mathbf{R}^{-1}\,, \\ \mathbf{R}M\mathbf{R}M = M\mathbf{R}M\mathbf{R} &,& \mathbf{R}^{-1}\bar{M}'\mathbf{R}^{-1}\bar{M}' = \bar{M}'\mathbf{R}^{-1}\bar{M}'\mathbf{R}^{-1}, \\ [\bar{M},\ M'] &=& 0\,, \end{array} \tag{5.1}$$

where as usual $M = M_1$ and $M' = M_2$ etc. Then, one can prove that the operators

$$\begin{array}{c} T(z,\ \bar{z}) = exp(zM)Texp(\bar{z}\bar{M}), \\ dT(z,\ \bar{z}) = dz(\partial_z T) + d\bar{z}(\partial_{\bar{z}} T) = dzMT + d\bar{z}T\bar{M} \end{array} \tag{5.2}$$

satisfy the commutation relations (2.8)-(2.10). The generators $\{e^i,\ (de^i)\}$ of the exterior algebra Ω_Z (2.1) for $c = q$ can be realized now as columns of the quantum matrices $T^i_j(z,\bar{z})$ and $dT^i_j(z,\bar{z})$. In this sense, we indeed can consider Eqs.(2.4),(2.5) as local co-transformations where $\{z,\ \bar{z}\}$ are coordinates of the space-time. We stress also that Eqs.(5.1) and (5.2) remind the formulas appeared in the framework of the Hamiltonian quantizing of the WZWN models (see e.g. [31] and references therein) and related toy model [32].

2. Another attractive possibility is the choice of noncommutative space-time isomorphic to the space of quantum group, e.g. $GL_q(N)$. In this case, it is tempting to explore monopole-instanton type gauge potential 1-forms

$$A^i_j = dT^i_k M^k_l(Z)(T^{-1})^l_j = dT^i_k(T^{-1})^l_j M^k_l(Z)\,, \tag{5.3}$$

where $Z = det_q T$ and $([M(Z),\ T] = 0,\ M(Z)\,dT = dT\,M(q^2Z))$. Substituting (5.3) in the anticommutation relations (2.23), we obtain that M satisfies the reflection equation

$$M(q^2Z)\mathbf{R}^{-1}M(Z)\mathbf{R}^{-1} - \mathbf{R}^{-1}M(q^2Z)\mathbf{R}^{-1}M(Z) = 0\,.$$

3. For arbitrary invertible Yang-Baxter R-matrix satisfying the characteristic equation (generalization of (2.3))

$$(\mathbf{R}-\lambda_1)(\mathbf{R}-\lambda_2)\cdots(\mathbf{R}-\lambda_m) = 0,\ \ (\lambda_i \neq \lambda_j\ \ if\ \ i \neq j) \tag{5.4}$$

one can introduce [33] a set of quantum hyperplanes and covariant differential calculi on them. Namely, for each eigenvalue λ_k we define the exterior algebra $\{e,\ (de)\}$ with the commutation relations [33]

$$\begin{array}{rcl} \prod_{j\neq k}\frac{(\mathbf{R}-\lambda_j)}{(\lambda_k-\lambda_j)}ee' &\equiv& \mathbf{P}_k ee' = 0 \\ \mathbf{R}(de)e' &=& -\lambda_k e(de)'\,, \\ \mathbf{R}(de)(de)' &=& \lambda_k(de)(de)', \end{array} \tag{5.5}$$

We choose two variants of the hyperplanes related to the eigenvalues λ_k and λ_i for which the projectors P_k and P_i are q-analogs of a symmetrizer and an antisymmetrizer (fermionic and bosonic hyperplanes). Then we deduce the commutation relations for T and dT substituting the transformations (2.4), (2.5) into these two variants of relations (5.5). Surprisingly, these relations coincide with the relations (2.8)-(2.10) for $\lambda_k\lambda_i = 1$ and, as it can be easily shown, such a differential complex is not consistent for $m > 2$, e.g., for quantum groups such as $SO_q(N)$ and $SP_q(2N)$ for which $m = 3$. Our conjecture is that the consistent differential complex for quantum groups with general R-matrices satisfying (5.4) can be represented in the form (cf. with formulas presented in [6])

$$\mathbf{R}TT' = TT'\mathbf{R}\,, \tag{5.6}$$

$$T(dT)' = \sum_{k,j=1}^{m}\alpha_{kj}\mathbf{P}_k(dT)T'\mathbf{P}_j - (dT)T'\,, \tag{5.7}$$

$$\sum_{k,j=1}^{m}\alpha_{kj}\mathbf{P}_k(dT)(dT)'\mathbf{P}_j = 0\,. \tag{5.8}$$

Here the differential d satisfies the undeformed graded Leibnitz rule, the coefficients $\alpha_{kj} = 0, 1$ $(k \neq j)$ and $\alpha_k \equiv \alpha_{kk}$ have to be fixed from the diamond condition (the unique lexicographic ordering of cubic monomials) for the algebra (5.6)-(5.8). In particular, one can deduce the following condition on α_k

$$[X(\Omega),\ \mathbf{R}] = 0$$

where $X(\Omega) = (1 - \sum_k \alpha_k\mathbf{P}'_k)\Omega_1 + \sum_{k,l}\alpha_k\alpha_l\mathbf{P}'_k\mathbf{P}_l\Omega_1\mathbf{P}_l\mathbf{P}'_k$. Note that the algebra (5.6)-(5.8) is an exterior Hopf algebra with the structure maps defined in (2.11).

4. Now we make some notes about Brzezinski theorem [16] and its application to the construction of quantum group covariant noncommutative geometry.

   Let $(\mathcal{A},\ \Delta,\ \mathcal{S}, \epsilon)$ be a Hopf algebra and $(\Gamma,\ d)$ - first order differential calculus on $\mathcal{A}$, where Γ is a space of 1-forms on $\mathcal{A}$, while d is a differential mapping which is nilpotent $d^2 = 0$ and satisfies graded Leibnitz rule. Denote the basic elements of $\mathcal{A}$ (including unity) as $\{t_i,\ t_0 = 1\}$ and define

$$t_i t_j = m^k_{ij} t_k\,, \tag{5.9}$$

$$\Delta(t_i) = \Delta_i^{kj} t_k \otimes t_j \,, \tag{5.10}$$

$$\mathcal{S}(t_i) = S_i^j t_j \,. \tag{5.11}$$

The comultiplication Δ is a homomorphic mapping for the algebra (5.9) and, therefore, we have the following condition on the structure constants:

$$\Delta_i^{kn} \Delta_j^{ql} m_{kq}^p m_{nl}^r = m_{ij}^k \Delta_k^{pr} \,. \tag{5.12}$$

Let us choose in Γ the basis of independent 1-forms $\{\omega_\alpha\}$ defined by the relations

$$dt_i = (\chi^\alpha)_i^j \omega_\alpha t_j \,. \tag{5.13}$$

where χ^α are some numerical matrices. Each element in Γ can be uniquely represented in the form $\sum a_\alpha \omega_\alpha$ or $\sum \omega_\alpha b_\alpha$, $(a_\alpha,\ b_\alpha \in \mathcal{A})$ and therefore we have to be able to commute $\{t_m\}$ and $\{\omega_\alpha\}$:

$$t_n \omega_\beta = (F_\beta^\alpha)_n^k \omega_\alpha t_k \,, \tag{5.14}$$

where

$$(F_\beta^\alpha)_n^k = \eta_{\gamma\beta} \left((\chi^\alpha)_r^l m_{nj}^r (\chi^\gamma)_l^j \delta_0^k - Tr(\chi^\gamma)(\chi^\alpha)_n^k \right),$$

$$(\eta_{\alpha\beta}\eta^{\beta\gamma} = \delta_\alpha^\gamma \text{ and } \eta^{\alpha\beta} = Tr(\chi^\alpha \chi^\beta))$$

are again some invertible numerical matrices. The corresponding commutation relations for the basis of 1-forms (in other words the definition of the exterior product $\omega \wedge \omega$) can be easily deduced by the differentiation of Eq.(5.14)

$$[\chi^\alpha \omega_\alpha \,,\ F_\beta^\gamma \omega_\gamma]_+ = \left(F_\beta^\alpha f_\alpha^{\gamma'\delta} - f_\beta^{\alpha\xi} F_\alpha^{\gamma'} F_\xi^\delta \right) \omega_{\gamma'} \omega_\delta \,. \tag{5.15}$$

One can guarantee that there are no other quadratic relations on ω_α since we choose these 1-forms as independent. We imply in Eq.(5.15) the exterior products of the differential forms and introduce structure constants $f_\gamma^{\alpha\beta}$ appeared in the Maurer-Cartan equation

$$d\omega_\alpha = f_\alpha^{\beta\gamma} \omega_\beta \wedge \omega_\gamma \,. \tag{5.16}$$

Comparing this relation with the differential of Eq.(5.13) one can express $f_\alpha^{\beta\gamma}$ in terms of the matrices χ^γ.

Relations (5.9), (5.14) and (5.15) are defining relations for the exterior algebra $\Omega = \bigoplus_{n=0} \Omega^{(n)}$ of $\mathcal{A}$. Here $\Omega^{(0)} = \mathcal{A}$, $\Omega^{(1)} = \Gamma$ and $\Omega^{(n)}$ denotes the space of n-forms. Now let us consider the mapping Δ': $\Omega \to \Omega \otimes \Omega$ where $\otimes$ is a graded tensor product and $\Delta'(\mathcal{A}) \equiv \Delta(\mathcal{A})$. Define the action of d on $\Omega \otimes \Omega$ as

$$d(\Omega^{(n)} \otimes \Omega^{(k)}) = d\Omega^{(n)} \otimes \Omega^{(k)} + (-1)^n \Omega^{(n)} \otimes d\Omega^{(k)}.$$

Our proposition is that if the mapping Δ' (coaction) commutes with d:

$$d\Delta' = \Delta' d \tag{5.17}$$

and the relations (5.14) are covariant under the coaction Δ', then the differential complex (5.9), (5.14) and (5.15) defines the exterior Hopf algebra of $\mathcal{A}$.

Proof: First, we note that from the condition (5.17) we obtain the explicit definition of Δ':

$$\begin{aligned} \Delta'(t_i) &= \Delta(t_i) \\ \Delta'(dt_i) &= d\Delta'(t_i) = \Delta_i^{kj}(dt_k \otimes t_j + t_k \otimes dt_j) \,. \end{aligned} \tag{5.18}$$

The coaction on the higher differential forms $\Omega^{(n)}$ can be derived from (5.18). From the covariance of the relations (5.14) it is not hard to show (applying the Leibnitz rule and condition (5.17)) that the relations (5.15) are also covariant under the coaction (5.18). The co-associativity of Δ

$$\Delta_i^{kj} \Delta_j^{ln} = \Delta_i^{jn} \Delta_j^{kl}$$

leads to the coassociativity of Δ' (5.18). Thus, Δ' is a coproduct for $\mathcal{A} \oplus \Gamma$ and, therefore, for Ω. Finally, we define the extended versions of the antipode $\mathcal{S}'$ and the counit ϵ' for the exterior algebra Ω by means of the relations

$$\begin{aligned} \mathcal{S}'(t_i) &= \mathcal{S}(t_i), \quad \mathcal{S}'(dt_i) = d\mathcal{S}(t_i), \\ \epsilon'(t_i) &= \epsilon(t_i), \quad \epsilon'(dt_i) = d\epsilon(t_i) = 0. \end{aligned} \tag{5.19}$$

All axioms for $\mathcal{S}'$ and ϵ' follow from the corresponding axioms for $\mathcal{S}$ and ϵ.

This proposition immediately implies Brzezinski's theorem [16] since the bicovariance for $(\Gamma,\ d)$ is nothing but the covariance of the relations (5.9), (5.14) and (5.15) with respect to the left Φ_L and right Φ_R coactions on $\mathcal{A} \oplus \Gamma$

$$\begin{aligned} \Phi_{L,R}(t_i) &= \Delta(t_i) \\ \Phi_L(dt_i) = \Delta_i^{kj} t_k \otimes dt_j \,, &\Phi_R(dt_i) = \Delta_i^{kj} dt_k \otimes t_j \,, \end{aligned} \tag{5.20}$$

and, therefore, relations (5.14) are also covariant under the coaction (5.18).

Now we consider the left coaction of the exterior Hopf algebra Ω on a left comodule represented by some exterior algebra Ω_Z:

$$\begin{aligned} x_\alpha &\to (C^i)_\alpha^\beta t_i \otimes x_\beta, \\ dx_\alpha &\to (C^i)_\alpha^\beta (dt_i \otimes x_\beta + t_i \otimes dx_\beta). \end{aligned} \tag{5.21}$$

Here $\{x_\alpha,\ dx_\alpha\}$ are generators of Ω_Z and matrices C^i represent the dual object: $(C^i)_\alpha^\beta (C^j)_\beta^\gamma = \Delta_k^{ij} (C^k)_\alpha^\gamma$. If we extend the algebra $\Omega_Z \to \Omega_Z$ by adding new generators A_α^β such that $A_\alpha^\beta \in \Omega_Z^{(1)}$ and introduce a new differential $\nabla x_\alpha = dx_\alpha - A_\alpha^\beta x_\beta$ transformed covariantly under (5.21):

$$\nabla x_\alpha \to (C^i)_\alpha^\beta t_i \otimes \nabla x_\beta, \tag{5.22}$$

then we interpret A_α^β as connection 1-forms. The definition of the curvature 2-forms is evident. One can try to construct the cross-product of the algebras Ω and $\Omega_{\bar{Z}}$ and obtain a new exterior Hopf algebra G for which Ω will be a Hopf subalgebra. In this case A_α^β could be realized as right-covariant 1-forms on G. Just this realization has been done in Sect.3 where $\Omega \equiv \Omega_{GL_q(N)}$ and $G \equiv \Omega_{GL_q(N+1)}$. So we see that, in principle, the algebraical constructions of Sections 2 and 3 could be adapted to the case of arbitrary exterior Hopf algebra.

5. Finally, we would like to stress that there are many variants of quantum group covariant commutation relations for connections, curvatures, veilbeins etc. For each variant (and for the same quantum group of covariance) one can obtain different noncommutative geometries. Therefore, the structure co-group (the group of covariance) does not define noncommutative geometry uniquely. Indeed, we can embed the structure quantum group Ω in various large algebras G and correspondingly to obtain various geometrical structures. For example, one can consider the embedding of the structure group $\Omega = \Omega_{GL_q(N)}$ in the arbitrary group $\Omega_{GL_q(M)}$ for $M > N+1$. Obviously this will be the generalization of noncommutative geometry for $M = (N+1)$ presented in Sect.3.

It seems that all these ideas are very closely related to the concept of noncommutative geometry on the quantum principal fibre bundles [11]. However, we stress that we have not done sequential analyses of these relations. It would be very interesting to interpret quantum group covariant noncommutative geometries as geometries on noncommutative principal fibre bundles.

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