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FROM ANTIBRACKET TO EQUIVARIANT CHARACTERISTIC CLASSES

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1 Introduction

Equivariant cohomology (see e. g. [1]-[3]) is presently attracting much interest in physics. It is stimulated by the applications of localization formulas to evaluation of path integrals in various field-theoretical models, *e. g.* topological field theories [4], [5], and supersymmetric [6] theories. In series of papers (see e. g. [7], [8] and refs. therein) A. Niemi et al. developed new approach for the evaluation of path integrals for the wide class of Hamiltonian systems based on the localization formulas.

The field-theoretical models are usually described by the degenerate Lagrangians. So it is necessary to construct the localization formulas on the surfaces in manifolds. In fact, this problem is equivalent to construction of equivariant cohomology classes on the surfaces. The equivariant Cartan and Euler classes are known (for a review see, [3]).

In this paper we construct more general family of equivariant generalizations of the Euler classes on the surfaces.

For this purpose we firstly construct the odd symplectic structure on the space of differential forms $\Lambda(M)$ on the Riemann manifold M. The Lie derivative of this symplectic structure along the vector field, corresponded to S^1- equivariant transformation (where S^1- group action on M is defines by the Killing vector) is *S^1-equivariant even (pre)symplectic structure* (*Section 2*).

Its Poincare- Cartan like invariants [13] define the set of equivariant cohomology classes generalizing the known equivariant Euler classes [3] (*Section 3*).

Notice, that initial object in our considerations : the odd symplectic structure is used mainly in Batalin-Vilkovisky quantization scheme [9]. However, the recent investigations of its geometry [10] allow to assume that the odd symplectic structure plays the essential role in problems connected with the integration over (super)surfaces.

Notice also, that in [11] it was demonstrated that the odd symplectic structure, constructed on the space of a differential form on the symplectic manifold, naturally describes the equivariant cohomologies on symplectic manifold and establishes the correspondence of the equivariant cohomologies to the bi-Hamiltonian supersymmetric dynamics (with even and odd symplectic structures) [12].

2 Odd and Even Symplectic Structures

In this Section we construct the odd symplectic structure and then – the even S^1-invariant (pre)symplectic one on the space of differential forms on the Riemann manifold M.

Let (M,g) be Riemann manifold and ξ – its Killing vector defining S^1 action. Let $\Lambda(M)$ be the space of differential forms on M. It can be parametrized by the local coordinates $z^A = (x^i, \theta^i)$, where x^i denote the local coordinates on M and θ^i – the basic 1-forms dx^i, $p(\theta^i) = 1$.

Consider the vector fields $\hat{X}$ and $\hat{E}$ on $\Lambda(M)$:

$$\hat{X} = \xi^i \frac{\partial}{\partial x^i} + \xi^i_{,k}\theta^k \frac{\partial}{\partial \theta^i}, \quad \hat{E} = \xi^i \frac{\partial}{\partial \theta^i} + \theta^i \frac{\partial}{\partial x^i} : \quad [\hat{E}, \hat{E}]_+ = 2\hat{X}. \tag{2.1}$$

It is obvious that $\hat{X}$ corresponds to the Lie derivative of differential forms on M along ξ: $\hat{X} \to L_\xi$, and $\hat{E}$ corresponds to the ξ-equivariant (S^1-equivariant) differentiation on

M : $\hat{E} \to d_\xi = d + \imath_\xi$. The last expression in (2.1) corresponds to homotopy formula $L_\xi = d\imath_\xi + \imath_\xi d$.

Below we consider $\Lambda(M)$ as a supermanifold with the $z^A = (x^i, \theta^i)$ and denote by $\mathcal{L}$ and d correspondingly the Lie derivative and exterior differentiation on $\Lambda(M)$.

It is easy to see that Berezin integration on $\Lambda(M)$ leads to the integration of differential forms on M.

Let construct on $\Lambda(M)$ the odd symplectic structure taking in the coordinates (x^i, θ^i) the form

$$\Omega_1 = dx^i \wedge d(g_{ij}\theta^j) = g_{ij} dx^i \wedge D\theta^j, \quad D\theta^i = d\theta^i + \Gamma^i_{kl}\theta^k dx^l, \tag{2.2}$$

where Γ^i_{kl} are Cristoffel symbols for the metric g_{ij}.

The corresponding odd Poisson bracket (antibracket) is :

$$\{f, g\}_1 = g^{ij}(\nabla_i f \frac{\partial g}{\partial \theta^j} - \frac{\partial f}{\partial \theta^i}\nabla_j g), \quad \nabla_i = \frac{\partial}{\partial x^i} - \Gamma^j_{ik}\theta^k \frac{\partial_l}{\partial \theta^j}, \tag{2.3}$$

It satisfies the conditions :

$$\{f, g\}_1 = -(-1)^{(p(f)+1)(p(g)+1)}\{g, f\}_1 \quad (\text{"antisymmetricity"}),$$
$$(-1)^{(p(f)+1)(p(h)+1)}\{f, \{g, h\}_1\}_1 + \text{cycl.perm.(f,g,h)} = 0 \quad (\text{Jacobi id.}).$$

The odd symplectic structure (2.2) is $\hat{X}$-invariant , so $\hat{X}$ can be presented in the Hamiltonian form :

$$\hat{X} = \{Q_1, .\}_1, \quad \text{where} \quad Q_1 = \xi_i \theta^i. \tag{2.4}$$

But it is not $\hat{E}$-invariant:

$$\mathcal{L}_E \Omega_1 = \tilde{\Omega}_0 \neq 0.$$

Here

$$\Omega_0 = \frac{1}{2}(\xi_{i;j} + g_{in} R^n_{jkl}\theta^k\theta^l) dx^i \wedge dx^j + g_{ij} D\theta^i \wedge D\theta^j \tag{2.5}$$

(R^n_{jkl} is the curvature tensor on M), being E-invariant (i.e S^1- equivariant) even closed 2-form :

$$p(\Omega_0) = 0, \; d = \mathcal{L}_E d\Omega_1 = 0, \; \mathcal{L}_E \Omega_0 = \mathcal{L}_E \mathcal{L}_E \Omega_1 = 2\mathcal{L}_X \Omega_1 = 0.$$

Therefore, the vector fields (2.1) are Hamiltonian ones under Ω_0. The corresponding Hamiltonians are following

$$\mathcal{H} \equiv \mathcal{L}_E Q_1 = \xi^i g_{ij} \xi^j - \xi_{i;j}\theta^i\theta^j, \quad Q_2 = \xi^i \xi_{i;j}\theta^j. \tag{2.6}$$

The potential 1-form $\mathcal{A}$: $d\mathcal{A} = \Omega_0$ is

$$\mathcal{A} = \Omega_1(\hat{E}, ...) = \xi_i dx^i + \theta^i g_{ij} D\theta^j. \tag{2.7}$$

Thus, starting from the odd symplectic structure we constructed the S^1-equivariant even (pre)symplectic structure on the space of differential forms on a Riemann manifold.

3 Equivariant Characteristic classes

In this Section we construct the equivariant characteristic classes for the surfaces in $\Lambda(M)$.

Let $\Gamma \subset \Lambda(M)$ be a surface and Ω_0 be nondegenerate on it. Let Γ is parametrized by the equations $z^A = z^A(w)$, where w^μ are local coordinates of Γ.
Thus the following density is correctly define on Γ

$$\mathcal{D}_\Gamma(w) = \sqrt{\mathrm{Ber}\;\Omega_0|_\Gamma} \equiv \sqrt{\mathrm{Ber}\frac{\partial_r z^A}{\partial w^\mu}\Omega_{(0)AB}\frac{\partial_l z^B}{\partial w^\nu}}, \qquad (3.1)$$

and it is invariant under canonical transformations of the presymplectic structure (2.5) [13]. So, this density is S^1-equivariant too.
Hence, *the functional*

$$Z^\lambda(\Gamma, F) = \int_\Gamma e^{F-\lambda \hat{E}\Psi}\mathcal{D}_\Gamma[dw], \qquad (3.2)$$

where $F(z)$ and $\Psi(z)$ are correspondingly the even $\hat{E}$- and odd $\hat{X}$-invariant functions

$$\hat{E}F = 0, p(F) = 0, \quad \hat{X}\Psi = 0, p(\Psi) = 1, \qquad (3.3)$$

is S^1-equivariant for any compact Γ. Therefore it is λ-independent.

Let the 2-form (2.5) be nondegenerate one on $\Lambda(M)$ and the surface Γ is defined by the equations $f^a(z) = 0$, $a = 1, \ldots \mathrm{codim}\Gamma$. In this case the functional (3.2) can be presents in the form (compare with [14])

$$Z^\lambda(\Gamma, F) = \int_{\Lambda(M)} e^{F(z)-\lambda(\hat{E}\Psi)}\delta(f^a)\sqrt{\mathrm{Ber}\{f^a, f^b\}_0}\mathcal{D}_0 dz, \qquad (3.4)$$

where

$$\{f(z), g(z)\}_0 = \nabla_i f(z)(\xi_{i;j} + R_{ijkl}\theta^k\theta^l)^{-1}\nabla_j g(z) + \frac{1}{2}\frac{\partial_r f(z)}{\partial\theta^i}g^{ij}\frac{\partial_l g(z)}{\partial\theta^j}, \qquad (3.5)$$

is the Poisson bracket, corresponding to (2.5) and $\mathcal{D}_0(z) \equiv \mathcal{D}_{\Lambda(M)}(z) = \sqrt{Ber\Omega_{(0)AB}}$.
This functional is invariant both under reparametrization of $\Lambda(M)$, and choice of the restriction functions f^a.

The functional $Z^\lambda(\Gamma, 0)$ is invariant under smooth deformations of Γ (if basic manifold N of Γ is compact one) [14], i. e. it is a topological invariant of Γ.
For example, $Z^\lambda(\Lambda(M), 0)$ coincide with the Euler number of M. In the limit $\lambda \to 0$ it gives the Poincare-Hopf formula, and in the limit $\lambda \to \infty$ (where we substitute $\Psi = Q_1 = \xi_i\theta^i$) – Gauss–Bonnet one for the Euler number of M [8].
Therefore, (3.1) defines the S^1-equivariant characteristic class of Γ.

Example : Let $\Gamma \subset \Lambda(M)$ is associated with the vector bundle $V(N)$: $V(N) \subset T(M), N \subset M$.
Let it is parametrized by the equations :

$$x^i = x^i(y^a), \quad \theta^i = P^i_\alpha(y)\eta^\alpha, \qquad (3.6)$$

where $w^\mu = (y^a, \eta^\alpha)$ are local coordinates of Γ, $p(y) = 0, p(\eta) = 1$ (y^a are local coordinates of N) .
Thus

$$\Omega_0|_\Gamma = \frac{1}{2}(\xi_{[a,b]} + g_{\alpha\delta}R^\delta_{\beta ab}\eta^\alpha\eta^\beta)dy^a \wedge dy^b + g_{\alpha\beta}D\eta^\alpha \wedge D\eta^\beta. \qquad (3.7)$$

Here we introduced the notations :

$$\xi_{[a,b]} = \xi_{i;j}\frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}; \quad g_{\alpha\beta} = P^i_\alpha g_{ij}P^j_\beta; \quad D\eta^\alpha = d\eta^\alpha + A^\alpha_{a\beta}\eta^\beta dy^a,$$

where

$$A^\alpha_{a\beta} = g^{\alpha\delta}P^i_\delta g_{ij}\left(P^j_{\beta,a} + \Gamma^j_{lk}P^k_\beta\frac{\partial x^l}{\partial y^a}\right)$$

is the induced connection on $V(N)$ (compatible with $g_{\alpha\beta}$) and $R^\delta_{\beta ab}$ is its curvature tensor.
Hence

$$\mathcal{D}_\Gamma(w) = \sqrt{\mathrm{Ber}\Omega_0|_\Gamma} = \left(\frac{\det(\xi_{[a,b]} + g_{\alpha\delta}R^\delta_{\beta ab}\eta^\alpha\eta^\beta)}{\det g_{\alpha\beta}}\right)^{\frac{1}{2}} \qquad (3.8)$$

defines the family of equivariant characteristic classes of N.

In the case $\Gamma = \Lambda(N)$, i. e. $P^i_\alpha = \frac{\partial x^i}{\partial y^a}$, (3.8) coincides with known expression of equivariant Euler classes on N [3], [8].

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